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The Lorenz attractor exists. (English. English, French summary)

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In a famous paper [J. Atmospheric Sci. 20 (1963), no. 2, 130–141], the meteorologist E. N. Lorenz proposed a very simplified model for thermal fluid convection. This model, a quadratic vector field in 3 dimensions:

$$(*) \qquad X(x,y,z) = \begin{cases} \dot{x} = -\alpha x + \alpha y \\ \dot{y} = \beta x - y - xz \\ \dot{z} = -\gamma z + xy, \end{cases}$$

and the values $\alpha = 10$, $\beta = 28$, and $\gamma = 8/3$ that he chose for the parameters, were motivated by an attempt to understand the foundations of weather forecasting. While it is unclear whether this simple system of differential equations does model the features of thermal convection, the observations of Lorenz came to attract a good deal of attention from mathematicians and experimentalists alike. Indeed, his numerical integration of equations $(*)$ showed that the solutions exhibit chaotic behavior, i.e., they are sensitive with respect to the initial conditions. Moreover, such a behavior seemed to be robust, meaning that it persists for all nearby parameter values.

However, Lorenz's equations proved to be very resistant to rigorous analysis, and also presented very serious obstacles to rigorous numerical study. A very successful approach was taken by V. S. Afraimovich, V. V. Bykov and L. P. Shilnikov [Dokl. Akad. Nauk SSSR 234 (1977), no. 2, 336–339; MR 57#2150] and by J. Guckenheimer and R. F. Williams [Inst. Hautes Etudes Sci. Publ. Math. No. 50, (1979), 59–72; MR 82b:58055a] independently: they constructed so-called geometric models for the behavior observed by Lorenz. These models are flows in 3 dimensions for which one can rigorously prove the existence of an attractor (a bounded region in phase-space, invariant under time evolution, such that the forward trajectories of most or even all points nearby converge to it) that contains an equilibrium point of the flow, together with regular solutions. Moreover, for almost every pair of nearby initial conditions the corresponding solutions move away from each other exponentially fast as they converge to the attractor. Most remarkably, this attractor is robust: it cannot be destroyed by any small perturbation of the original flow. It remained to prove that the original equations do contain such a strange (sensitive) robust attractor.

Another approach was through rigorous numerics. In this way, it could be proved, by B. D. Hassard et al. [Appl. Math. Lett. 7 (1994), no. 1, 79–83; MR 96d:58082], S. P. Hastings and W. C. Troy [Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 2, 298–303; MR 93f:58150] and K. Mischaikow and M. Mrozek [Bull. Amer. Math. Soc. (N.S.) 32 (1995), no. 1, 66–72; MR 95e:58121; Math. Comp. 67 (1998), no. 223, 1023–1046; MR 98m:58095], that the equations exhibit a suspended Smale horseshoe [S. Smale, Bull. Amer. Math. Soc. 73 (1967), 747–817; MR 37#3598]. In particular, they have infinitely many closed solutions. However, proving the existence of a strange attractor as in the geometric models is an even harder task, because one cannot avoid the main numerical difficulty posed by Lorenz's equations, which arises from the very presence of an equilibrium point: solutions slow down as they pass near the origin, which means unbounded return times and, thus, unbounded integration errors.

So, after numerous attempts for the last quarter of a century or so, it seemed that the problem of establishing Lorenz's observations on a rigorous basis remained out of reach. Yet, rather surprisingly, a positive solution to this problem is announced by Tucker, in the paper under review. Using a combination of rigorous numerics and normal form theory, he proves that the Lorenz equations (∗) support a robust strange attractor $\mathcal{A}$, for the classical parameter values $\alpha = 10$, $\beta = 28$, $\gamma = 8/3$. Furthermore, the flow admits a unique Sinai-Ruelle-Bowen (SRB) measure μ_X with $\operatorname{supp}(\mu_X) = \mathcal{A}$. This result corresponds to his Ph.D. thesis under the guidance of L. Carleson.

