Functional Analysis F3/F4/NVP (2005)
Homework assignment 1

All students should solve the following problems:

1. The *boundary* of a set $A \subset (X, d)$ is defined in Section 1.3, Problem 11 (p.24). Show that the boundary of an arbitrary set is a closed set.

2. Section 1.6, Problem 10.

3. Section 2.3: Problem 2.

4. Section 2.5: Problem 2.

Students taking Functional Analysis as a 6 point course should also solve the following problems:

5. Section 2.1: Problem 14 and Section 2.3, Problem 14.

6. Section 2.5: Problem 4.

Solutions should be handed in by Tuesday, February 1, 16.00. (Either give the solutions to me directly or put them in my mailbox, third floor, House 3, Polacksbacken.)

Functional Analysis F3/F4/NVP

Comments to homework assignment 1

(Notation as in my solutions.)

1. Note that to prove that a set is *closed* it is *not* sufficient to prove that it is not open. In fact, in a metric space (X, d) there are often many sets which are *neither* open nor closed. (There are also sets which are *both* open and closed; for example, in every metric space (X, d), the subsets X and $\emptyset$ (the empty set) are both open and closed.)

3. A common mistake: Since $x_j \to x$ we have, for each n, $\lim_{j\to\infty} \xi_{n,j} = \eta_n$. Hence we may write:

$$\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \big(\lim_{j\to\infty} \xi_{n,j} \big).$$

So far it is correct! But it is *not* acceptable to change the order of limits here (to get "$= \lim_{j\to\infty} \big(\lim_{n\to\infty} \xi_{n,j} \big) = \lim_{j\to\infty} 0 = 0$") without careful motivation!

Here is one example that shows why this is not possible: Let

$$\begin{aligned} x_1 &= (1,0,0,0,0,0,...) \\ x_2 &= (1,1,0,0,0,0,...) \\ x_3 &= (1,1,1,0,0,0,...) \\ x_4 &= (1,1,1,1,0,0,...) \\ &\text{etc.} \end{aligned}$$

This sequence *does not* converge in ℓ^∞. Furthermore we have $\lim_{j\to\infty} \xi_{n,j} = 1$ for every n, hence $\lim_{n\to\infty} \big(\lim_{j\to\infty} \xi_{n,j} \big) = 1$, whereas $\lim_{j\to\infty} \big(\lim_{n\to\infty} \xi_{n,j} \big) = 0$. This indicates that to motivate the desired change of order of limits we must make *further use of the fact that $x_j \to x$ in ℓ^∞.*

5. In Problem 2.1; 14, one has to prove that the stated operations on cosets are *well-defined*. Logically this should be done *before* one proves anything else about the operations (since it is only after we have proved "well-definedness" that we truly know that the operations "exist"). [But I did not give minus score for proving well-defined in the end of solution.]

Another common mistake: As a step in trying to prove that $|| \cdot ||_0$ satisfies (N4), many students have claimed

$$\inf_{y \in Y} \big(||x + y|| + ||w + y|| \big) = \inf_{y \in Y} ||x + y|| + \inf_{y \in Y} ||w + y||$$

(or $*** \leq ***$). This is in general *not* true: The right hand side is in general *smaller* than the left hand side since $\inf_{y \in Y} ||x + y||$ and $\inf_{y \in Y} ||w + y||$ may be attained at completely *different* points $y \in Y$. (In fact we even do not know if the infima are attained in general.)

Another common mistake: Many students seem to use laws like $\hat{x} + \hat{w} = \{x + w \mid x \in \hat{x}, w \in \hat{w}\}$ in the second half of the problem, without ever proving this. Note that we have no right to assume (without proof) that this agrees with the definition of "+" given in the problem! (Cf. the "alternative proof that addition and multiplication of cosets are well-defined" in the solution below.)

Functional Analysis F3/F4/NVP
Solutions to homework assignment 1

1. Let A be any subset of a metric space X. Let ∂A be the boundary of A. We wish to prove that ∂A is closed.

Let $x_1, x_2, \ldots$ be any sequence of points in ∂A such that $x_n \to x$ for some point $x \in X$.

