Necessary and sufficient condition for the global stability of a delayed discrete-time single neuron model

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Abstract

We consider the global asymptotic stability of the trivial fixed point of the difference equation $x_{n+1} = mx_n - \alpha\varphi(x_{n-1})$, where $(\alpha, m) \in \mathbb{R}^2$ and φ is a real function that satisfies $0 \le x\varphi(x) \le x^2$ for all $x \in \mathbb{R}$. We show that $(\alpha, m) \in (|m| - 1, 1/(1 + |m|)) \times (-1, 1)$ is a sufficient condition for the global asymptotic stability of 0. As our main result, we prove that if $\varphi(x) \equiv \tanh(x)$, then the condition $(\alpha, m) \in [|m| - 1, 1] \times [-1, 1]$, $(\alpha, m) \neq (0, -1), (0, 1)$ is necessary and sufficient for global asymptotic stability.

Keywords: global stability; rigorous numerics; Neimark–Sacker bifurcation; strong resonance; graph representations; interval analysis; neural networks

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1 Introduction

Consider the difference equation given by

$$x_{n+1} = mx_n - \alpha\varphi(x_{n-1}), \tag{1.1}$$

where$(\alpha, m) \in \mathbb{R}^2$ and φ is a real function which satisfies

$$0 \le x\varphi(x) \le x^2 \text{ for all } x \in \mathbb{R}. \tag{1.2}$$

This means visually that the graph of $\varphi(x)$ is between the x-axis and the line $y = x$. This situation is depicted on Figure 1. Note that (1.2) implies, that φ is continuous at 0, $\varphi(0) = 0$ and $0 \le \liminf_{x\to 0} \frac{\varphi(x)-\varphi(0)}{x} \le \limsup_{x\to 0} \frac{\varphi(x)-\varphi(0)}{x} \le 1$.

Equation (1.1) can be interpreted as a discrete-time single neuron model with delay or as a discrete-time version of the Krisztin–Walther equation [16], as well. For a comprehensive description of the global dynamics of the delayed, continuous Krisztin–Walther equation see papers of Cao, Krisztin and Walther [4, 14, 15, 16] and the monograph of Krisztin, Walther and Wu [17].

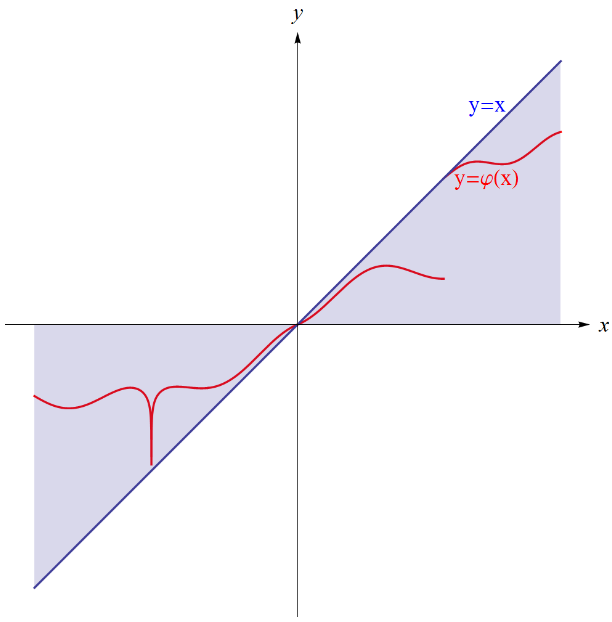

Figure 1: The graph of $\varphi(x)$ lies between the x-axis and the line $y = x$

We consider (1.1) as the two dimensional map $F_{\alpha,m} : (x,y) \mapsto (y, my - \alpha\varphi(x))$ that has a fixed point in the origin. We shall investigate the global asymptotic stability (GAS) of this fixed point. Besides of that from a mathematical point of view, global stability of a unique equilibrium point is always a fundamental topic, in neural networks it is also important in solving optimization and signal processing problems. We have applied the method that was developed to analyze the two dimensional Ricker map in our previous work [3]. The main goal of this paper is to demonstrate, on a special case of (1.1), that this technique is easily applicable for other problems. In addition, we give a new sufficient condition for the global asymptotic stability of the trivial fixed point of equations of the class (1.1) in Theorem 3.6.

There is a vast number of papers giving sufficient conditions for global stability of more complicated models of neural networks (see in [6, 11, 20, 28, 29] and the references therein, without attempting to be comprehensive), but to the best of our knowledge, none of those are claimed to be necessary. We also call the reader's attention to the monograph of Kocić and Ladas [13] (and in particular to Section 2.1) in which the authors present some very interesting results on the global stability of the trivial fixed point of a delay difference equation, similar to (1.1). Although the models in the aforementioned works are more general in some sense, their results do not apply directly to (1.1). Moreover, Theorem 3.6 yields new parameter regions of global stability for our equation (1.1) even in those cases when the above mentioned general results may be applied.

In our main result, Theorem 4.3 we give a necessary and sufficient condition for the global stability of our model difference equation

$$x_{n+1} = mx_n - \alpha \tanh(x_{n-1}), \tag{1.3}$$

with the same assumptions on m and α as in the general case (1.1). The tanh function is one of the most common examples for a sigmoid-type feedback function occurring in neural network models.

We shall see that it suffices to concentrate on parameter values satisfying $(\alpha, m) \in [0,1]^2$. Figure 2 gives an overview of this region. Note that the formerly mentioned results may be applied for equation (1.3). The statement is elementary to prove on the triangle marked with A and is also a consequence of e.g. [11]. The theorem of Kocić and Ladas [13] establishes that the fixed point is GAS in the triangle labelled with B. Theorem 3.6 will cover the area marked with C. Finally, we deal with D and E using computational tools in the proof of Theorem 4.3 presented in Section 4.

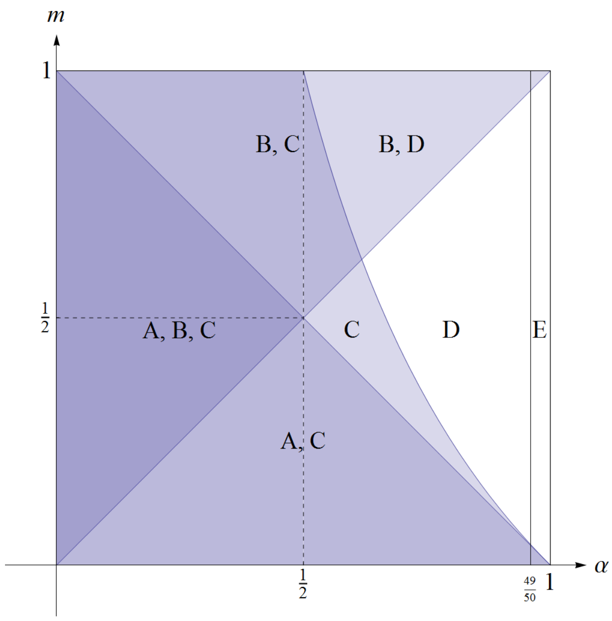

Figure 2: The parameter region $(\alpha, m) \in [0,1]^2$

Our proof is a combination of analytical and computer-aided tools and is based on a technique presented in our previous work [3]. The term computer-aided refers to that we do our calculations using a computer program that gives validated results, every possible numerical error is controlled. This allows us to prove *mathematical theorems* from the obtained outputs. For more information about computer-aided proofs and rigorous numerics, the reader is referred to Moore [21], Alefeld [2], Tucker [25, 26], and Nedialkov et al. [22]. We shall use graph representations, whereas we model our function on a grid, resulting in a directed graph. This concept has been utilized both in rigorous and non-rigorous computations for analyzing maps by Dellnitz, Hohmann and Junge [8, 9], Galias [10], Luzzatto and Pilarczyk [19] and for studying the attractor of a differential equation by Wilczak [27].

The article has the following structure. Section 2 contains the definitions and notations used in this paper. In Section 3 we give the proof of Theorem 3.6, and in addition, we construct two invariant and attracting sets with a compact intersection S, having $(\alpha, m) \in [0,1]^2$. To do the latter, we assume that $\varphi(x)$ is bounded and continuous, which is satisfied by our model equation (1.3). In Section 4 we turn our attention to (1.3), and prove that the trivial fixed point is GAS for the parameter values $(\alpha, m) \in [\frac{1}{m+1}, 1] \times [0,1]$. We do this by showing the property on the regions marked with D and E on Figure 2. Combining this with Theorem 3.6, Remarks 4.1 and 4.2 completes the proof of Theorem 4.3. As the first part of the proof, we derive a compact neighbourhood of the origin that lies entirely in the basin of attraction of $(0,0)$. It is an important feature of this neighbourhood $U(\alpha)$ that for every parameter value $(\alpha,m) \in [\alpha] \times [m] \subseteq [\frac{1}{m+1},1] \times [0,1]$, where $[\alpha]$ and $[m]$ are intervals, it contains the same closed disc around the origin, thus its complement is uniformly bounded away from zero. As the system undergoes a Neimark–Sacker bifurcation at $\alpha = 1$ and a strong 1:4 resonance occurs at $(\alpha,m) = (1,0)$, we obtain $U(\alpha)$ by studying the corresponding normal form and the linearized equation at the fixed point. We finish our proof in Part II by using a rigorous computer program that analyzes different graph representations of the two dimensional map corresponding to (1.3). The computations show that for any given pair $(\alpha,m) \in [\frac{1}{m+1},1] \times [0,1]$, every trajectory starting from S, that was constructed in Section 3, eventually enters the compact neighbourhood $U(\alpha)$, thus the origin is globally attracting.

Our further research interests in the topic include the application of our method for higher dimensional maps. This should involve a center manifold reduction, that gives birth to new technical challenges. A method for automatized generation of S, together with a recipe-like algorithm for finding $U(\alpha)$ is amongst our plans as well.

