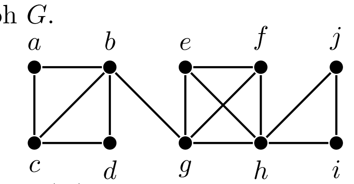
UPPSALA UNIVERSITET
Matematiska institutionen
Anders Johansson

Mathematics
Frist, KandMa, IT
Graph Theory
2010–12–20

Written exam in Mathematics: Graph Theory

Use short but full motivations. Those who has participated in the problem sessions should not do the first question, since full points are credited.

1. Consider the following graph G.

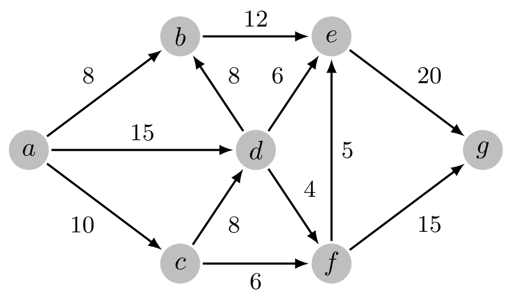


(a) State the clique-number $\omega(G)$ and the chromatic number $\chi(G)$.

(b) Draw the tree $B(G)$ of blocks and cut-vertices. (Biconnected components and articulation points).

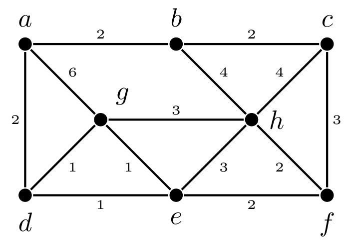
(c) Give the number of spanning trees in G.

2. Consider the following weighted network $G = (V, E, c)$.



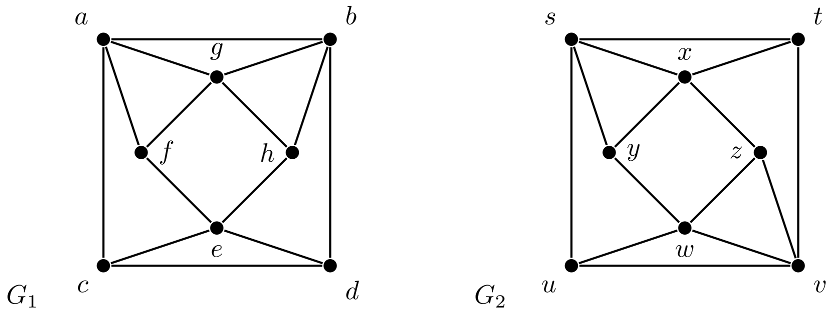
Interpret the weights as capacities for a flow network and find a maximum flow from node a to node g together with a minimum cut.

3. Find two minimum spanning trees in the following weighted graph, using a greedy algorithm.



4. For the alphabet $\mathcal{A} = \{A, B, C, D, E, F\}$ with weights $\{8, 9, 12, 34, 2, 39\}$, construct a corresponding Huffman code $\varphi : \mathcal{A} \to \mathbb{Z}_2^*$.

5. Consider the following two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.



(a) Define a proper 3-colouring $\sigma : V(G_1) \to \{1, 2, 3\}$.

(b) Determine whether G_1 and G_2 are isomorphic graphs.

6. Show that there is a tree with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ *if and only if* $d_1 \geq 1$ and

$$\sum_{i=1}^{n} d_i = 2(n-1).$$

7. (a) For each n, give an example of a *color-critical* graph $G = (V, E)$ with $|V| = n$, i.e. a graph G such that $\chi(G - v) < \chi(G)$ for all $v \in V$.

(b) If G is color-critical, show that $d(v, G) \geq \chi(G) - 1$ for all $v \in V$.

8. Assume that G is a simple graph containing a cycle $C \subset G$ and that two vertices $x, y \in V(C)$ on C are connected in G by a path P of length k. Show that G contains a cycle of length at least $\sqrt{2k}$.

Terminology

Auto-morphism For a simple graph $G = (V, E)$, a bijective map $\phi : V \to V$ such that $\{x, y\} \in E$ if and only if $\{\phi(x), \phi(y)\} \in E$ is called an *auto-morphism.*

Decomposition A set of edge-disjoint subgraphs which together cover all edges in the given graph.

Degree and Neighbourhood The degree, $d(v)$, of a vertex v is the number of edges with which it is incident. Two vertices are adjacent if they are incident to a common edge. The set of neighbours (neighbourhood), $N(v)$, of a vertex v is the set of vertices which are adjacent to v. For a simple graph, the degree of a vertex is also the cardinality of its neighbour set.