Let $\varepsilon > 0$ be an arbitrary number. Then, since $x_n \to x$, there is some index N such that $d(x_N, x) < \varepsilon/10$.

But $x_N \in \partial A$, hence by the definition of ∂A (problem 11, p.24) every neighborhood of x_N contains points of A as well as points not belonging to A. In particular this holds for the neighborhood $B(x_N, \varepsilon/10)$, i.e. there is a point $a \in B(x_N, \varepsilon/10)$ which lies in A, and there is another point $b \in B(x_N, \varepsilon/10)$ which does not lie in A.

By the triangle inequality, using $d(x_N, x) < \varepsilon/10$ and $a \in B(x_N, \varepsilon/10)$, we get

$$d(x, a) \leq d(x, x_N) + d(x_N, a) < \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon.$$

Similarly:

$$d(x, b) \leq d(x, x_N) + d(x_N, b) < \frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon.$$

Hence both a and b lie in the ball $B(x, \varepsilon)$, meaning that $B(x, \varepsilon)$ contains a point of A as well as a point not belonging to A.

But recall that ε was an *arbitrary* positive number; hence we have proved that *every* neighborhood of x contains points of A as well as points not belonging to A. By the definition of ∂A (problem 11, p.24) this means that $x \in \partial A$.

Since this is true for *every* point $x \in X$ which is a limit point of a sequence of points $x_1, x_2, \ldots$ in ∂A, it follows from Theorem 1.4-6(b) that the set ∂A is closed.

Alternative solution, not using Theorem 1.4-6. Let A be any subset of a metric space X. Let ∂A be the boundary of A. We wish to prove that ∂A is closed. In other words (see Def 1.3-2), we wish to prove that the complement set $X - \partial A$ is open.

Let x be an arbitrary point in $X - \partial A$. Then since $x \notin \partial A$, by the definition of ∂A in problem 11, p.24, there exists some $\varepsilon > 0$ such that the ε-neighborhood $B(x, \varepsilon)$ only contains points of A, *or* only contains points not belonging to A, that is,

$$(*) \qquad B(x, \varepsilon) \subset A \qquad \text{or} \qquad B(x, \varepsilon) \subset X - A.$$

We now claim that in fact

$$(**) \qquad B(x,\varepsilon) \subset X - \partial A.$$

To prove this, let y be an arbitrary point in $B(x,\varepsilon)$. Then since $B(x,\varepsilon)$ is open (known from problem 1, p.23), there is some $r > 0$ such that $B(y,r) \subset B(x,\varepsilon)$. Combining this with (*), we see that:

$$B(y,r) \subset A \qquad \text{or} \qquad B(y,r) \subset X - A.$$

This means that y has a neighborhood which only contains points of A, *or* only contains points not belonging to A. By the definition of ∂A (problem 11, p.24), this means that $y \notin \partial A$, i.e. $y \in X - \partial A$. But recall that y was an *arbitrary* point in $B(x,\varepsilon)$. This means that (**) is true!

Now recall that x was an *arbitrary* point in $X - \partial A$, i.e. (**) says $X - \partial A$ contains a ball about each of its points. Hence $X - \partial A$ is open, by Def 1.3-2.

Hence ∂A is closed.

Alternative solution (which uses more facts from the book, and gives extra useful information about ∂A). It follows from the definition of ∂A in problem 11, p.24 that

$$(*) \qquad \partial A = \Big\{x \in X \mid \forall \varepsilon > 0 : \big[B(x,\varepsilon) \cap A \neq \emptyset \text{ and } B(x,\varepsilon) \cap (X - A) \neq \emptyset\big]\Big\}.$$

We recall the definition of the interior of A (see p.19):

$$A^\circ = \Big\{x \in X \mid \exists \varepsilon > 0 : B(x,\varepsilon) \subset A\Big\}.$$

This implies that

$$\begin{aligned} X - A^\circ &= \Big\{x \in X \mid \forall \varepsilon > 0 : B(x,\varepsilon) \not\subset A\Big\} \\ &= \Big\{x \in X \mid \forall \varepsilon > 0 : B(x,\varepsilon) \cap (X - A) \neq \emptyset\Big\}. \end{aligned}$$