2 Definitions and notations

Let us define some notations that we shall use in this paper. We denote by $\mathbb{N}, \mathbb{N}_0, \mathbb{R}$ and $\mathbb{C}$ the set of positive integers, nonnnegative integers, reals and complex numbers respectively. The open ball in the maximum norm with radius $\delta > 0$ around $0 \in \mathbb{R}^n$ is denoted by K_δ. The open disk on the complex plain with radius $\delta > 0$ is denoted by $\mathrm{B}_\delta = \{z \in \mathbb{C} : \; |z| < \delta\}$, where $|z|$ denotes the absolute value of $z \in \mathbb{C}$. For $R \subseteq \mathbb{R}^2$, let $\mathrm{bd}(R)$ and $\mathrm{cl}(R)$ denote the topological boundary and the closure of the set R, respectively. It is unambiguous whether a vector in a formula is a row or a column vector, therefore we omit the usage of the transpose. For $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$ let $\langle \xi, \zeta \rangle$ denote the scalar product of them defined by $\langle \xi, \zeta \rangle = \overline{\xi_1}\zeta_1 + \overline{\xi_2}\zeta_2$. For a bounded function $\psi : \mathbb{R} \to \mathbb{R}$, let $M_\psi = \sup_{x\in\mathbb{R}} |\psi(x)|$. For a real function $\gamma : \mathbb{R} \to \mathbb{R}$ and for $c \in \mathbb{R}$, let $\delta_{\inf}(\gamma; c) := \liminf_{x\to c} \frac{\gamma(x)-\gamma(c)}{x-c}$ and $\delta_{\sup}(\gamma; c) := \limsup_{x\to c} \frac{\gamma(x)-\gamma(c)}{x}$.

Given a number or set X, by $[X]$ we denote an interval enclosure of X. With the usage of this notation, we emphasize always, that even though we might obtain $[X]$ from a computation, $X \subseteq [X]$ is always satisfied. Any subsequent computations will result in validated results due to the proper usage of interval analysis.

Consider the continuous map $f : \mathscr{D}_f \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. For $k \in \mathbb{N}_0$, f^k denotes the k-fold composition of f, i.e., $f^{k+1}(x) = f(f^k(x))$, and $f^0(x) = x$.

Definition 2.1. *The point $x^* \in \mathscr{D}_f$ is called a* fixed point *of f if $f(x^*) = x^*$. The point $q \in \mathscr{D}_f$ is a* non-wandering point *of f if for every neighbourhood U of q and for any $M \geq 0$, there exists an integer $m \geq M$ such that $f^m(U \cap \mathscr{D}_f) \cap U \cap \mathscr{D}_f \neq \emptyset$.*

A fixed point $x^* \in \mathscr{D}_f$ of f is called locally stable *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - x^*\| < \delta$ implies $\|f^k(x) - x^*\| < \varepsilon$ for all $k \in \mathbb{N}$, where $\|.\|$ denotes the Euclidean norm. We say that the fixed point x^** attracts *the region $U \subseteq \mathscr{D}_f$ if for all points $u \in U$, $\|f^k(u) - x^*\| \to 0$ as $k \to \infty$. The fixed point x^* is* globally attracting *if it attracts all of $\mathscr{D}_f$, and it is* globally asymptotically stable *(GAS) if it is locally stable and globally attracting.*

We shall associate directed graphs with f. The vertices of these graphs are sets and the edges correspond to transitions between them. These graphs reflect the behaviour of the map, if for every point (x,y) and its image $f(x,y)$, it is satisfied that there is an edge going from any vertex containing (x,y) to any vertex containing $f(x,y)$. We give the necessary definitions here, the reader is referred to [3] for a more detailed overview of graph representations.

Definition 2.2. *$\mathscr{P}$ is called a* partition *of $\mathscr{D} \subseteq \mathbb{R}^2$ if it is a collection of closed subsets of $\mathbb{R}^2$ such that $|\mathscr{P}| := \cup_{p\in\mathscr{P}} p = \mathscr{D}$ and $\forall p_1, p_2 \in \mathscr{P} : p_1 \cap p_2 \subseteq \mathrm{bd}(p_1) \cup \mathrm{bd}(p_2)$. We define the* diameter *of the partition $\mathscr{P}$ by*

$$\mathrm{diam}(\mathscr{P}) = \sup_{p\in\mathscr{P}} \sup_{x,y\in p} \|x-y\|.$$

Let $f : \mathscr{D}_f \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $\mathscr{D} \subseteq \mathscr{D}_f$, and $\mathscr{P}$ be a partition of $\mathscr{D}$. We say that the directed graph $\mathscr{G}(\mathscr{V},\mathscr{E})$ is a graph representation of f on $\mathscr{D}$ with respect to $\mathscr{P}$, *if there exists a bijection $\iota : \mathscr{V} \to \mathscr{P}$ such that the following implication is true for all $u, v \in \mathscr{V}$:*

$$f(\iota(u) \cap \mathscr{D}) \cap \iota(v) \cap \mathscr{D} \neq \emptyset \Rightarrow (u,v) \in \mathscr{E}.$$

We take the liberty to handle the elements of the cover as vertices and vice versa, omitting the usage of ι.

Let us now define the following 2-dimensional map, corresponding to equation (1.1)

$$F : \mathbb{R}^2 \to \mathbb{R}^2,\ F(x,y) = F_{\alpha,m}(x,y) = (y, my - \alpha\varphi(x)). \tag{2.1}$$

For $(x,y) \in \mathbb{R}^2$ and $k \in \mathbb{N}_0$, we shall use notation $(x_k, y_k) = F^k(x,y)$.

3 Preliminaries and a sufficient condition for global stability

Even though φ is not assumed to be differentiable at 0, one may characterize the local stability of the origin by the following *generalized multipliers* of the map F at $(0,0)$

$$\mu_{1,2}(\lambda) = \mu_{1,2}(\alpha, m; \lambda) = \tfrac{m\pm\sqrt{m^2-4\lambda\alpha}}{2},$$

where $\lambda \in [\delta_{\inf}(\varphi;0), \delta_{\sup}(\varphi;0)]$ is an accumulation point of $\frac{\varphi(x)-\varphi(0)}{x} = \frac{\varphi(x)}{x}$ as $x \to 0$. Recall from our initial observations, that (1.2) implies $\lambda \in [0,1]$. It is easy to see, that $\max|\mu_{1,2}(\lambda)| \leq 1$ is satisfied only if $m \in [-2,2]$ and $\lambda\alpha \in [|m|-1, 1]$ hold and we have equality if and only if $\lambda\alpha = 1$ or $\lambda\alpha = |m|-1$. Consequently, the global stability of the zero solution may hold only if both $\delta_{\inf}(\varphi;0)\alpha \in [|m|-1,1]$ and $\delta_{\sup}(\varphi;0)\alpha \in [|m|-1,1]$. Our first goal in this section is to give a region on the parameter plane (α,m) where, with some exceptions, the global asymptotic stability of the trivial fixed point is guaranteed without any further assumptions. Note that in the case of $\delta_{\sup}(\varphi;0) = 1$ or in the special case when φ is differentiable at the origin and $\varphi'(0) = 1$, local stability may only hold if $\alpha \in [|m|-1,1]$ is satisfied. Observe that if $(\alpha,m) = (0,1)$, then every $c \in \mathbb{R}$ is an equilibrium point of equation (1.1). On the other hand, if $(\alpha,m) = (0,-1)$, then $\{c,-c\}$ is a period two orbit for every $c \in \mathbb{R}$, $c \neq 0$. According to these simple observations and to the following lemma, we shall restrict our attention in the sequel to the parameter range

$$(\alpha, m) \in \mathscr{R} := \mathrm{cl}(\mathscr{R}_0) \setminus \{(0,-1),(0,1)\},$$

where $\mathscr{R}_0$ is the open set $(|m|-1,1) \times (-1,1)$. These regions are depicted on Figure 3.

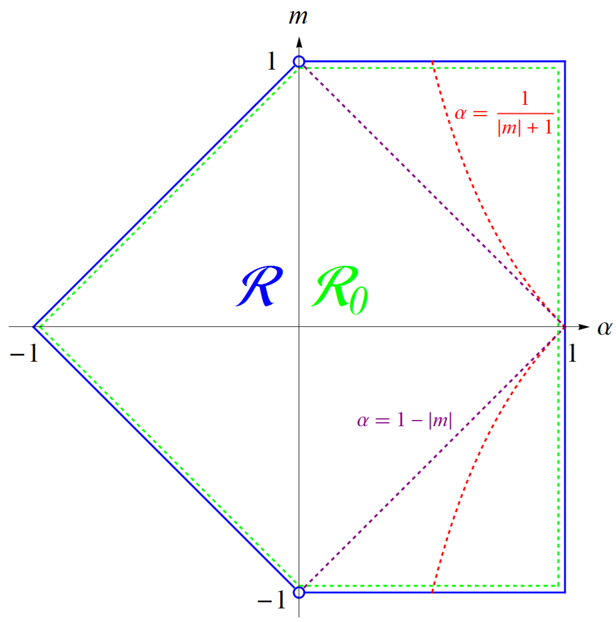

Figure 3: The solid blue and dashed green lines represent the sets $\mathscr{R}$ and $\mathscr{R}_0$, respectively. The dashed red and purple lines correspond to the curves $\alpha = \frac{1}{1+|m|}$ and $\alpha = 1-|m|$, respectively.

Remark 3.1. *We will see that in order to show the global asymptotic stability of* $(0,0)$ *for parameter pairs in* $\mathrm{bd}(\mathscr{R}_0)$*, we shall need additional information on* φ*. For practical reasons we have chosen to assume that* φ *is continuous and* $0 < x\varphi(x) < x^2$ *for* $x \neq 0$.

After stating a sufficient condition for GAS in Theorem 3.6, we shall restrict our attention to $m \in [0,1]$ and φ being bounded and as in Remark 3.1. Having these assumptions, we construct a compact, invariant and attracting region of the plane for (1.1).