Maximal, average and minimal degree The average degree of a graph G is

$$\bar{d}(G) := \frac{1}{|V|} \sum_{v \in V} d(v, G).$$

The minimal degree is $\delta(G) := \min_v d(v, G)$ and the maximal degree is $\Delta(G) := \max_v d(v, G)$. Clearly, $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$.

Spanning tree A tree is a connected graph without cycles. Every connected graph on n vertices has at least one tree as a spanning subgraph — a spanning tree.

Internally disjoint Paths are internally vertex disjoint if the corresponding vertex-sets only intersect at end-points.

Induced subgraphs For a set of vertices X, we use $G[X]$ to denote the induced subgraph of G whose vertex set is X and whose edge set is the subset of $E(G)$ consisting of those edges with both ends in X. For a set S of edges, we use $G[S]$ to denote the edge induced subgraph of G whose edge set is S and whose vertex set is the subset of $V(G)$ consisting of those vertices incident with any edge in S. If Y is a subset of $V(G)$, we write $G - Y$ for the subgraph $G[V(G) - Y]$.

Paths and cycles and more A walk is an alternating sequence of vertices and edges, with each edge being incident to the vertices immediately preceeding and succeeding it in the sequence. A trail is a walk with no repeated edges. A path is a walk with no repeated vertices. A walk is closed if the initial vertex is also the terminal vertex. A cycle is a closed trail with at least one edge and with no repeated vertices except that the initial vertex is the terminal vertex. We refer to paths and cycles, also identified the *subgraphs* spanned by the edges occurring.

Clique, $\omega(G)$ A (sub-)graph graph is complete, or a clique, if every pair of distinct vertices is adjacent. We write K_m for the (isomorphism class of) complete graph on m vertices. We write $\omega(G)$ for the largest clique of a graph. (The clique-number of G.)

Hamiltonian and Eulerian A Hamiltonian graph is a graph that has a Hamilton-cycle. An Eulerian graph is one that admits an Euler-cycle, i.e. connected and all vertices have even degree.

Colouring and proper colouring A colouring simply means an assignment $\sigma : V \to S$, where S is a finite set of colours. A *proper* colouring is such that adjacent vertices receives different colours. A proper edge-colouring is a mapping $\sigma : E \to S$ such that no two incident edges obtain the same colour.

k**-factor** A k-factor of a graph $G = (V, E)$ is a spanning subgraph F which is k-regular, i.e. $d(x, F) = k$ for all $x \in V$. For balanced bipartite graphs, a one-factor is also called a *perfect matching.*

k-regular A graph is k-regular if every vertex has degree k.

Orienting a graph An undirected graph can be *oriented*, i.e. made into a digraph $\overrightarrow{G}$, by choosing, for each edge $e = \{u, v\}$ one of (u, v) or (v, u) as the corresponding edge in $\overrightarrow{G}$.

Bipartite graph A bipartite graph is a graph such that V is composed of two non-empty disjoint parts X and Y and all edges connects vertices in X with vertices in Y. A bipartite graph is *balanced* if $|X| = |Y|$.

Number of components of a graph G is denoted by $c(G)$.

Oriented cycle In a digraph an *oriented walk* is a sequence $v_1e_1v_2 \cdots v_ke_kv_{k+1}$, of vertices v_i and edges e_j, such that either $e_i = v_iv_{i+1}$ or $e_i = v_{i+1}v_i$; i the second case we say the edge is oppositely oriented. Oriented trails, circuits, paths and cycles are defined in an analogue manner.

Directed cycle A normal, i.e. not oriented, cycle in a digraph.

Strongly connected A digraph is strongly connected if for any pair of vertices (i, j) there is a directed path from i to j. (And thus also a directed path from j to i.)

Cut-set, k-edge-connected,bridge A cut-set in a connected graph G is a set of edges W such that $G - W$ contains at least two components. A graph is k-edge connected if every cut-set has at least k edges. A bridge is a cut-set of one edge.

Cut-vertex, block A cut-vertex is a vertex v such that $G - v$ have more components than G. A *block* a is a maximal (induced) subgraph without any cut-vertex, a maximal 2-connected graph or a bridge.

k-connected, $\kappa(G)$ A connected graph G is *k-connected* (or k-vertex connected) if $|V(G)| \geq k+1$ and $G-S$ is connected for any set $S \subset V(G)$ of at most $k-1$ vertices. In other words, no two vertices are separated by a set of $k - 1$ vertices. The largest k, such that G is k-connected, is called the (vertex-) connectivity of G and is denoted $\kappa(G)$.

Tournament A tournament is a complete oriented graph, i.e. a simple directed graph such that the underlying undirected graph is isomorphic to K_n for some n.