

# Combination of exhaustive search and continuation method for the study of sinks in the Hénon map

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**Abstract**—The problem of existence of stable periodic orbits (sinks) for the Hénon map in a neighborhood of classical parameter values is studied numerically. Several parameter values which sustain a sink are found. It is shown rigorously that the sinks exist. Regions of existence in the parameter space of the sinks are located using the continuation method.

## I. INTRODUCTION

The Hénon map [1] is a two-parameter, invertible map

$$h(x, y) = (1 + y - ax^2, bx), \quad (1)$$

displaying a wide array of dynamical behaviors as its parameters are varied. For the classical parameter values,  $a = 1.4$ ,  $b = 0.3$  the so-called Hénon attractor is observed (see Fig. 1).

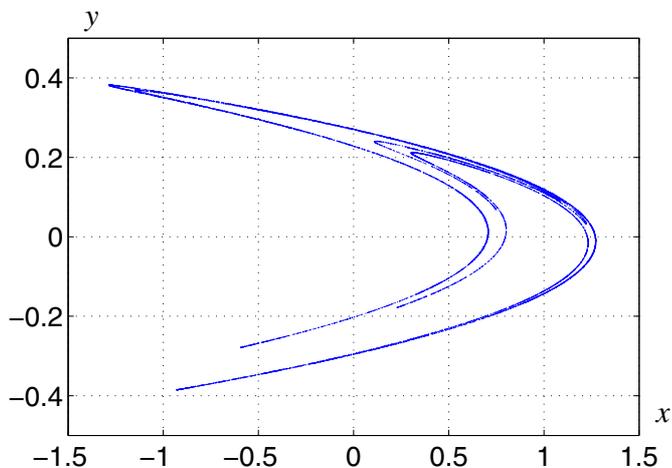


Fig. 1.  $a = 1.4$ ,  $b = 0.3$ , trajectory of the Hénon map composed of 10000 points

The Hénon map is a prominent example of a chaotic map and has many potential applications in engineering [2], [3]. In these applications it is quietly assumed that the map is chaotic (trajectories belonging to the attractor are aperiodic and sensitive to initial conditions). However, the question whether the Hénon attractor is indeed chaotic remains open. If the Hénon map is chaotic, we will perhaps be never able to prove this fact. It is known [4] that there is a set of parameters (near  $b = 0$ ) with positive Lebesgue measure for which the

Hénon map has a strange (chaotic) attractor. However, this set is believed to be densely filled with regions, where the attractor is periodic. It means that given a specific point  $(a, b)$  in the parameter space, it is not possible to prove that the dynamics of the map generates a strange attractor.

The second possibility is that the Hénon attractor is periodic, i.e. that what we observe in computer simulations is actually a transient behaviour to the periodic steady state or that we observe a periodic orbit with a very long period. If this second possibility is true then theoretically, we are able to prove this fact. Proving the existence of a sink involves finite computations, and all necessary conditions are robust (there exists an open set in the parameter space in which all conditions remain true).

In the setting we are considering, it is known [5] that when a saddle point generates a homoclinic intersection, a cascade of (periodic) sinks will occur. Furthermore, there will be nearby parameter values for which the map has infinitely many coexisting sinks. Nevertheless, coexisting sinks for the Hénon map appear to be very elusive; due to the dissipative nature of the map, the regions in the parameter space for which sinks appear simultaneously are extremely small.

In this work, we report results of the numerical search for parameter values — close to the classical ones — for which there exists a sink. We locate a number of such parameter values in a neighborhood of  $(a, b) = (1.4, 0.3)$ . Our study shows that close to the classical case, the regions of existence of sinks are very narrow and finding them is not a trivial numerical task. We present examples confirming that in many cases where there appears to be a strange attractor, the true underlying dynamics is in fact governed by a periodic sink. Using the continuation method, the regions of existence of found sinks are extended, which makes it possible to move closer to the classical case.