Lemma 3.2. *Assume that* φ *is bounded. If* $|m| > 1$*, then the zero fixed point of* (2.1) *is not GAS.*

Proof. Indeed, if $|m| > 1$ then we readily get that $\min\{|x_0|,|y_0|\} > \frac{M_\varphi|\alpha|}{|m|-1}$ implies $\min\{|x_1|,|y_1|\} > \frac{M_\varphi|\alpha|}{|m|-1}$, excluding the global stability of the fixed point $(0,0)$ of F in this case. □

Lemma 3.3. *The fixed point* $(0,0)$ *is globally asymptotically stable if*

(a) $(\alpha, m) \in \mathscr{R}_0$ *with* $\alpha < 1-|m|$ *or*

(b) φ *is as in Remark 3.1 and* $(\alpha, m) \in \mathscr{R}$ *with* $\alpha < 1-|m|$.

See Figure 3 for an image of these regions.

Proof. We prove statement *(a)* first. For any point $(x_0,y_0)\in\mathbb{R}^2$, we have $|y_1|\le|m||y_0|+|\alpha||\varphi(x_0)|$, thus the inequalities $\max\{|x_1|,|y_1|\}\le\max\{|x_0|,|y_0|\}$ and $\max\{|x_2|,|y_2|\}\le(|m|+|\alpha|)\max\{|x_0|,|y_0|\}$ are satisfied. Since $|m|+|\alpha|<1$, by induction we obtain that $\max\{|x_{2k+1}|,|y_{2k+1}|\}\le\max\{|x_{2k}|,|y_{2k}|\}$ for $k\in\mathbb{N}$ and $\lim_{k\to\infty}\max\{|x_{2k}|,|y_{2k}|\}=0$, thus global asymptotic stability holds in this case.

Consider now statement *(b)*. For $\alpha\in(|m|-1,1-|m|)$ the same argument works, thus let $\alpha=|m|-1\in[-1,0)$ and $m\in(-1,1)$. In the same way, for a point $(x_0,y_0)\in\mathbb{R}^2$, we obtain that $\max\{|x_{2k}|,|y_{2k}|\}$ is decreasing, thus $\max\{|x_{2k}|,|y_{2k}|\}\to c\ge 0$. In order to show that $c=0$, due to the continuity of φ, it is enough to establish that, for any (x_0,y_0) such that $\max\{|x_0|,|y_0|\}=c>0$, the orbit satisfies $\limsup_{k\to\infty}\max\{|x_{2k}|,|y_{2k}|\}<c$. This easily follows from the condition on φ and $m\in(-1,1)$, since for any point $(x,y)\neq(0,0)$, we have $|my-\alpha\varphi(x)|<\max\{|x|,|y|\}$. □

Even though Theorem 3.6 will supply a stronger condition, note that using an analogous argument, GAS is easily shown for $(\alpha,m)\in\{1-|m|\}\times(-1,1)$ if we assume that φ fulfils the conditions in Remark 3.1.

Let us define the following sets for $a,b\in(0,\infty]$

$$
\begin{aligned}
H_1(a,b)&=\{(x,y):0\le x\le a;\ 0<y\le b\},\\
H_2(a,b)&=\{(x,y):0<x\le a;\ -b\le y\le 0\},\\
H_3(a,b)&=\{(x,y):-a\le x\le 0;\ -b\le y<0\},\\
H_4(a,b)&=\{(x,y):-a\le x<0;\ 0\le y\le b\}
\end{aligned}
$$

and $H_i=H_i(\infty,\infty)$ for $i\in\{1,2,3,4\}$. Figure 4 shows these four sets for a pair of values (a,b).

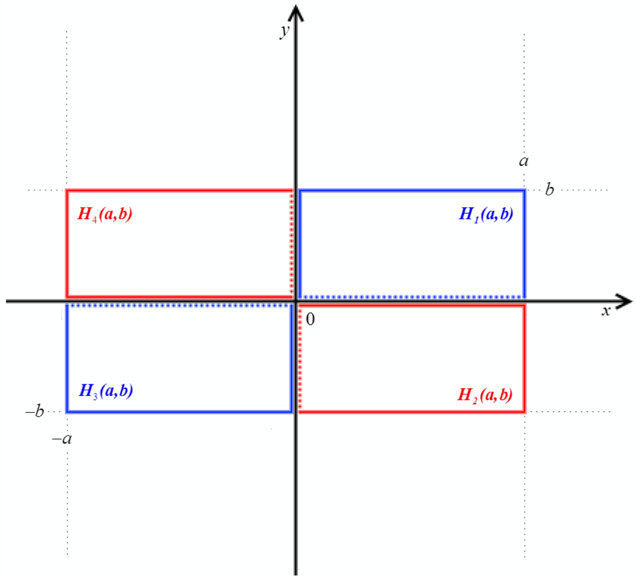

Figure 4: The sets $H_1(a,b),H_2(a,b),H_3(a,b)$ and $H_4(a,b)$

Proposition 3.4. *For $m\in[0,1]$ and $\alpha\in[0,1]$ the following statements hold.*

(i) If $(x_0,y_0)\in H_1(a,b)$, then $(x_1,y_1)\in H_1(b,mb)\cup H_2(b,\alpha a)$.

(ii) If $(x_0,y_0)\in H_2(a,b)$, then $(x_1,y_1)\in H_3(b,mb+\alpha a)$.

(iii) *If* $(x_0, y_0) \in H_3(a,b)$*, then* $(x_1, y_1) \in H_3(b, mb) \cup H_4(b, \alpha a)$*.*

(iv) *If* $(x_0, y_0) \in H_4(a,b)$*, then* $(x_1, y_1) \in H_1(b, mb + \alpha a)$*.*

(v) *Having* $m \in [0,1)$ *and* $(x_k, y_k) \in H_1$ *or* $(x_k, y_k) \in H_3$ *for all* $k \in \mathbb{N}_0$ *implies* $\lim_{k\to\infty}(x_k, y_k) = (0,0)$*. In addition, if* φ *is as in Remark 3.1, then the claim holds for* $m = 1$ *as well.*

Proof. The proof is elementary. We denote the first point with (x_0, y_0). To see that statement *(i)* holds, first note that $0 < x_1 = y_0 \leq b$. Since we have $y_1 = my_0 - \alpha\varphi(x_0)$ and $0 \leq x\varphi(x) \leq x^2$, therefore $(x_0, y_0) \in H_1(a,b)$ readily implies $-\alpha a \leq y_1 \leq mb$, resulting in $(x_1, y_1) \in H_1(b, mb) \cup H_2(b, \alpha a)$. Statements *(ii)*–*(iv)* can be proven in a similar manner.

To prove statement *(v)*, let us suppose that the point $(x_0, y_0) \in H_1$ is such that $(x_k, y_k) \in H_1$ holds for all $k \in N_0$. Using the notation $a = \max\{x_0, y_0\} > 0$ and statement *(i)*, we obtain by induction that

$$(x_{2k}, y_{2k}) \in H_1(m^k a, m^k a),\ (x_{2k+1}, y_{2k+1}) \in H_1(m^k a, m^{k+1} a) \text{ and } 0 < y_{2k+1} \leq y_{2k}$$

hold for all $k \in \mathbb{N}_0$ implying that

$$\lim_{k\to\infty} \max\{x_k, y_k\} = \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = c \geq 0,$$

which results in $c = 0$ if $m \in [0,1)$. We finish our argument by noting, that for $m = 1$, the continuity of φ implies that it is enough to show that (c,c) cannot be a fixed point for $c > 0$. This easily follows from $0 < \varphi(c)$.

The case of $(x_k, y_k) \in H_3$ is analogous. □

We may formulate similar statements for $m \in [-1, 0]$.

Proposition 3.5. *For* $m \in [-1,0]$ *and* $\alpha \in [0,1]$ *the following statements hold.*

(i) *If* $(x_0, y_0) \in H_1(a,b)$*, then* $(x_1, y_1) \in H_2(b, |m|b + \alpha a)$*.*

(ii) *If* $(x_0, y_0) \in H_2(a,b)$*, then* $(x_1, y_1) \in H_3(b, \alpha a) \cup H_4(b, |m|b)$*.*

(iii) *If* $(x_0, y_0) \in H_3(a,b)$*, then* $(x_1, y_1) \in H_4(b, |m|b + \alpha a)$*.*

(iv) *If* $(x_0, y_0) \in H_4(a,b)$*, then* $(x_1, y_1) \in H_1(b, \alpha a) \cup H_2(b, |m|b)$*.*

The proof is analogous to what we have seen at Proposition 3.4. Now we are ready to state one of the main results of this section.

Theorem 3.6. *The fixed point* $(0,0)$ *is globally asymptotically stable, if any of the following conditions is satisfied*

(a) $(\alpha, m) \in \mathscr{R}_0 \setminus [\frac{1}{1+|m|}, 1] \times (-1,1)$*, or*

(b) φ *is as in Remark 3.1 and* $(\alpha, m) \in \mathscr{R} \setminus (\frac{1}{1+|m|}, 1] \times [-1,1]$*.*

See Figure 3 for a visualization of these parameter regions.

