Correction to "Convergence rates in precise asymptotics"

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Abstract

We provide a correction note to our paper with the above title.

According to a careful scrutiny of our manuscript [3], which helped us to correct an earlier mistake, we were also led to reconsider—and correct our paper [2]. In particular, Lemma 2.3 there has to be modified as follows:

Lemma 2.3 *We have, as* $n \to \infty$,

$$\sum_{j=1}^{n} j^\gamma = \begin{cases} \frac{n^{\gamma+1}}{\gamma+1} + \frac{n^\gamma}{2} + \mathcal{O}\big(n^{\gamma-1}\big), & \text{for} \quad \gamma > 1, \\ \frac{n^2}{2} + \frac{n}{2}, & \text{for} \quad \gamma = 1, \\ \frac{n^{\gamma+1}}{\gamma+1} + \frac{n^\gamma}{2} + \mathcal{O}(1), & \text{for} \quad 0 < \gamma < 1, \\ n, & \text{for} \quad \gamma = 0, \\ \frac{n^{\gamma+1}}{\gamma+1} - \kappa_\gamma + \mathcal{O}\big(n^\gamma\big), & \text{for} \quad -1 < \gamma < 0, \end{cases}$$

where, in the last case, $0 < -\frac{\gamma}{\gamma+1} < \kappa_\gamma \le \frac{1}{\gamma+1}$.

The flaw in [2] occurs for the case $-1 < \gamma < 1$, where a more careful application of the Euler-MacLaurin summation formula provides the above approximations (cf., e.g., [1], p. 124).

A direct consequence of this is that Lemma 3.1 in [2] must be modified into

Lemma 3.1 *In the notation of Subsection 2.1 we have, as* $\varepsilon \searrow 0$,

$$\lambda_{r,p}(\varepsilon) = \begin{cases} \frac{p}{r-p} \cdot A_{r,p}(\varepsilon) + \frac{1}{2} \cdot A^{(1)}_{r,p}(\varepsilon) + \mathcal{O}\big(A^{(2)}_{r,p}(\varepsilon)\big), & \text{for} \quad r > 3p, \\ \frac{1}{2} \cdot A_{r,p}(\varepsilon) + \frac{1}{2} \cdot A^{(1)}_{r,p}(\varepsilon), & \text{for} \quad r = 3p, \\ \frac{p}{r-p} \cdot A_{r,p}(\varepsilon) + \frac{1}{2} \cdot A^{(1)}_{r,p}(\varepsilon) + \mathcal{O}(1), & \text{for} \quad 2p < r < 3p, \\ A_{r,p}(\varepsilon), & \text{for} \quad r = 2p, \\ \frac{p}{r-p} \cdot A_{r,p}(\varepsilon) - \kappa_{(r/p)-2} + \mathcal{O}(\varepsilon) + \mathcal{O}\big(A^{(1)}_{r,p}(\varepsilon)\big), & \text{for} \quad 2 \le r < 2p, \end{cases}$$

where, in the last case, $0 < -\frac{r-2p}{r-p} < \kappa_{(r/p)-2} \le \frac{p}{r-p}$.

This, finally leads to the following modification of Proposition 1.1 of [2]:

Proposition 1.1 *Let* $0 < p < 2$ *and* $r \ge 2$, *and suppose that* $Y, X_1, X_2, \ldots$ *are i.i.d. normal random variables with mean* 0 *and variance* $\sigma^2 > 0$, *and set* $S_n = \sum_{k=1}^n X_k$, $n \ge 1$.
(i) *If* $r < 2p$, *then*

$$\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \ge \varepsilon n^{1/p}) - \frac{p}{r-p} \cdot \varepsilon^{-\frac{2(r-p)}{2-p}} E|Y|^{\frac{2(r-p)}{2-p}} \Big) = -\kappa_{(r/p)-2} \,.$$