## II. LOCATING SINKS IN THE PARAMETER SPACE

The most natural method to find a sink is to follow a trajectory and monitor whether it converges to a periodic orbit. First, a number of iterates are computed in the hope that a trajectory reaches a steady-state. The number of iterations which are discarded is usually chosen by trial-and-error. It depends on Lyapunov exponents along the orbit, the size and

shape of its basin of attraction. Next, we take the current iterate as the new initial point and check if the trajectory periodically returns very close to this point.

Once an approximate position of a sink is known, to be sure that the orbit exists we have to prove its existence. It is possible that an observed periodic behaviour is an artifact caused by rounding errors. The existence can be proved using methods from interval analysis [6], [7]. These methods allow us to obtain rigorous results using computers. Interval methods provide simple tests for the existence and uniqueness of zeros of a map within a given interval vector. To investigate zeros of  $F$  in the interval vector  $\mathbf{v}$  one evaluates an interval operator, for example the interval Newton operator [7], over  $\mathbf{v}$ :

$$N(\mathbf{v}) = \hat{v} - F'(\mathbf{v})^{-1}F(\hat{v}), \quad (2)$$

where  $\hat{v} \in \mathbf{v}$ , and  $F'(\mathbf{v})$  is an interval matrix containing the Jacobian matrices  $F'(v)$  for all  $v \in \mathbf{v}$ . The main theorem on the interval Newton operator states that if  $N(\mathbf{v}) \subset \mathbf{v}$ , then  $F$  has exactly one zero in  $\mathbf{v}$ .

In order to study the existence of period- $p$  orbits of  $h$ , we construct the map  $F$  defined by  $[F(v)]_k = z_{(k+1) \bmod p} - h(z_k)$  for  $0 \leq k < p$ , where  $v = (z_0, z_1, \dots, z_{p-1})$ . It is clear that  $v$  is a zero of  $F$  if and only if  $z_0$  is a fixed point of  $h^p$ . To prove the existence of a periodic orbit in a neighborhood of the computer generated trajectory  $v = (z_0, z_1, \dots, z_{p-1})$ , we choose the radius  $r$ , construct the interval vector  $\mathbf{v} = (\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_{p-1})$ , where  $\mathbf{z}_k = [z_k - r, z_k + r]$  and verify whether  $N(\mathbf{v}) \subset \mathbf{v}$ . If the existence condition does not hold we may choose different  $r$  and try again (see [8] for details). This method combined with the bisection technique has been used to find all short periodic orbits for discrete systems including the Hénon map [9] and continuous systems [8], [10].

Stability of the orbit  $v = (z_0, z_1, \dots, z_{p-1})$  depends on the eigenvalues  $\lambda_i$  of the Jacobian matrix  $J = (h^p)'(z_0) = h'(z_{p-1}) \cdots h'(z_1) \cdot h'(z_0)$ . If all eigenvalues lie within the unit circle, i.e.  $|\lambda_i| < 1$  then the orbit is asymptotically stable. If at least one eigenvalue lies outside the unit circle ( $|\lambda_i| > 1$ ) then the orbit is unstable.

When a point in the parameter space with a sink is found one may use the continuation method to find a connected region in the parameter space for which this sink exists. The border of the region is defined by two conditions. The first condition is that the periodic orbit exists, and the second one is that one of the eigenvalues of the Jacobian matrix has the absolute value 1. Note that it is not necessary to run long computations to find the steady state for new test points in the parameter space. The position of the sink for a new test point is found using the standard (real valued) Newton method started at the position of the orbit for the current point in the parameter space. This method works because positions of periodic orbits change continuously with the parameters.

In order to find the border of the existence region, first, we continue along a straight line, starting from the point for which the existence of the sink has been verified. This gives us two points belonging to the border of the existence region. For each of the two points we use the simplex continuation method [11].

In this method, a sequence of triangles is constructed such that each triangle has non-empty intersection with the border. Corners of the triangles are located on a regular grid. As the first triangle we choose the one containing the border point found in the initial step. Assuming that two edges of this triangle has nonempty intersection with the existence region we continue in two directions. In each direction, the initial triangle is replaced by the triangle defined by two corners of the chosen edge, and the third corner is a grid point located symmetrically to the third point of the initial triangle. This process is repeated and a sequence of triangles containing the border is found.