Proof. For simplicity, we restrict our attention to the case when $m \geq 0$. The other case can be treated similarly. For $\alpha \in [m-1, 1-m)$, the statement follows from Lemma 3.3. Let $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$ be an arbitrary point. From Proposition 3.4 it is clear that either $(x_k, y_k) \to (0,0)$ as $k \to \infty$ or there exists a sequence of integers

$$0 \leq k_{1,1} < k_{1,2} < k_{1,3} = k_{1,2} + 1 < k_{1,4} < k_{2,1} = k_{1,4} + 1 < \cdots k_{2,4} < \cdots$$

such that $(x_{k_{n,i}}, y_{k_{n,i}}) \in H_i$ for all $n \in \mathbb{N}$ and $i \in \{1,2,3,4\}$. In the former case our claim trivially holds, thus we may assume the latter. Now we may suppose without loss of generality that $k_{1,1} = 0$, that is $(x_0, y_0) \in H_1(a,a)$ for some positive constant a. Statement *(i)* of Proposition 3.4 implies that $(x_k, y_k) \in H_1(a,a)$ for $0 \leq k < k_{1,2}$ and $(x_{k_{1,2}}, y_{k_{1,2}}) \in H_2(a, \alpha a)$. From statement *(ii)* of Proposition 3.4

$$\begin{aligned}(x_{k_{1,3}}, y_{k_{1,3}}) = (x_{k_{1,2}+1}, y_{k_{1,2}+1}) &\in H_3(\alpha a,\ (m+1)\alpha a) \\ &\subseteq H_3((m+1)\alpha a,\ (m+1)\alpha a)\end{aligned}$$

follows. Similarly, from statement *(iii)* we obtain that for $k_{1,3} \leq k < k_{1,4}$,

$$(x_k, y_k) \in H_3((m+1)\alpha a,\ (m+1)\alpha a)$$

holds. Moreover, statements *(iv)* and *(i)* of Proposition 3.4 imply

$$(x_{k_{1,4}}, y_{k_{1,4}}) \in H_4((m+1)\alpha a,\ (m+1)\alpha^2 a)$$

and

$$(x_{k_{1,4}+1}, y_{k_{1,4}+1}) = (x_{k_{2,1}}, y_{k_{2,1}}) \in H_1(((m+1)\alpha)^2 a,\ ((m+1)\alpha)^2 a).$$

Note that $0 \leq |y_{k_{2,1}}| \leq |y_{k_{1,1}}|$. We obtain by induction that

$$\lim_{n\to\infty} \max\{|x_{k_{n,1}}|, |y_{k_{n,1}}|\} = c \geq 0.$$

If $0 \leq 1-m \leq \alpha < \frac{1}{1+m}$, then $c = 0$ follows immediately. Since the origin is LAS, this implies $\lim_{k\to\infty}(x_k, y_k) = (0,0)$. Now, let us consider the case $\alpha = \frac{1}{m+1}$. The continuity of φ implies that in order to obtain $c = 0$, it is enough to show that for any point $(x'_0, y'_0) \in H_1$ such that $\max\{|x'_0|, |y'_0|\} = c > 0$ and $((x'_k, y'_k))_{k=0}^{\infty}$ visits every H_i infinitely many times, inequality $\max\{|x'_{k_{2,1}}|, |y'_{k_{2,1}}|\} < c$ is satisfied (by keeping the above meaning of the indexes). We finish by noting that the condition on φ implies that there exists $0 < c' < c$ such that $(x'_{k_{1,3}}, y'_{k_{1,3}}) \in H_3(c', c') \subset H_3(c,c)$ implying $(x'_{k_{2,1}}, y'_{k_{2,1}}) \in H_1(c', c')$. □

In the remaining part of the section we limit our analysis to the case when $m \in [0,1]$, φ is bounded and satisfies the conditions in Remark 3.1. Let $M \geq 0$ and consider the sets $\mathbb{T}(M,m)$ and $\mathscr{S}(M,m)$ given by

$$\mathbb{T}(M,m) = \begin{cases} \mathbb{R}^2, & \text{for } m = 0, \\ H_1(\frac{2M}{m}, \frac{2M}{m}) \cup H_2(\frac{M}{m}, \frac{M}{m}) \cup H_3(\frac{2M}{m}, \frac{2M}{m}) \cup H_4(\frac{M}{m}, \frac{M}{m}) \cup \{(0,0)\}, & \text{for } m \in (0,1], \end{cases}$$

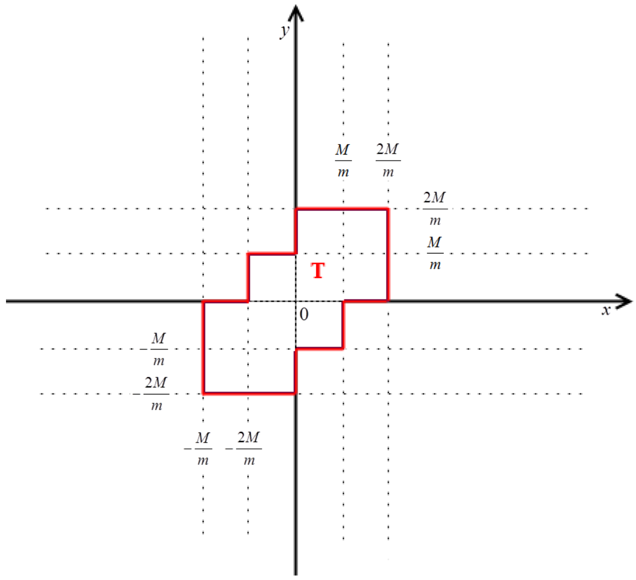

Figure 5: The set $\mathbb{T}(M,m)$ for $m>0$

and

$$\mathscr{S}(M,m)=\begin{cases}[-\frac{2M}{1-m},\ \frac{2M}{1-m}]^2, & \text{for } m\in[0,1),\\ \mathbb{R}^2, & \text{for } m=1.\end{cases}$$

We sketched $\mathbb{T}(M,m)$ on Figure 5 for $m\neq 0$.

The following proposition has an essential role in the proof of our main result, Theorem 4.3.

Proposition 3.7. *Assume that $(\alpha,m)\in[0,1]^2$, φ is as in Remark 3.1, bounded and let $M=M_\varphi$. Then the following statements hold.*

(i) $(x_0,y_0)\in\mathbb{T}(M,m)$ *implies* $(x_1,y_1)\in\mathbb{T}(M,m)$*, moreover, for* $(x_0,y_0)\in\mathbb{R}^2$*, there exists* $k\in\mathbb{N}_0$ *such that* $(x_k,y_k)\in\mathbb{T}(M,m)$ *is satisfied.*

(ii) $(x_0,y_0)\in\mathscr{S}(M,m)$ *implies* $(x_1,y_1)\in\mathscr{S}(M,m)$*, moreover, for* $(x_0,y_0)\in\mathbb{R}^2$*, there exists* $k\in\mathbb{N}_0$ *such that* $(x_k,y_k)\in\mathscr{S}(M,m)$ *holds.*

Proof. Let m and M be fixed and let us use notations $\mathbb{T}=\mathbb{T}(M,m)$ and $\mathscr{S}=\mathscr{S}(M,m)$.

(i) The case $m=0$ is trivial, therefore we may assume $m\in(0,1]$. First, let us show the second part of the statement. Let $(x_0,y_0)\in\mathbb{R}^2$ be an arbitrary point. According to Proposition 3.4 either there exists $k_0\in\mathbb{N}_0$ such that $(x_{k_0},y_{k_0})\in H_1$ or we have $(x_k,y_k)\to(0,0)$ as $k\to\infty$, which implies $(x_k,y_k)\in\mathbb{T}$ for large enough values of k. Thus we may assume $(x_0,y_0)\in H_1$.

a) If $0<y_0\leq\frac{M}{m}$, then we readily get that $(x_1,y_1)\in H_1(\frac{M}{m},\frac{M}{m})\cup H_2(\frac{M}{m},\frac{M}{m})\subset\mathbb{T}$.

b) $y_0>\frac{M}{m}$ leads to $0<x_1=y_0$ and $0<y_1=my_0-\alpha\varphi(x_0)\leq my_0\leq y_0$. Now if $y_1\leq\frac{M}{m}$, then we are in case a). Otherwise $y_1>\frac{M}{m}$ and $(x_1,y_1)\in H_1$. We obtain by induction, that either there exists $k_0\in\mathbb{N}$ such that $0<y_{k_0}\leq\frac{M}{m}$ and $x_{k_0}>\frac{M}{m}$ and the claim follows from case a), or $(x_k,y_k)\in H_1\setminus H_1(\frac{M}{m},\frac{M}{m})$ for all $k\in\mathbb{N}$. In the latter case, Proposition 3.4 leads to $\lim_{k\to\infty}(x_k,y_k)=(0,0)$, implying a contradiction.

Now, we may prove the first part of statement *(i)*. The argument above also shows that for $(x_0,y_0) \in H_1(\frac{2M}{m},\frac{2M}{m})$, $(x_1,y_1) \in \mathbb{T}$ is guaranteed. For $(x_0,y_0) \in H_2(\frac{M}{m},\frac{M}{m})$, statement *(ii)* of Proposition 3.4 with $a = b = \frac{M}{m}$ yields $(x_1,y_1) \in \mathbb{T}$. A similar argument can be applied to show that for $(x_0,y_0) \in H_3(\frac{2M}{m},\frac{2M}{m}) \cup H_4(\frac{M}{m},\frac{M}{m})$, $(x_1,y_1) \in \mathbb{T}$ holds, which completes the proof of *(i)*.

(ii) The statement is trivial for $m = 1$, thus we may assume that $m \in [0,1)$. To prove the first part, let us suppose that $(x_0,y_0) \in \mathscr{S}$. Then $|x_1| = |y_0| \le \frac{2M}{1-m}$ together with $|y_1| \le m|y_0| + \alpha M \le m\frac{2M}{1-m} + M < \frac{2M}{1-m}$ yields $(x_1,y_1) \in \mathscr{S}$.

To prove the second part of the statement let us assume that $(x_0,y_0) \notin \mathscr{S}$.

a) If $|y_0| \ge \frac{2M}{1-m}$, then $|x_1| = |y_0| \ge \frac{2M}{1-m}$ and $|y_1| \le m|y_0| + M \le \frac{m+1}{2}|y_0| < |y_0|$. By induction we get a geometrically decreasing series y_k, thus there exists $k_0 \in \mathbb{N}$ such that $|x_{k_0}| \ge \frac{2M}{1-m}$ and $|y_{k_0}| < \frac{2M}{1-m}$. Now $|x_{k_0+1}| = |y_{k_0}| < \frac{2M}{1-m}$ and $|y_{k_0+1}| \le m|y_{k_0}| + M < m\frac{2M}{1-m} + M < \frac{2M}{1-m}$, thus $(x_{k_0+1}, y_{k_0+1}) \in \mathscr{S}$.

b) If $|y_0| < \frac{2M}{1-m}$, then $(x_0,y_0) \notin \mathscr{S}$ implies $|x_0| > \frac{2M}{1-m}$ which reduces to case a) and makes our proof complete.