Furthermore the closure of A is (see p.21, easy reformulation):

$$\overline{A} = \Big\{x \in X \mid \forall \varepsilon > 0 : B(x,\varepsilon) \cap A \neq \emptyset\Big\}.$$

From the last two formulas we see that

$$\begin{aligned} &(X - A^\circ) \cap \overline{A} \\ &= \Big\{x \in X \mid \forall \varepsilon > 0 : \big[B(x,\varepsilon) \cap A \neq \emptyset \text{ and } B(x,\varepsilon) \cap (X - A) \neq \emptyset\big]\Big\}. \end{aligned}$$

Hence by (*),

$$\partial A = (X - A^\circ) \cap \overline{A}.$$

But we know from p. 19 that A° is open; hence $X - A^\circ$ is closed. We also know from p. 21 that $\overline{A}$ is closed. Hence, since every intersection of closed sets is closed[1], $\partial A = (X - A^\circ) \cap \overline{A}$ is closed.

(Remark: The formula $\partial A = (X - A^\circ) \cap \overline{A}$ can also be written: $\partial A = \overline{A} - A^\circ$.)

2. Assume that $x_1, x_2, ...$ and $x'_1, x'_2, ...$ are sequences in X and that $x_n \to \ell$ and $x'_n \to \ell$ for some point $\ell \in X$. Then $\lim_{n\to\infty} d(x_n, \ell) = \lim_{n\to\infty} d(x'_n, \ell) = 0$. But note that by the triangle inequality,

$$\forall n: \qquad 0 \le d(x_n, x'_n) \le d(x_n, \ell) + d(\ell, x'_n) = d(x_n, \ell) + d(x'_n, \ell).$$

Here $d(x_n, \ell) + d(x'_n, \ell) \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

Alternative solution using Lemma 1.4-2. Since $x_n \to \ell$ and $x'_n \to \ell$, Lemma 1.4-2(b) yields $\lim_{n\to\infty} d(x_n, x'_n) = d(\ell, \ell)$. But $d(\ell, \ell) = 0$ by (M2) in Def 1.1-1. Hence: $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

3. We will use Theorem 1.4-6(b) to prove that c_0 is closed.

Let $x_1, x_2, ...$ be any sequence of vectors in c_0 such that $\lim_{j\to\infty} x_j = x$ for some vector $x \in \ell^\infty$. By definition, x and each x_j is a sequence of complex (or real) numbers, say $x = (\eta_1, \eta_2, \eta_3, ...)$ and $x_j = (\xi_{1,j}, \xi_{2,j}, \xi_{3,j}, ...)$, where all η_n and all $\xi_{n,j}$ are complex numbers.

We wish to prove that $\lim_{n\to\infty} \eta_n = 0$. Let $\varepsilon > 0$. Then since $\lim_{j\to\infty} x_j = x$ there is a number J such that

$$\forall j \ge J: \qquad ||x - x_j|| < \varepsilon/10.$$

In particular we have $||x - x_J|| < \varepsilon/10$. Recall that $||\cdot||$ is the ℓ^∞-norm; hence the last inequality can be written more explicitly as:

$$(*) \qquad \forall n \ge 1: \qquad |\eta_n - \xi_{n,J}| < \varepsilon/10.$$

But $x_J = (\xi_{1,J}, \xi_{2,J}, \xi_{3,J}, ...) \in c_0$, hence $\lim_{n\to\infty} \xi_{n,J} = 0$. Hence there is a number N such that

$$(**) \qquad \forall n \ge N: \qquad |\xi_{n,J} - 0| < \varepsilon/10.$$

Combining (*) and (**) and using the triangle inequality for complex numbers, we obtain:

$$\forall n \ge N: \qquad |\eta_n - 0| \le |\eta_n - \xi_{n,J}| + |\xi_{n,J} - 0| < \varepsilon/10 + \varepsilon/10 < \varepsilon.$$

[1]This is a useful fact to learn! It follows from p. 19 (T2) together with Def 1.3-2. Namely, if C_1, C_2 are two closed subsets of X then $C_1^C = X - C_1$ and C_2^C are open, hence by (T2), $C_1^C \cup C_2^C$ is open; hence $C_1 \cap C_2 = (C_1^C \cup C_2^C)^C$ is closed. The same type of argument shows that the intersection of *any* family of closed sets is closed.