Even if there is a single attractor for the system, it may take quite a long time for a trajectory to converge to it. In this context, a very important notion is the so-called *immediate basin size* of the attractor. It is defined as the largest number  $r_\varepsilon$  such that all trajectories starting closer than  $r_\varepsilon$  from the attractor do not escape further than  $\varepsilon$  from the attractor, and converge to it. If the immediate basin size of an attractor is very small then it will usually take very long time to converge to it. What we observe in such a case is a very long transient, which sometimes is misidentified as a chaotic trajectory.

### III. RESULTS

In order to locate sinks for parameter values close to the classical ones  $(a, b) = (\bar{a}, \bar{b}) = (1.4, 0.3)$ , we carried out the following search in the parameter space. The fixed value of the parameter  $b$  was used ( $b = 0.3$ ).  $n_{\text{par}} = 10^6$  values of the parameter  $a$  from the interval [1.3999, 1.4001] were chosen. For each parameter value,  $n_{\text{init}} = 5000$  random initial points were selected, and for each initial point the search for a sink was performed. In the search procedure  $n_{\text{skip}} = 10^5$  iterations were computed and skipped to reach the steady state and the following  $n_{\text{iter}} = 10^4$  iterations were used to verify whether the trajectory in the steady state is periodic.

We found  $n_{\text{sink}} = 57$  (out of  $n_{\text{par}} = 10^6$ ) parameter values with a sink. When for two adjacent parameter values a sink with the same period is detected, we consider that these two parameter values belong to the same periodic window. In these computations,  $n_{\text{win}} = 11$  periodic windows have been found.

Note that during the whole search procedure for each of the  $10^6$  values of  $a$  considered, more than  $10^9$  iterations of the Hénon map were computed. When the number of iterations was too low, the number of parameter values with a sink detected was significantly smaller.

TABLE I  
COMPUTATION DETAILS OF SEARCH FOR SINKS FOR  $a$  CLOSE TO 1.4  $b = 0.3$

$a$	$n_{\text{par}}$	$n_{\text{init}}$	$n_{\text{skip}}$	$n_{\text{iter}}$	$n_{\text{sink}}$	$n_{\text{win}}$
[1.3999, 1.4001]	$10^6$	5000	$10^5$	$10^4$	57	11
[1.39999, 1.40001]	$10^6$	1000	$5 \cdot 10^5$	$10^4$	8	3
[1.399999, 1.400001]	$10^6$	1000	$10^5$	$10^4$	0	0
[1.399999, 1.400001]	$2 \cdot 10^6$	2000	$10^5$	$10^4$	1	1
[1.3999999, 1.4000001]	$10^6$	2000	$10^5$	$10^4$	0	0

Similar computations were performed for various intervals centered at the point  $a = 1.4$ . Descriptions of the tests carried

out and the number  $n_{\text{sink}}$  of parameter values for which a sink was found and the number  $n_{\text{win}}$  of periodic windows are collected in Table I.

Let us note that for the interval  $[1.3999, 1.4001]$  of length  $2 \cdot 10^{-4}$  we found 57 parameter values with a sink. There are 8 such parameter values for the interval of length  $2 \cdot 10^{-5}$ , only 1 for the interval of length  $2 \cdot 10^{-6}$ . In fact, this sink was only found when the number of test points was increased to  $2 \cdot 10^6$ ; for  $10^6$  test points no sinks were found (compare Table I). For the interval of length  $2 \cdot 10^{-7}$  no sinks were found.

Before carrying out the computation, we tested whether one should compute one very long trajectory or many shorter trajectories starting from random initial conditions. The tests showed that what matters is the total number of iterations (i.e.  $n_{\text{init}} \cdot n_{\text{skip}}$ ). The search procedure with many randomly chosen initial conditions was used because it was easier to parallelize.

Results on periodic windows found are shown in Table II. We report the value of the parameter  $a$  for which the existence of a sink was proved, the period  $p$  of the sink, the width  $d$  of the periodic window, the immediate basin size  $r_{\varepsilon}$ , and the largest Lyapunov exponent  $\lambda_1$ .