□

Corollary 3.8. *Let us assume that $(\alpha,m) \in [0,1]^2$, φ is continuous and bounded. Given $M = M_\varphi$, the sets $\mathbb{T}(M,m)$ and $\mathscr{S}(M,m)$ are well defined. Their intersection $S = \mathbb{T}(M,m) \cap \mathscr{S}(M,m)$ is compact. Moreover, S is invariant and attracting for F. In addition, the following inclusion holds*

$$S = \mathbb{T}(M,m) \cap \mathscr{S}(M,m) \subseteq [-\tfrac{2M}{\max\{m,1-m\}}, \tfrac{2M}{\max\{m,1-m\}}]^2 \subseteq [-4M,4M]^2.$$

4 Main result: necessary and sufficient condition for global stability

In this section we restrict our attention to equation (1.3), namely

$$x_{n+1} = mx_n - \alpha \tanh(x_{n-1}),$$

where $(\alpha,m) \in \mathbb{R}^2$, that is (1.1) with $\varphi(x) \equiv \tanh(x)$.

Remark 4.1. *Note that, the function* tanh *is as in Remark 3.1. In addition, it is bounded ($M_{\tanh} = 1$) and* $\tanh'(0) = 1$.

These observations imply, in accordance with the results of Section 3, that GAS of the zero solution may only hold when

$$(\alpha,m) \in \mathscr{R} = [|m|-1,1] \times [-1,1] \setminus \{(0,-1),(0,1)\}.$$

Remark 4.2. *Using the substitution $y_k := (-1)^k x_k$ and the fact that* tanh *is an odd function one obtains $y_{n+1} = (-m)y_n - \alpha \tanh(y_{n-1})$.*

Notice that at $\alpha = 1$ a Neimark–Sacker bifurcation takes place with a 1:4 strong resonance at $(\alpha, m) = (1, 0)$. Our main result is that we show in Theorem 4.3 that condition $(\alpha, m) \in \mathscr{R}$ is not only necessary but also sufficient for global asymptotic stability of the origin. Keeping the notations of the previous section we get that

$$F : \mathbb{R}^2 \to \mathbb{R}^2,\ F(x,y) = F_{\alpha,m}(x,y) = (y, my - \alpha \tanh(x)). \tag{4.1}$$

Theorem 4.3. *The fixed point* $(0,0)$ *of the map* F *is globally asymptotically stable if and only if* $(\alpha, m) \in \mathscr{R}$.

Note that due to Theorem 3.6 and the symmetry described in Remark 4.2, it is sufficient to consider $(\alpha, m) \in [\frac{1}{m+1}, 1] \times [0, 1]$. The proof of global stability in this region consists of two parts. In Part I, for every such pair (α, m), we obtain a compact neighbourhood $U(\alpha)$ inside the basin of attraction of $(0,0)$, such that for any parameter interval $[\alpha] \times [m] \subseteq [1/(1+m), 1] \times [0, 1]$, the set $U([\alpha]) := \cap_{\alpha \in [\alpha]} U(\alpha)$ contains a closed disk around the origin. For this we shall study the linearized equation and the 1:4 resonant normal form of the Neimark–Sacker bifurcation. After we have derived this neighbourhood and the compact set $S = \mathbb{T}(M_{\tanh}, m) \cap \mathscr{S}(M_{\tanh}, m)$ from Section 3, in Part II we analyze the equation using a rigorous computer program. The results prove that having $(\alpha, m) \in [\alpha] \times [m]$, every trajectory starting in S will enter $U([\alpha]) \subseteq U(\alpha)$.

Part I: Obtaining the compact neighbourhood $U(\alpha)$

Linearizing F at the $(0,0)$ fixed point yields

$$(x,y) \mapsto F(x,y) = A(\alpha, m)(x,y)^T + f_{\alpha,m}(x,y) \tag{4.2}$$

where the linear part is

$$A(\alpha, m) = \begin{pmatrix} 0 & 1 \\ -\alpha & m \end{pmatrix},$$

and the remainder is given by

$$f_\alpha(x,y) = \begin{pmatrix} 0 \\ \alpha x - \alpha \tanh(x) \end{pmatrix}.$$

First recall that the eigenvalues of $A(\alpha, m)$ are $\mu_{1,2}(\alpha, m) = \frac{m \pm i\sqrt{4\alpha - m^2}}{2} \in \mathbb{C}$. Let $\mu = \mu_1(\alpha, m)$ and q denote the eigenvector $q = q(\alpha, m) = \left(\frac{m - i\sqrt{-m^2 + 4\alpha}}{2\alpha}, 1\right)^T \in \mathbb{C}^2$. Let also $p = p(\alpha, m) \in \mathbb{C}^2$ denote the eigenvector of $A(\alpha, m)^T$ corresponding to $\overline{\mu}$ such that $\langle p, q \rangle = 1$. This results in

$$p = \left(-\frac{i\alpha}{\sqrt{4\alpha - m^2}}, \frac{1}{2} + \frac{im}{2\sqrt{4\alpha - m^2}}\right). \tag{4.3}$$

We shall introduce the complex variable

$$z = z(x, y, \alpha, m) = \langle p, (x,y) \rangle = \frac{\alpha\left(mx - 2y - ix\sqrt{4\alpha - m^2}\right)}{m^2 - 4\alpha - im\sqrt{4\alpha - m^2}}. \tag{4.4}$$

The inverse of the transformation may also be given by

$$(x,y) = zq + \overline{zq} = \left(\frac{1}{\alpha}\left(-iz\sqrt{4\alpha - m^2} + \left(m + i\sqrt{4\alpha - m^2}\right)\mathrm{Re}z\right), 2\mathrm{Re}z\right). \tag{4.5}$$

System (4.1) is now transformed into the complex system

$$\begin{aligned} z \mapsto G(z) = G(z, \bar{z}, \alpha, m) &= \langle p,\, A(\alpha, m)(zq + \overline{zq}) + f_{\alpha,m}(zq + \overline{zq})\rangle \\ &= \mu z + g(z, \bar{z}, \alpha, m), \end{aligned} \tag{4.6}$$

where g is a complex valued smooth function of $z, \bar{z}, \alpha$ and m defined by

$$\begin{aligned} g(z, \bar{z}, \alpha, m) =& 2\alpha\left(m\mathrm{Re}z + \sqrt{4\alpha - m^2}\mathrm{Im}z - \alpha\tanh\left(\tfrac{m\mathrm{Re}z + \sqrt{4\alpha - m^2}\mathrm{Im}z}{\alpha}\right)\right) \\ &\cdot \left(4\alpha - m^2 + im\sqrt{4\alpha - m^2}\right)^{-1}. \end{aligned} \tag{4.7}$$

It is also clear that for fixed α and m, g is an analytic function of z and $\bar{z}$. Calculating the Taylor expansion of g around 0 with respect to z and $\bar{z}$ we get that it has only cubic and higher order terms (due to the fact that $\tanh''(0) = 0$). That is,

$$g(z, \bar{z}, \alpha, m) = \sum_{k+l=3} \frac{g_{kl}}{k!l!} z^k \bar{z}^l + R_1(z), \quad \text{with } k, l \in \{0, 1, 2, 3\}, \tag{4.8}$$

where $g_{kl} = g_{kl}(\alpha, m) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} g(z, \bar{z}, \alpha, m)\Big|_{z=0}$ for $k+l = 3$, $k, l \in \{0, 1, 2, 3\}$ and $R_1(z) = R_1(z, \bar{z}, \alpha, m) = O(|z|^4)$ for fixed (α, m).

Theorem 4.4. *Let $\alpha \in [\frac{1}{2}, 1)$ and $m \in [0, 1]$. If $(x_0, y_0) \in U(\alpha) = \mathrm{K}_{\varepsilon(\alpha)}$, where*

$$\varepsilon(\alpha) = \sqrt[4]{\tfrac{27}{800}}\sqrt{1 - \sqrt{\alpha}},$$

then $\lim_{k\to\infty}(x_k, y_k) = (0, 0)$.

Proof. Let us study our map in the form (4.6). Let also $(x, y) \in \mathrm{K}_{\varepsilon(\alpha)} \setminus \{(0, 0)\}$ be an arbitrary point and $z = z(x, y, \alpha, m)$ be defined by (4.4). We are going to show, that $|G(z, \bar{z}, \alpha, m)| < |z|$ if $z \neq 0$. Using equations (4.4) and (4.5) it can be easily shown that for all $\alpha \in [\frac{1}{2}, 1]$ and $m \in [0, 1]$

$$\begin{gathered} \max\{|x|, |y|\} \le \tfrac{2|z|}{\sqrt{\alpha}} \le 2\sqrt{2}|z|, \quad \text{and} \\ |z| \le \sqrt{\tfrac{\alpha(1+\alpha+m)}{4\alpha - m^2}}\max\{|x|, |y|\} \le \tfrac{\sqrt{5}}{2}\max\{|x|, |y|\} \end{gathered} \tag{4.9}$$

hold with $z = z(x, y, \alpha, m)$. Using the Taylor expansion of the tanh function, inequality $\varepsilon(\alpha) < 1$ and that $\max_{|x|\le 1}\left\{\left|\frac{\mathrm{d}^3}{\mathrm{d}x^3}\tanh(x)\right|\right\} = 2$, we get that

$$\begin{aligned} |g(z, \bar{z}, \alpha, m)| &= \left|\left\langle p(\alpha, m), f_{\alpha,m}\left(zq(\alpha, m) + \overline{zq(\alpha, m)}\right)\right\rangle\right| \\ &= \sqrt{\tfrac{\alpha}{4\alpha - m^2}}\alpha^2|x - \tanh(x)| \\ &\le \sqrt{\tfrac{\alpha}{4\alpha - m^2}}\frac{\alpha^2}{6}\max_{|x|\le\varepsilon(\alpha)}\left\{\left|\tfrac{\mathrm{d}^3}{\mathrm{d}x^3}\tanh(x)\right|\right\}|x|^3 \\ &= \sqrt{\tfrac{\alpha}{4\alpha - m^2}}\frac{\alpha^2}{3}|x|^3. \end{aligned}$$

Now, by the first inequality in (4.9) and equation $|\mu| = \sqrt{\alpha}$, we obtain

$$\begin{aligned} |G(z,\bar{z},\alpha,m)| &\leq \sqrt{\alpha}|z| + \sqrt{\tfrac{\alpha}{4\alpha-m^2}\tfrac{\alpha^2}{3}}(2\sqrt{2})^3|z|^3 \\ &= |z|\cdot\left(\sqrt{\alpha}+\sqrt{\tfrac{\alpha}{4\alpha-m^2}\tfrac{\alpha^2}{3}}16\sqrt{2}|z|^2\right). \end{aligned}$$