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More precisely, if $2r-5p+2<0$, *then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n|\geq \varepsilon n^{1/p}) - \frac{p}{r-p}\cdot\varepsilon^{-\frac{2(r-p)}{2-p}}E|Y|^{\frac{2(r-p)}{2-p}} = -\kappa_{(r/p)-2}+\mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon\searrow 0\,,$$

if $2r-5p+2=0$, *then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n|\geq \varepsilon n^{1/p}) - \frac{p}{r-p}\cdot\varepsilon^{-\frac{2(r-p)}{2-p}}E|Y|^{\frac{2(r-p)}{2-p}} = -\kappa_{(r/p)-2}+\mathcal{O}\big(\varepsilon\log(1/\varepsilon)\big) \quad \text{as} \quad \varepsilon\searrow 0\,,$$

and, if $2r-5p+2>0$, *then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n|\geq \varepsilon n^{1/p}) - \frac{p}{r-p}\cdot\varepsilon^{-\frac{2(r-p)}{2-p}}E|Y|^{\frac{2(r-p)}{2-p}} = -\kappa_{(r/p)-2}+\mathcal{O}\big(\varepsilon^{\frac{2(2p-r)}{2-p}}\big) \quad \text{as} \quad \varepsilon\searrow 0\,.$$

(ii) *If* $r=2p$, *then*

$$\lim_{\varepsilon\searrow 0}\Big(\sum_{n=1}^{\infty} P(|S_n|\geq \varepsilon n^{1/p}) - \varepsilon^{-\frac{2p}{2-p}}E|Y|^{\frac{2p}{2-p}}\Big) = -\frac{1}{2}\,;$$

More precisely, if $p\geq 2/3$, *then*

$$\sum_{n=1}^{\infty} P(|S_n|\geq \varepsilon n^{1/p}) - \varepsilon^{-\frac{2p}{2-p}}E|Y|^{\frac{2p}{2-p}} = -\frac{1}{2}+\mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon\searrow 0\,,$$

and, if $p<2/3$, *then*

$$\sum_{n=1}^{\infty} P(|S_n|\geq \varepsilon n^{1/p}) - \varepsilon^{-\frac{2p}{2-p}}E|Y|^{\frac{2p}{2-p}} = -\frac{1}{2}+\mathcal{O}\big(\varepsilon^{\frac{2p}{2-p}}\big) \quad \text{as} \quad \varepsilon\searrow 0\,.$$

(iii) *If* $2p<r<3p$, *then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n|\geq \varepsilon n^{1/p}) - \frac{p}{r-p}\cdot\varepsilon^{-\frac{2(r-p)}{2-p}}E|Y|^{\frac{2(r-p)}{2-p}} = \mathcal{O}(1) \quad \text{as} \quad \varepsilon\searrow 0\,.$$

(iv) *If* $r=3p$, *then*

$$\sum_{n=1}^{\infty} nP(|S_n|\geq \varepsilon n^{1/p}) - \frac{1}{2}\varepsilon^{-\frac{4p}{2-p}}E|Y|^{\frac{4p}{2-p}} = \mathcal{O}(1) \quad \text{as} \quad \varepsilon\searrow 0\,.$$

(v) *If* $r>3p$, *then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n|\geq \varepsilon n^{1/p}) - \frac{p}{r-p}\cdot\varepsilon^{-\frac{2(r-p)}{2-p}}E|Y|^{\frac{2(r-p)}{2-p}} = \mathcal{O}\big(\varepsilon^{-\frac{2(r-3p)}{2-p}}\big) \quad \text{as} \quad \varepsilon\searrow 0\,.$$

The revised proof of the proposition amounts to replacing the old Lemma 3.1 with the above one whenever there is a discrepancy between the two versions.

Parts of Corollary 1.1 must consequently be modified accordingly.

References

[1] Cramér, H. (1946). *Mathematical Methods of Statistics.* Princeton University Press. Princeton, N.J.

[2] Gut, A., and Steinebach, J. (2012). Convergence rates in precise asymptotics. *J. Math. Anal. Appl.* **390**, 1-14.

[3] Gut, A. and Steinebach, J. (2013). Convergence rates in precise asymptotics II. *Annales Univ. Sci. Budapest., Sect. Comp.* **39**, 95-110.

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