TABLE II  
PERIODIC WINDOWS FOR  $a$  CLOSE TO 1.4  $b = 0.3$

$a$	$p$	$d$	$r_{\varepsilon}$	$\lambda_1$
1.399922051	25	$5.522 \cdot 10^{-12}$	$2.473 \cdot 10^{-12}$	-0.00132
1.39997174948	30	$1.354 \cdot 10^{-11}$	$3.561 \cdot 10^{-12}$	-0.01887
1.3999769102	18	$3.207 \cdot 10^{-09}$	$1.014 \cdot 10^{-09}$	-0.05306
1.39998083519	24	$1.703 \cdot 10^{-11}$	$7.384 \cdot 10^{-12}$	-0.02819
1.399984477	20	$8.875 \cdot 10^{-10}$	$4.076 \cdot 10^{-10}$	-0.05099
1.39999492185	22	$3.686 \cdot 10^{-11}$	$1.531 \cdot 10^{-11}$	-0.09600
1.3999964733062	39	$2.784 \cdot 10^{-13}$	$1.115 \cdot 10^{-13}$	-0.03547
1.399999486944	33	$1.110 \cdot 10^{-12}$	$6.901 \cdot 10^{-13}$	-0.01843
1.40000929916	25	$1.118 \cdot 10^{-11}$	$5.128 \cdot 10^{-12}$	-0.08379
1.4000227433	21	$2.262 \cdot 10^{-10}$	$7.901 \cdot 10^{-11}$	-0.05612
1.40002931695	27	$5.782 \cdot 10^{-11}$	$2.646 \cdot 10^{-11}$	-0.01140
1.40006377472	27	$8.692 \cdot 10^{-11}$	$3.810 \cdot 10^{-11}$	-0.05636
1.40006667358	24	$6.278 \cdot 10^{-11}$	$2.646 \cdot 10^{-11}$	-0.01112
1.4000843045	27	$9.400 \cdot 10^{-11}$	$4.572 \cdot 10^{-11}$	-0.06870
1.40009110518	22	$3.493 \cdot 10^{-11}$	$1.531 \cdot 10^{-11}$	-0.02157
1.4000967515	26	$2.463 \cdot 10^{-10}$	$1.365 \cdot 10^{-10}$	-0.13233

First, let us note that widths of periodic windows are in some cases extremely small. In general, the windows for longer orbits are smaller. The smallest width  $d = 2.784 \cdot 10^{-13}$  corresponds to the period-39 sink, the longest one detected. This shows that it is necessary to make a very fine sampling of the parameter space to find periodic windows corresponding to longer orbits.

The periodic window of the period-33 sink containing the point  $a = 1.399999486944$  is the one closest to classical parameter values. The distance is less than  $5.14 \cdot 10^{-7}$ .

Fig. 2(a) shows a trajectory of the Hénon map with  $a = 1.399999486944$ ,  $b = 0.3$  obtained by starting at the initial condition  $(x, y) = (0.1, 0.1)$ , skipping  $5 \cdot 10^9$  iterations and plotting the next 10000 iterations. The plot looks like the Hénon attractor observed for  $a = 1.4$ ,  $b = 0.3$ . In spite of the fact that a huge number of iterations have been skipped the trajectory has not reached the steady state yet. The steady

state — a period-33 sink — obtained after  $6 \cdot 10^9$  iterations is shown in Fig. 2(b). This example shows that it might be necessary to wait very long until the steady state is observed. This long transient is related to the very small immediate basin size of the sink. A chaotic transient has very low probability of falling into the immediate basin, although in case of a single attractor, eventually it will happen with probability one.

It follows that what we observe in many examples reported in the literature, and what is claimed to be a chaotic trajectory, might in fact be a transient to a periodic steady state.