As $\sqrt{\frac{\alpha}{4\alpha-m^2}\frac{\alpha^2}{3}} \leq \frac{1}{3\sqrt{3}}$ holds for all $\alpha \in [\frac{1}{2},1]$ and $m \in [0,1]$, thus $0 \neq |z| < \varepsilon_0(\alpha) = \sqrt[4]{\frac{27}{512}}\sqrt{1-\sqrt{\alpha}}$ guarantees $|G(z)| < |z|$. Using the second inequality of (4.9) yields that for $(x,y) \in \mathrm{K}_{\varepsilon(\alpha)}$, inequality $|z| = |z(x,y,\alpha,m)| < \varepsilon_0(\alpha)$ is satisfied. Now, we have $|G(z)| < |z|$ if $z \neq 0$. This implies $G^k(z) \to 0$ as $k \to \infty$. Thus the $(0,0) \in \mathbb{R}^2$ solution is asymptotically stable in $U(\alpha) = \mathrm{K}_{\varepsilon(\alpha)}$ for our original system $(x,y) \mapsto F(x,y)$. □

Theorem 4.5. *Let* $\alpha \in [0.98,1]$ *and* $m \in [0,1]$. *If* $(x_0,y_0) \in U(\alpha) = \mathrm{K}_{\varepsilon(\alpha)}$, *where*

$$\varepsilon(\alpha) = \tfrac{1}{6},$$

then $\lim_{k\to\infty}(x_k,y_k) = (0,0)$.

The proof is based on the argument applied in our previous work [3]. As already noted, at $\alpha = 1$, the dynamical system defined by $F_{\alpha,m}$ undergoes a Neimark–Sacker bifurcation. However, at $(\alpha,m) = (1,0)$, a strong $1:4$ resonance occurs. We shall transform our system into its $1:4$ resonant normal-form (according to Kuznetsov [18]) to prove the claim of the theorem, as the non-resonant normal form of the Neimark–Sacker bifurcation would not be not applicable near the parameter values $(\alpha,m) = (1,0)$. The reason for that is that we shall need, among others, uniform estimates on the transformation, which is impossible as the parameters tend to the critical pair $(1,0)$. However, the resonant normal form is applicable over the whole region $(\alpha,m) \in [0.98,1]\times[0,1]$. In the following proof we used the assistance of the symbolic toolbox of Wolfram Mathematica.

Proof of Theorem 4.5. In this proof, we shall present several estimations. The given bounds shall always be uniform, that is, they hold for all parameter values $\alpha \in [0.98,1]$ and $m \in [0,1]$.

Step 1: Transformation into the 1:4 resonant normal form

Let us consider our system in the form (4.6). We are looking for a smooth complex function $h = h_{\alpha,m} : \mathbb{C} \to \mathbb{C}$, which is defined and is invertible on a neighbourhood of $0 \in \mathbb{C}$ and which transforms our system (4.6) into the following normal form $w \mapsto G_{1:4}(w) = G_{1:4}(w,\overline{w},\alpha,m)$, where

$$G_{1:4}(w) = h^{-1}(G(h(w),\overline{h(w)},\alpha,m)) = \mu w + c(\alpha,m)w^2\overline{w} + d(\alpha,m)\overline{w}^3 + R_2(w), \tag{4.10}$$

and $R_2(w) = R_2(w,\overline{w},\alpha,m) = O(|w|^4)$ for (α,m) fixed. One can find such a function h by assuming it to be a polynomial of w and $\overline{w}$ with at most cubic terms. This results in

$$h(w) = h(w,\overline{w},\alpha,m) = w + \tfrac{h_{30}}{6}w^3 + \tfrac{h_{12}}{2}w\overline{w}^2, \tag{4.11}$$

and

$$h^{-1}(z) = h^{-1}(z,\bar{z},\alpha,m) = z - \tfrac{h_{30}}{6}z^3 - \tfrac{h_{12}}{2}z\bar{z}^2 + R_3(z), \tag{4.12}$$

where

$$h_{30} = h_{30}(\alpha, m) = \frac{g_{30}}{\mu(\mu^2-1)}, \quad h_{12} = h_{12}(\alpha, m) = \frac{g_{12}}{2\mu(\overline{\mu}^2-1)}$$

and $R_3(z) = R_3(z, \bar{z}, \alpha, m) = O(|z|^4)$ for (α, m) fixed. The domains of h and h^{-1} are to be defined later.

Our aim is now to find $\varepsilon_0 > 0$ such that for all $(x, y) \in \left[-\frac{1}{6}, \frac{1}{6}\right]^2$, $|w| < \varepsilon_0$ is satisfied and for all $0 \neq |w| < \varepsilon_0$ the following inequality holds

$$|G_{1:4}(w)| = |\mu w + c(\alpha, m) w^2 \overline{w} + d(\alpha, m) \overline{w}^3 + R_2(w, \overline{w}, \alpha, m)| < |w|. \tag{4.13}$$

To find ε_0, we need several uniform estimations on the higher order (error) terms R_1, R_2 and R_3, on the transformations h and $(x, y) \mapsto z$ and their inverses and on the functions g, c and d, as well.

Step 2: Estimations

Estimation of g and R_1

First of all, it can be easily shown from equations (4.4) and (4.5) that the following inequalities hold

$$\begin{aligned} \max\{|x|, |y|\} &\leq \frac{2|z|}{\sqrt{\alpha}} \leq 2.03|z| \quad \text{and} \\ |z| &\leq \sqrt{\frac{\alpha(1+\alpha+m)}{4\alpha-m^2}} \max\{|x|, |y|\} \leq 1.01 \cdot \max\{|x|, |y|\}, \end{aligned} \tag{4.14}$$

for all $\alpha \in [0.98, 1]$, $m \in [0, 1]$. Now, it is clear from the Taylor expansion of the tanh function and from equations (4.2), (4.3) and (4.5) that

$$|R_1(z)| \leq \left|\frac{1}{2} + \frac{im}{2\sqrt{4\alpha - m^2}}\right| \cdot \frac{\alpha}{120} \cdot \max_{|x| \leq \frac{1}{6}} \left\{\left|\frac{d^5}{dx^5} \tanh(x)\right|\right\} \cdot |x|^5 = \sqrt{\frac{\alpha}{4\alpha - m^2}} \frac{2\alpha^2}{15} \cdot |x|^5$$

is satisfied if $z = z(x, y, \alpha, m)$. Now, using rigorous estimations, from the first inequality of (4.14) and from $|x| \leq \frac{1}{6}$ it can be readily shown that

$$|R_1(z)| \leq 0.22|z|^4. \tag{4.15}$$

We have the following explicit formulae for the third order terms of g

$$\begin{aligned} g_{30} &= \frac{2i\alpha - m\left(im + \sqrt{4\alpha - m^2}\right)}{\alpha\sqrt{4\alpha - m^2}}, & g_{21} &= -\frac{2i}{\sqrt{4\alpha - m^2}}, \\ g_{12} &= \frac{2\left(m + i\sqrt{4\alpha - m^2}\right)}{4\alpha - m^2 + im\sqrt{4\alpha - m^2}}, & g_{03} &= \frac{\left(m + i\sqrt{4\alpha - m^2}\right)^3}{2\alpha\left(4\alpha - m^2 + im\sqrt{4\alpha - m^2}\right)}. \end{aligned} \tag{4.16}$$

By symbolic calculations, we obtain that

$$\sum_{k+l=3} \frac{|g_{kl}|}{k!l!} = \frac{8}{3\sqrt{4\alpha - m^2}} < 1.57, \text{ with } k, l \in \{0, 1, 2, 3\}. \tag{4.17}$$

Inequalities (4.15) and (4.17) together with equations (4.6) and (4.8) yield in particular that

$$|G(z)| \leq |z| + 1.57|z|^3 + 0.22|z|^4. \tag{4.18}$$

A region where transformation h is valid, and estimation of h

We are going to show that the transformation h, defined by equation (4.11), is injective on $\overline{\mathrm{B}_{1/2}} \subset \mathbb{C}$ and that its inverse h^{-1} is defined on $\overline{\mathrm{B}_{1/3}}$ and has the form (4.12).

The following equations and upper bound can be easily obtained

$$|h_{30}| = |h_{12}| = \frac{2}{\sqrt{\alpha(4\alpha-m^2)((1+\alpha)^2-m^2)}} < 0.7. \tag{4.19}$$

Let

$$H_z = H_{\alpha,m,z} : \mathbb{C} \ni w \mapsto w + z - h(w) \in \mathbb{C}.$$

By this notation, $H_z(w) = w$ holds if and only if $h(w) = z$. Let us make the following observation.

$$\begin{aligned}|H_z(w_1) - H_z(w_2)| =& |w_1 - h(w_1) - w_2 + h(w_2)| \\ \leq& \tfrac{|h_{30}|}{6}\left|w_1^3 - w_2^3\right| + \tfrac{|h_{12})|}{2}\left|\overline{w}_1|w_1|^2 - \overline{w}_2|w_2|^2\right|.\end{aligned}$$

Note also that

$$\begin{aligned}\left|\overline{w}_1|w_1|^2 - \overline{w}_2|w_2|^2\right| \leq& \left|\overline{w}_1|w_1|^2 - \overline{w}_1|w_2|^2\right| + \left|\overline{w}_1|w_2|^2 - \overline{w}_2|w_2|^2\right| \\ =& |w_1|\left(|w_1|^2 - |w_2|^2\right) + |w_2|^2|\overline{w}_1 - \overline{w}_2| \\ \leq& |w_1|(|w_1| - |w_2|)(|w_1| + |w_2|) + |w_2|^2|w_1 - w_2| \\ \leq& |w_1 - w_2|\left(|w_1|^2 + |w_1||w_2| + |w_2|^2\right).\end{aligned}$$

Now, if $w_1, w_2 \in \mathrm{B}_{1/2}$ are arbitrary and $z \in \mathrm{B}_{1/3}$ is fixed, then we have the following estimations

$$\begin{aligned}|H_z(w_1) - H_z(w_2)| \leq& |w_1 - w_2| \cdot \left(\tfrac{|h_{30}|}{6} + \tfrac{|h_{12}|}{2}\right)\left(|w_1|^2 + |w_1||w_2| + |w_2|^2\right) \\ \leq& 0.47|w_1 - w_2| \cdot 3 \cdot \tfrac{1}{4} \leq |w_1 - w_2|,\end{aligned}$$

and

$$|H_z(w)| \leq |z| + |w - h(w)| \leq |z| + \left(\tfrac{|h_{30}|}{6} + \tfrac{|h_{12}|}{2}\right)|w|^3 \leq \tfrac{1}{3} + 0.47 \cdot 2 \cdot \tfrac{1}{8} < \tfrac{1}{2}.$$

We obtained that $H_{\alpha,z} : \overline{\mathrm{B}_{1/2}} \to \overline{\mathrm{B}_{1/2}}$ is a contraction. Hence for all fixed $z \in \overline{\mathrm{B}_{1/3}}$ there exists exactly one $w = w(z) \in \overline{\mathrm{B}_{1/2}}$ such that $H_z(w(z)) = w(z)$, that is $h(w(z)) = z$. This means that h^{-1} can be defined on $\overline{\mathrm{B}_{1/3}}$.