But ε was arbitrary; hence we have now proved that $\lim_{n\to\infty} \eta_n = 0$. In other words, $x = (\eta_1, \eta_2, \eta_3, ...) \in c_0$.

Since this is true for *every* point $x \in \ell^\infty$ which is a limit point of a sequence of points in c_0, it follows from Theorem 1.4-6(b) that c_0 is closed.

4. Let X be a discrete metric space consisting of infinitely many points.

Since X is an infinite set there exists an infinite sequence of *distinct* points $x_1, x_2, x_3, ...$ in X; that is, $x_j \neq x_k$ whenever $j \neq k$.

Let $x_{j_1}, x_{j_2}, x_{j_3}, ...$ be an arbitrary subsequence of the sequence $x_1, x_2, x_3, ...$ (here $1 \leq j_1 < j_2 < j_3 < ...$). Then $x_{j_n} \neq x_{j_k}$ for all $n \neq k$, and hence $d(x_{j_n}, x_{j_k}) = 1$ for all $n \neq k$, by the definition of a discrete metric space (Def 1.1-8). Hence we do *not* have $d(x_{j_n}, x_{j_k}) \to 0$ as $n, k \to \infty$, i.e. the sequence $x_{j_1}, x_{j_2}, x_{j_3}, ...$ is *not* Cauchy. Hence by Theorem 1.4-5, $x_{j_1}, x_{j_2}, x_{j_3}, ...$ is *not* a convergent sequence.

We have proved that the sequence $x_1, x_2, x_3, ...$ in X does not have any convergent subsequence. Hence, by Def 2.5-1, X is *not* compact.

Alternative solution, not using the Cauchy criterion:

Let X be a discrete metric space consisting of infinitely many points.

Since X is an infinite set there exists an infinite sequence of *distinct* points $x_1, x_2, x_3, ...$ in X; that is, $x_j \neq x_k$ whenever $j \neq k$.

Let $x_{j_1}, x_{j_2}, x_{j_3}, ...$ be an arbitrary subsequence of the sequence $x_1, x_2, x_3, ...$ (here $1 \leq j_1 < j_2 < j_3 < ...$), and let x be any point in X. Then there is at most one index k such that $x_{j_k} = x$, and hence for all sufficiently large indices n we have $d(x_{j_n}, x) = 1$. Hence the sequence $x_{j_1}, x_{j_2}, x_{j_3}, ...$ does not converge to x.

Since this is true for every $x \in X$ and every subsequence of $x_1, x_2, x_3, ...$, it follows that the sequence $x_1, x_2, x_3, ...$ in X does not have any convergent subsequence. Hence, by Def 2.5-1, X is *not* compact.

5. We first solve problem 14 on p.57. Let Y be a subspace of a vector space X. For every $x \in X$ we define (as in the problem formulation) the *coset of* x (with respect to Y) to be the set

$$(*) \qquad x + Y = \{v \mid v = x + y, y \in Y\} = \{x + y \mid y \in Y\}.$$

We first have to prove that the distinct cosets form a partition[2] of X, i.e. that every element of X belongs to one coset and that distinct cosets are disjoint.

Clearly, for every $x \in X$ we have $x \in x+Y$, since $x = x+0$ and $0 \in Y$. Hence every element of X belongs to at least one coset.

Now let $x_1 + Y$ and $x_2 + Y$ be two arbitrary cosets which are *not* disjoint, i.e. $(x_1+Y) \cap (x_2+Y) \neq \emptyset$. Let v be any element in this intersection; then by the definition (*) there is some $y_1 \in Y$ such that $x_1 + y_1 = v$, and there is some $y_2 \in Y$ such that $x_2 + y_2 = v$. It follows that $x_1 - x_2 = y_2 - y_1 \in Y$. Hence for every $y \in Y$ we have $(x_1 - x_2) + y \in Y$ and thus $x_1 + y = x_2 + (x_1 - x_2) + y \in x_2 + Y$. Thus:

$$x_1 + Y \subset x_2 + Y.$$

Similarly, using $x_2 - x_1 = y_1 - y_2 \in Y$, one proves

$$x_2 + Y \subset x_1 + Y.$$

Hence

$$x_1 + Y = x_2 + Y.$$

This proves that any two cosets which are *not* disjoint are in fact *equal*. In other words, any two distinct cosets are disjoint.