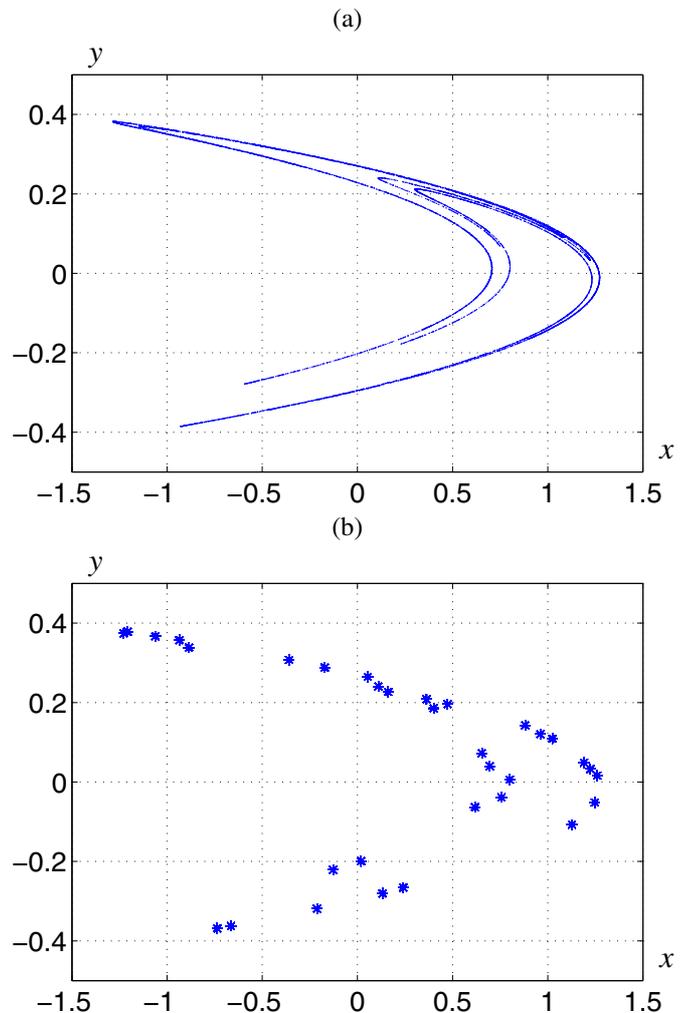


Fig. 2.  $a = 1.399999486944$ ,  $b = 0.3$ , trajectory of the Hénon map composed of 10000 points (a)  $5 \cdot 10^9$  iterations skipped, (b)  $6 \cdot 10^9$  iterations skipped

Clearly, the convergence time depends on the initial point. For example, when the initial point is  $(x, y) = (0, 0)$ , the convergence time is  $1.63 \cdot 10^8$ . To compute the average convergence time we performed the following test. 100000 random initial conditions were selected, and in each case the convergence time was recorded. The shortest convergence time was 26971. In two cases the convergence time exceeded  $10^{11}$ . From this data set one can compute the number of iterations  $n_{\text{conv}}(p)$  which are required to converge to the sink with probability  $p$ . For example  $n_{\text{conv}}(0.5) \approx 1.66 \cdot 10^9$ ,  $n_{\text{conv}}(0.9) \approx$

$5.51 \cdot 10^9$ . The number  $N_k$  of initial points with the convergence time in the interval  $[2^{k-1}, 2^k)$  is shown in Fig. 3. The maximum is for  $k = 32$ , which means that most trajectories converge after  $[2^{31}, 2^{32})$  iterations.

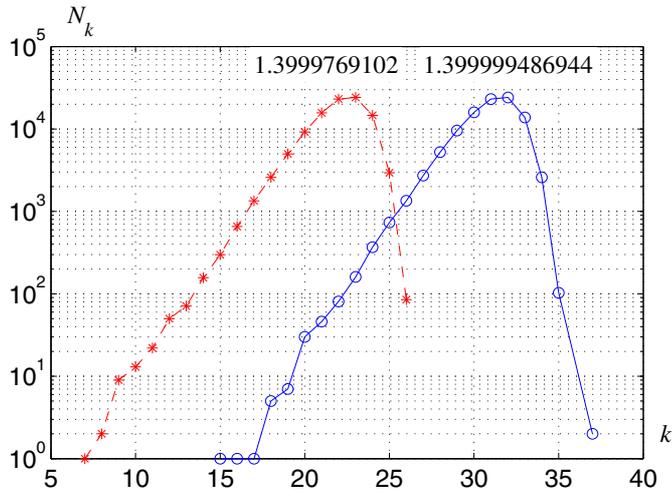


Fig. 3. The number  $N_k$  of random initial points with the convergence time in the interval  $[2^{k-1}, 2^k)$ , (a)  $a = 1.399999486944$  with a period-33 sink, the symbol “○”, (b)  $a = 1.3999769102$  with a period-18 sink, the symbol “\*”