An SRB measure μ is characterized as being the limit time average for a set of initial points x_0 with positive volume (Lebesgue measure):

$$\mu = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \delta_{x_t}\,dt,$$

where δ_p is the Dirac measure on a point p and $t \mapsto x_t$ is the solution with initial condition x_0.

Sketch of the proof. Tucker's proof is through a computer algorithm (put into effect via a C language program) that estimates convenient solutions of (∗), keeping rigorous bounds on the errors. A successful termination of the algorithm proves the presence of a robust strange attractor in the equations. Existence of an SRB measure then follows along well-understood lines. The algorithm incorporates two main kinds of ingredients: a numerical integrator, which is used to compute good approximations of trajectories of the flow far from the equilibrium point sitting at the origin; and quantitative results from normal form theory, which make it possible to handle trajectories close to the origin.

The paper under review is an announcement of this result, containing only an outline of main ideas. The source codes and list of initial data used by the computer program are available on the author's web page for his Ph.D. thesis [http://www.math.uu.se/~warwick/thesis.html].

The author promises a complete proof in a paper that, in the meantime, has appeared as a preprint ["A rigorous ODE solver and Smale's 14th problem", http://www.math.uu.se/~warwick/rodes.html]. In what follows we give a brief sketch of Tucker's strategy, based on the present paper, as well as on a survey of this and related topics that has recently been published by M. Viana [Math. Intelligencer 22 (2000), no. 3, 6–19]. I am thankful to Viana for showing me his paper before publication, and for helpful conversations on this topic.

To start with, Tucker rewrites $(*)$ in new coordinates x_1, x_2, x_3, related to the original ones by a linear map and chosen so that the expression of $DX(0)$ in these new coordinates is diagonal. Also, the local stable manifold $W^s(0)$ of the origin is contained in the plane $x_1 = 0$. Then he fixes a cross-section $\Sigma \subset \{x_3 = 27\}$ for the flow, and restates the problem in terms of a first return map of the flow to the cross-section Σ. This return map P is not defined in $\Gamma = \Sigma \cap W^s(0)$. There are three fundamental facts to be proved: (a) There is $N \subset \Sigma$ with $P(N \setminus \Gamma) \subset \mathrm{int}(N)$, where $\mathrm{int}(A)$ means the interior of A. (b) The first return map P admits a forward invariant cone field $\mathcal{C}$, i.e., for all $x \in N$, $DP(x)\mathcal{C}(x) \subset \mathcal{C}(P(x))$. (c) There are constants $a > 0$ and $\lambda > 1$ such that for every $v \in \mathcal{C}(x)$ and $n \geq 1$, $\|DP^n(x)v\| \geq a\lambda^n\|v\|$.

Indeed, (a), (b), and (c) imply that the flow has a strange attractor. Actually, one also needs a lower bound for the value of λ, which is also provided by the computer program, in order to conclude that the attractor is dynamically indecomposable, that is, it contains dense orbits.

Let us detail the main ingredients of the algorithm a bit more.

Near the equilibrium. Here "near the equilibrium" means "inside a cube C of size 1/5 around the origin". If a trajectory hits C, Tucker's algorithm estimates the exit point directly, using normal form theory (instead of step-by-step integration), as follows.

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the original vector field at the origin are far from being resonant: $n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3$ are not zero, or too close to zero, for many values of $n_1 > 0$, $n_2 > 0$, $n_3 > 0$. Although the set of triples (n_1, n_2, n_3) for which this fact is needed is infinite, it can be dealt with through only a finite number of computations. Those computations are done by an auxiliary computer program.