It is also clear from equation (4.11) and inequality (4.19) that

$$|w| - 0.47|w|^3 \leq |h(w)| \leq |w| + 0.47|w|^3. \tag{4.20}$$

Estimation of h^{-1}

Using inequalities (4.20) and assuming $w \in \mathrm{B}_{1/5}$, $z = h(w)$ yield the following inequality

$$|w| \leq 1.02|h^{-1}(z)|. \tag{4.21}$$

In order to have a similar upper estimation on its inverse h^{-1}, as well, we need to estimate the remainder term R_3. Let us assume that $z \in \mathrm{B}_{1/3}$. Since h^{-1} is defined on $\mathrm{B}_{1/3}$, hence there exists exactly one number w in $\mathrm{B}_{1/2}$ such that $z = h(w)$. Now, we have

$$R_3(z) = R_3(h(w)) = h^{-1}(h(w)) - h(w) + \tfrac{h_{30}}{6}(h(w))^3 + \tfrac{h_{12}}{2}h(w)\left(\overline{h(w)}\right)^2,$$

a polynomial of w and $\overline{w}$ having only fourth to ninth order terms. Assuming now $w \in \mathrm{B}_{1/5}$ and using inequalities (4.19) and (4.21) we obtain that

$$R_3(z) \leq 0.14|w|^4 < 0.16|z|^4$$

is satisfied for $z = h(w)$. This inequality combined with equation (4.12) and inequality (4.19) yields that if $w \in \mathrm{B}_{1/5}$ and $z = h(w)$, then

$$\left|h^{-1}(z)\right| \leq |z| + 0.47|z|^3 + 0.16|z|^4 \tag{4.22}$$

holds.

Estimation of R_2

Now, we are ready to estimate R_2. Let us define the following three polynomials

$$\begin{aligned} h^{-1;\max}(s) &= s + 0.47s^3 + 0.16s^4, \\ G^{\max}(s) &= s + 1.57s^3 + 0.22s^4, \\ h^{\max}(s) &= s + 0.47s^3. \end{aligned} \tag{4.23}$$

Let also $Q(s) = \sum_{k=1}^{48} q_k s^k = h^{-1;\max} \circ G^{\max} \circ h^{\max}(s)$. It is obvious from our previous estimations that for $0 \neq w \in \mathrm{B}_{1/5}$, we have $|R_2(w)| < \sum_{k=4}^{48} q_k |w|^4 \left(\frac{1}{5}\right)^{k-4}$, which leads to

$$|R_2(w)| < 1.59|w|^4. \tag{4.24}$$

Step 3: A region of attraction for the fixed point 0 of system (4.10)

From equations (4.10),(4.11), (4.12) and (4.16) one can readily derive the formulae

$$c = c(\alpha, m) = -\frac{i}{\sqrt{4\alpha - m^2}}, \quad d = d(\alpha, m) = \frac{\left(m + i\sqrt{4\alpha - m^2}\right)^3}{12\alpha\left(4\alpha - m^2 + im\sqrt{4\alpha - m^2}\right)}. \tag{4.25}$$

Let

$$\beta = \beta(\alpha, m) = \frac{|\mu(\alpha,m)|}{\mu(\alpha,m)} c(\alpha, m) = \frac{-im - \sqrt{4\alpha - m^2}}{2\sqrt{\alpha}\sqrt{4\alpha - m^2}}$$

and let $\gamma = \gamma(\alpha)$ denote the real part of β, which is $\gamma = -\frac{1}{2\sqrt{\alpha}}$. Using these notations and inequality (4.24) we obtain that for all $0 \neq w \in \mathrm{B}_{1/5}$ we have

$$\begin{aligned} |G_{1:4}(w)| =& \left|\mu w + c(\alpha, m) w^2 \overline{w} + d(\alpha, m)\overline{w}^3 + R_2(w)\right| \\ \leq& |w| \left(\left|\mu + c|w|^2\right| + |d||w|^2\right) + |R_2(w)| \\ =& |w| \left(\left||\mu| + \beta|w|^2\right| + |d||w|^2\right) + |R_2(w, \overline{w}, \alpha)| \\ <& |w| \left(\left|\sqrt{\alpha} + \beta|w|^2\right| + |d||w|^2 + 1.59|w|^3\right) \\ \leq& |w| \left(\left|\sqrt{\alpha} + \gamma|w|^2\right|\right) \\ &+ |w| \Big(\left|\left|\sqrt{\alpha} + \beta|w|^2\right| - (\sqrt{\alpha} + \gamma|w|^2)\right| + |d||w|^2 + 1.59|w|^3\Big). \end{aligned} \tag{4.26}$$

Note that $-1 < -\frac{5}{7\sqrt{2}} \leq \gamma \leq -\frac{1}{2}$. Now supposing $0 \neq w \in \mathrm{B}_{1/5}$ yields the following

$$\begin{aligned} \left|\left|\sqrt{\alpha} + \beta|w|^2\right| - (\sqrt{\alpha} + \gamma|w|^2)\right| &= \left|\sqrt{\alpha + 2\sqrt{\alpha}\gamma|w|^2 + |\beta|^2|w|^4} - (\sqrt{\alpha} + \gamma|w|^2)\right| \\ &= \left|\frac{(|\beta|^2 - \gamma^2)|w|^4}{\sqrt{\alpha + 2\sqrt{\alpha}\gamma|w|^2 + |\beta|^2|w|^4} + \sqrt{\alpha} + \gamma|w|^2}\right| \\ &\leq \frac{(|\beta|^2 - \gamma^2)|w|^4}{\sqrt{25\alpha|w|^2 + 2\sqrt{\alpha}\gamma|w|^2} + 5\sqrt{\alpha}|w| + \gamma|w|} \\ &\leq \frac{(|\beta|^2 - \gamma^2)}{\sqrt{25\alpha - 2} + 5\alpha - 1}|w|^3. \end{aligned}$$

Using the formulae for γ, β and d, one readily get that the following inequalities hold

$$\left| |\sqrt{\alpha} + \beta|w|^2| - (\sqrt{\alpha} + \gamma|w|^2) \right| < 0.02|w|^3, \tag{4.27}$$

and

$$|d| \leq \frac{5}{3\sqrt{73}} < \tfrac{1}{5}. \tag{4.28}$$

Combining inequalities (4.26), (4.27) and (4.28) we obtain that for $0 \neq w \in \mathrm{B}_{1/5}$ we have

$$\begin{aligned} |G_{1:4}(w)| &< |w| \left(1 - 0.5|w|^2 + 0.2|w|^2 + 1.61|w|^3\right) \\ &= |w| \left(1 - |w|^2(0.3 - 0.161|w|)\right) < |w|, \end{aligned}$$

provided that $|w| < \varepsilon_0 = \frac{0.3}{0.161}$. This proves the asymptotic stability of the 0 fixed point of system (4.10) in the region $\mathrm{B}_{\varepsilon_0}$.

Step 4: The 0 fixed point of system (4.1) is asymptotically stable in the region $[-\frac{1}{6}, \frac{1}{6}]^2$

Inequalities (4.14) and (4.21) imply that for all $(x,y) \in [-\frac{1}{6}, \frac{1}{6}]^2$, $w \in \mathrm{B}_{\varepsilon_0}$ is satisfied. This guarantees that given $(x_0, y_0) \in U(\alpha) = [-\frac{1}{6}, \frac{1}{6}]^2$, we have $\lim_{k\to\infty}(x_k, y_k) = (0,0)$ and completes our proof.

□

Figure 6 illustrates how $U(\alpha)$ changes with the parameter α.

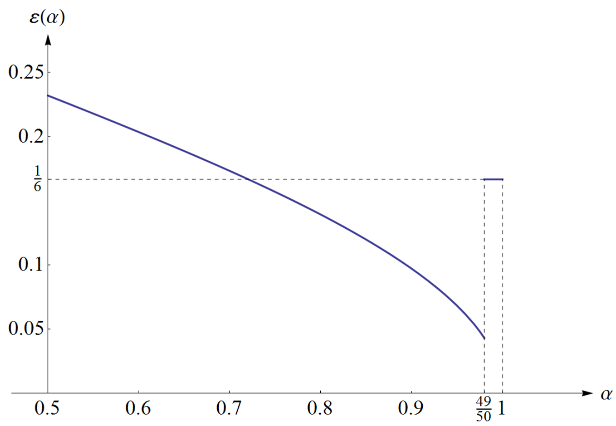

Figure 6: $U(\alpha)$ is the square with sides $2\varepsilon(\alpha)$, centred at 0.