Similar computations were performed for the parameter value  $a = 1.3999769102$ , for which a period-18 sink exists. In this case, the shortest and the longest convergence times are 82 and 61392634 iterations, respectively. The convergence probabilities  $n_{\text{conv}}(0.5) \approx 3.35 \cdot 10^6$ ,  $n_{\text{conv}}(0.9) \approx 1.11 \cdot 10^7$  are almost 500 times smaller than in the previous case. This is a consequence of a much larger immediate basin size,  $1.014 \cdot 10^{-9} > 6.901 \cdot 10^{-13}$  (compare Table II). Results on convergence times are shown in Fig. 3. The plot is similar in shape to the plot for  $a = 1.399999486944$ , however, it is shifted towards lower values of  $k$ . The maximum is for  $k = 23$ .

Using the continuation procedure presented in Section II, we found the existence region of the period-18 sink. The results are shown in Fig 4. The region is very thin. Its average width is approximately  $3 \cdot 10^{-9}$ . It was proved that the point  $(a, b) = (1.3999945292, 0.2999901796)$  belongs to this region. The Euclidean distance between this point and the point of classical parameter values  $(\bar{a}, \bar{b}) = (1.4, 0.3)$  is  $d \approx 1.12 \cdot 10^{-5}$  which is considerably smaller than the distance  $d \approx 2.31 \cdot 10^{-5}$  from the point, where the continuation procedure was initiated.

A local existence region was also found for the period-33 sink (compare Fig 4). It was verified that the point  $(a, b) = (1.399999566171, 0.299999814578)$  belongs to this region. The Euclidean distance between this point and the point  $(\bar{a}, \bar{b})$  is less than  $4.72 \cdot 10^{-7}$ .

#### IV. CONCLUSION

Several regions in the parameter space, for which the Hénon map has a sink, were located. It was shown that there exist low-period sinks extremely close to the classical case. The number of iterations which should be computed to ensure that with a certain probability a trajectory converges to a sink has

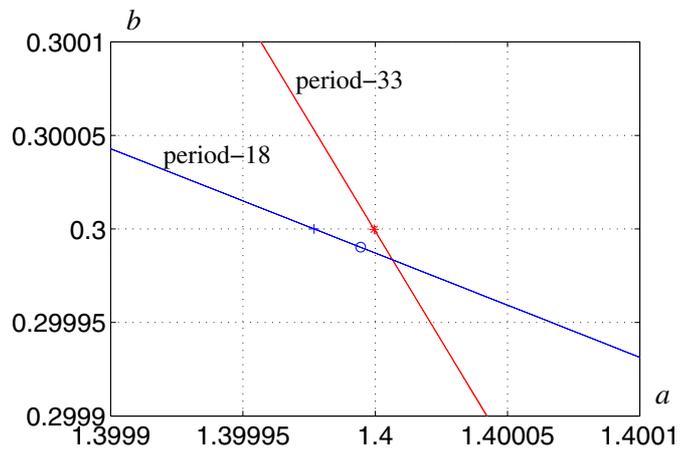


Fig. 4. Local existence regions of period-18 and period-33 sinks found using the continuation method; period-18 sink: “+” — the continuation method initial point, “○” — the point with the distance  $1.12 \cdot 10^{-5}$  from  $(\bar{a}, \bar{b})$ ; period-33 sink: “\*” — the point with the distance  $4.72 \cdot 10^{-7}$  from  $(\bar{a}, \bar{b})$

been estimated. We have shown that the convergence time could be very large. Based on these results we conclude that what in many research papers is claimed to be a chaotic trajectory of the Hénon map might be in fact a transient to a periodic steady state. In future, we plan to focus on multiprecision computations with the goal of finding longer sinks for parameter values even closer to the standard case.

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