Having this, Tucker can prove a precise normal form theory estimate for the vector field inside C: there is a change of variables $y = (y_1, y_2, y_3)$ which transforms the Lorenz equations $(*)$ into one very close to being linear: $(**)$ $X(y) = DX(0)y + G(y)$, with $|G(y)| \leq K|y_1|^{10}(|y_2| + |y_3|)^{10}$ and K a positive constant. Thus, in these coordinates solutions of the flow look very much like solutions of a linear flow (that can be obtained analytically), and so their exit point is well approximated by that of those linear solutions. Most important, $(**)$ also provides good rigorous bounds for the errors involved in this approximation.

Rigorous integrations. In order to prove statement (a), Tucker begins by finding a good candidate for the trapping region N, through non-rigorous computations of the image of Σ under the return map. He covers the approximate locations of this image by 350 adjacent small rectangles N_i of size $\delta_{\max} = 0.03$ inside Σ. The forward trajectories of each of these small rectangles are then going to be estimated, separately. The goal is to prove that, integration errors taken into account, those trajectories are bound to hit again inside the region N.

The algorithm deals with each N_i, separately, as follows. Using Euler's method, the trajectory of the center of the rectangle is integrated, until it hits another horizontal plane Σ_1, situated at distance $h = 10^{-3}$ underneath Σ. Taylor expansion gives an estimate of the position of the image inside Σ_1 of the whole N_i under the corresponding flow map, and one also has a rigorous upper bound for the integration error. Thus one may find some rectangle N_i' inside Σ_1 that surely contains the image of N_i. Typically, N_i' is much larger than N_i. The program subdivides it into subrectangles, small enough so that it can use Taylor expansion for each one of them, in the next integration step. Then it proceeds with each one of these rectangles in the place of N_i.

The whole algorithm consists in repeating this step, apart from the following two observations. As we already explained, stretches of solutions close to the origin are treated globally, not by step-by-step integration. Moreover, there are regions where the vector field is relatively close to the horizontal direction, which may cause large errors when one integrates from one horizontal plane to another. So, at each step wherever this happens, the program switches from horizontal cross-sections to vertical ones, with rigorous bounds on the errors involved in this. Later, it may go back to horizontal planes, if the vector field becomes nearly vertical again, and so on.

Integration proceeds until rectangles return to Σ. Of course, due to the subdivision, the program has to deal with an increasing number of rectangles. If they all come back inside N, this proves that N is indeed a trapping region as in (a).

An invariant expanding cone field. The verification of property (b) is similar, albeit more subtle. Tucker starts by choosing a cone field $\mathcal{C}$ on each rectangle N_i. Each cone C_i is represented by the two angles α_i^- and α_i^+ that its boundary vectors u^0 and v^0 make with the x_1-axis. He always takes $\theta^{(0)} = \alpha_i^+ - \alpha_i^- = \pi/18$.

Then, in parallel to finding a rigorous upper bound for the image of the rectangle N_i under the flow, as explained above, the program also computes a rigorous upper bound for the image of the cone C_i under the linearized flow. This linearized flow is described by a system of nine equations. Integrating the trajectories of u^0 and v^0, and taking integration errors into account, one obtains a new cone that certainly contains the image of C_i. Unlike the case in the previous part of the proof, here subdivision is not necessary.

Again, one knows that the cone field $\mathcal{C}$ is invariant if, upon return to Σ, every image cone lies inside the original one.

Finally, the program computes a rigorous lower bound for the expansion of tangent vectors inside each C_i. For this, the program computes lowest possible expansions at each integration step, in the following way. At each step where the cone becomes slimmer at that step, the least expansion takes place on the boundary of the cone, and so it may be estimated from the sizes of the boundary vectors and their images. In the opposite case, this estimate is corrected by a factor which differs from 1 by a term of second order on the size of the cone.

The program keeps track of these successive expansion lower bounds, so that property (c) can be readily checked at the time of return to Σ: the product is bigger than some constant $\lambda > 1$.

Successful verification of all the inequalities involved shows that Λ is a robust strange attractor for $(*)$, and thus Lorenz's conjecture is proved. *Maria Jose Pacifico* (BR-FRJ)