Part II: Rigorous computations

Consider now a pair of parameter values$(\alpha, m) \in [\frac{1}{m+1}, 1] \times [0,1]$. Given any starting point (x_0, y_0), the accumulation points of its orbit $((x_k, y_k))_{k=0}^{\infty}$ are non-wandering points of $F_{\alpha,m}$. In order to prove that the fixed point $(0,0)$ is globally attracting, it is enough to show that it is the only non-wandering point of $F_{\alpha,m}$. We know from Corollary 3.8, that all the non-wandering points are inside $S = [-4,4]^2$. We shall show that S lies entirely in the basin of attraction of $(0,0)$, or equivalently, S contains exactly one non-wandering point, and that is $(0,0)$.

In the remaining part of the paper we emphasize, that $[\alpha], [m], [S]$ and $[U]$ are quantities that are represented in the computer as intervals or interval boxes, while $F_{[\alpha],[m]}$ is an interval valued function. Even though the sets are handled numerically, they provide rigorous enclosures of the number or set between the brackets. For any $(\alpha, m) \in [\alpha] \times [m]$ and for any $(x, y) \in [S]$, we have $F_{\alpha,m}(x, y) \in F_{[\alpha],[m]}(x, y)$. This is achieved by using the CAPD Library [7] for validated computations.

To proceed with the proof, first we divide the parameter range into small interval boxes $[\alpha] \times [m]$. Given one small box and a starting resolution δ, we shall run the procedure `Global_Stability`, that appeared as Algorithm 3 together with a proof of its correctness in [3]. The algorithm uses partitions and graph representations. For a detailed introduction the reader is referred to [3].

1: **procedure** GLOBAL_STABILITY($[\alpha], [m], \delta$)
2: $\quad [S] \leftarrow [-4, 4]^2$
3: $\quad [U] \leftarrow \cap_{\alpha \in [\alpha]} U(\alpha)$ $\qquad \triangleright$ from Theorems 4.4 and 4.5
4: $\quad \mathscr{V} \leftarrow$ Partition($[S], \delta$) $\qquad \triangleright$ $\mathscr{V}$ is a partition of $[S]$, $\mathrm{diam}(\mathscr{V}) \leq \delta$
5: $\quad$ **repeat**
6: $\quad\quad \mathscr{E} \leftarrow$ Transitions($\mathscr{V}, F_{[\alpha],[m]}$)
7: $\quad\quad \mathscr{G} \leftarrow$ GRAPH($\mathscr{V}, \mathscr{E}$) $\qquad \triangleright$ $\mathscr{G}$ is a graph representation of $F_{[\alpha],[m]}$
8: $\quad\quad T \leftarrow \{v : v$ is in a directed cycle $\}$
9: $\quad\quad$ **for all** $v \in \mathscr{V}$ **do**
10: $\quad\quad\quad$ **if** $v \notin T$ **or** $v \subseteq [U]$ **or** $F_{[\alpha],[m]}(v) \subseteq [U]$ **then**
11: $\quad\quad\quad\quad$ **remove** v from $\mathscr{G}$
12: $\quad\quad\quad$ **end if**
13: $\quad\quad$ **end for**
14: $\quad\quad \delta \leftarrow \delta/2$
15: $\quad\quad \mathscr{V} \leftarrow$ Partition($|\mathscr{V}|, \delta$)
16: $\quad$ **until** $|\mathscr{V}| = \emptyset$
17: **end procedure**

To obtain a simple picture of what the algorithm does, notice that it utilizes graph representations of the function $F_{[\alpha],[m]}$ over nested compact sets and with respect to partitions of decreasing diameter. The next (smaller) compact set is obtained by removing certain partition elements in line 11. A vertex v is removed from the graph representation only when we manage to establish that either it does not contain any non-wandering point or it lies inside the basin of attraction of the origin.

If the procedure ends in finite time, that is, at one point $|\mathscr{V}| = \emptyset$ is satisfied, it implies that the origin is the only non-wandering point in $[S]$, thus it is globally attracting for all parameter pairs inside the given box $[\alpha] \times [m]$.

The code is implemented in C++. The CAPD Library [7] and the Boost Graph Library [23] were used for obtaining rigorous computations and handling directed graphs respectively. We used Tarjan's algorithm [24] in order to find the directed cycles. We used different sizes for the parameter intervals and ran the computations on a cluster of the NIIF HPC centre at the University of Szeged parallelizing it with OpenMP. We covered the region $(\alpha, m) \in [\frac{1}{m+1}, 1] \times [0, 1]$ using 6964 parameter intervals $[\alpha] \times [m]$ of size between 0.01×0.01 and 0.001×0.001. The iteration count (that is one cycle in program `Global_Stability`) varied from 10 to 25. The computation took 67 minutes and 54 seconds, while the total run time, summing for all the simultaneous processes was 11 hours 47 minutes and 3 seconds.

Proof of Theorem 4.3. The program `Global_Stability` ran successfully for every parameter box. Combining this with Theorem 4.4 and Theorem 4.5, proves that $(0,0)$ is globally attracting for $(\alpha, m) \in [\frac{1}{m+1}, 1] \times [0,1]$. The output of these computations can be found at [1]. These results, together with Theorem 3.6, Corollary 3.8, Remarks 4.1 and 4.2, prove the global attractivity of $(0,0)$ for $(\alpha, m) \in \mathscr{R}$, and thus complete the proof of Theorem 4.3. $\square$

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References

[1] `http://www.math.u-szeged.hu/~garab/discrete_krisztin-walther`.

[2] Goetz Alefeld. Introduction to interval analysis. *SIAM Rev.*, 53(2):380–381, 2011.

[3] Ferenc A Bartha, Ábel Garab, and Tibor Krisztin. Local stability implies global stability for the 2-dimensional Ricker map. *arXiv preprint arXiv:1209.2406*, 2012.

[4] Yulin Cao. Uniqueness of periodic solution for differential delay equations. *J. Differential Equations*, 128(1):46–57, 1996.

[5] CAPA: Computer-aided proofs in analysis group. `http://www2.math.uu.se/~warwick/CAPA/`. University of Uppsala, University of Bergen.

[6] Wu-Hua Chen, Xiaomei Lu, and Dong-Ying Liang. Global exponential stability for discrete-time neural networks with variable delays. *Physics Letters A*, 358(3):186–198, 2006.

[7] Computer Assisted Proofs in Dynamics group. CAPD Library. `http://capd.ii.uj.edu.pl`. a C++ package for rigorous numerics.

[8] Michael Dellnitz and Andreas Hohmann. A subdivision algorithm for the computation of unstable manifolds and global attractors. *Numer. Math.*, 75(3):293–317, 1997.

[9] Michael Dellnitz, Andreas Hohmann, Oliver Junge, and Martin Rumpf. Exploring invariant sets and invariant measures. *Chaos*, 7(2):221–228, 1997.

[10] Zbigniew Galias. Rigorous investigation of the Ikeda map by means of interval arithmetic. *Nonlinearity*, 15(6):1759–1779, 2002.

[11] Shangjiang Guo, Lihong Huang, and Lin Wang. Exponential stability of discrete-time Hopfield neural networks. *Comput. Math. Appl.*, 47(8-9):1249–1256, 2004.

[12] NIIF: National Information Infrastructure Development Institute. `http://www.niif.hu`.

[13] V. L. Kocić and G. Ladas. *Global behavior of nonlinear difference equations of higher order with applications*, volume 256 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[14] Tibor Krisztin. Periodic orbits and the global attractor for delayed monotone negative feedback. In *Proceedings of the 6th Colloquium on the Qualitative Theory of Differential Equations (Szeged, 1999)*, Proc. Colloq. Qual. Theory Differ. Equ., pages No. 15, 12 pp. (electronic). Electron. J. Qual. Theory Differ. Equ., Szeged, 2000.

[15] Tibor Krisztin. Unstable sets of periodic orbits and the global attractor for delayed feedback. In *Topics in functional differential and difference equations (Lisbon, 1999)*, volume 29 of *Fields Inst. Commun.*, pages 267–296. Amer. Math. Soc., Providence, RI, 2001.

[16] Tibor Krisztin and Hans-Otto Walther. Unique periodic orbits for delayed positive feedback and the global attractor. *J. Dynam. Differential Equations*, 13(1):1–57, 2001.

[17] Tibor Krisztin, Hans-Otto Walther, and Jianhong Wu. *Shape, smoothness and invariant stratification of an attracting set for delayed monotone positive feedback*, volume 11 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1999.

[18] Yuri A. Kuznetsov. *Elements of applied bifurcation theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1998.

[19] Stefano Luzzatto and Paweł Pilarczyk. Finite resolution dynamics. *Found. Comput. Math.*, 11(2):211–239, 2011.

[20] S. Mohamad and K. Gopalsamy. Exponential stability of continuous-time and discrete-time cellular neural networks with delays. *Appl. Math. Comput.*, 135(1):17–38, 2003.

[21] Ramon E. Moore. *Methods and applications of interval analysis*, volume 2 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1979.

[22] N. S. Nedialkov, K. R. Jackson, and G. F. Corliss. Validated solutions of initial value problems for ordinary differential equations. *Appl. Math. Comput.*, 105(1):21–68, 1999.

[23] Jeremy G. Siek, Lie-Quan Lee, and Andrew Lumsdaine. *The Boost Graph Library User Guide and Reference Manual (With CD-ROM)*. 2002.

[24] Robert Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1(2):146–160, 1972.

[25] Warwick Tucker. A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.*, 2(1):53–117, 2002.

[26] Warwick Tucker. *Validated numerics*. Princeton University Press, Princeton, NJ, 2011. A short introduction to rigorous computations.

[27] Daniel Wilczak. Uniformly hyperbolic attractor of the Smale-Williams type for a Poincaré map in the Kuznetsov system. *SIAM J. Appl. Dyn. Syst.*, 9(4):1263–1283, 2010. With online multimedia enhancements.

[28] Honglei Xu, Yuanqiang Chen, and Kok Lay Teo. Global exponential stability of impulsive discrete-time neural networks with time-varying delays. *Applied Mathematics and Computation*, 217(2):537–544, 2010.

[29] Qiang Zhang, Xiaopeng Wei, and Jin Xu. On global exponential stability of discrete-time Hopfield neural networks with variable delays. *Discrete Dyn. Nat. Soc.*, pages Art. ID 67675, 9, 2007.