This completes the proof that the distinct cosets form a partition of X.

We next prove that addition and multiplication of cosets are *well-defined* by the definitions in the problem formulation;

$$\begin{aligned}
&(w+Y) + (x+Y) := (w+x) + Y, \\
&\alpha(x+Y) = \alpha x + Y.
\end{aligned}$$

To show this, we assume $\alpha \in K$ and that w, x, w', x' are any elements in X such that $w + Y = w' + Y$ and $x + Y = x' + Y$; we then want to prove $(w+x) + Y = (w'+x') + Y$, and $\alpha x + Y = \alpha x' + Y$. But $w + Y = w' + Y$ and $x + Y = x' + Y$ imply that there exist vectors $y_1, y_2, y_3, y_4 \in Y$ such that $w + y_1 = w' + y_2$ and $x + y_3 = x' + y_4$. Now

$$\begin{aligned}
w + x &= (w' + y_2 - y_1) + (x' + y_4 - y_3) \\
&= (w' + x') + (y_2 - y_1 + y_4 - y_3),
\end{aligned}$$

and $y_2 - y_1 + y_4 - y_3 \in Y$; hence $w + x$ lies both in $(w+x) + Y$ and in $(w'+x') + Y$; hence these two cosets are not disjoint, and hence (by

[2]This statement is actually a well-known fact from group theory, if one notes that $\langle X, + \rangle$ is an abelian group and Y is a subgroup; similarly it is well-known that "+" is a well-defined operation on X/Y which makes X/Y into a group. However, we will give direct proofs of all these facts.

what we have proved earlier), $(w+x)+Y=(w'+x')+Y$. We also have

$$\alpha x=\alpha(x'+y_4-y_3)=\alpha x'+\alpha(y_4-y_3),$$

and by the same type of argument as above, this leads to $\alpha x+Y=\alpha x'+Y$. Hence the addition and multiplication of cosets are indeed well-defined by the given definitions.

Alternative proof that addition and multiplication of cosets are well-defined by the given definitions: It is natural to define operations addition and multiplication by scalar for *any* subsets $A,B\subset X$ as follows (in the present discussion we will write "$\tilde{+}$" and "$\tilde{\cdot}$" in order to distinguish these operations from the operations given in the problem):

$$A\tilde{+}B:=\{a+b\mid a\in A, b\in B\};$$
$$\alpha\,\tilde{\cdot}\,A:=\{\alpha a\mid a\in A\}\qquad(\alpha\in K).$$

These operations are *obviously* well-defined, and give subsets of X as results. Hence it suffices to prove that the operations given in the problem are simply *special cases* of the above operations (in particular it then follows that the $\tilde{+}$-sum of any two cosets is again a coset). To prove this, note that for any vectors $x,w\in X$ we have

$$\begin{aligned}(x+Y)\tilde{+}(w+Y)&=\{a+b\mid a\in x+Y,\ b\in w+Y\}\\&=\{a+b\mid a=x+y_1, b=w+y_2,\ y_1,y_2\in Y\}\\&=\{x+w+y_1+y_2\mid y_1,y_2\in Y\}\\&=\{x+w+y\mid y\in Y\}\\&=(x+w)+Y,\end{aligned}$$

where in the next to last step we used $\{y_1+y_2\mid y_1,y_2\in Y\}=Y=\{y\mid y\in Y\}$ which is true since Y is subspace of X. The above calculation shows that the operation "+" on cosets as defined in the problem is well-defined, and is a special case of $\tilde{+}$. Similarly, for any $x\in X$, $\alpha\in K$ we have, *if* α *is non-zero:*

$$\begin{aligned}\alpha\,\tilde{\cdot}\,(x+Y)&=\{\alpha a\mid a\in x+Y\}\\&=\{\alpha(x+y)\mid y\in Y\}\\&=\{\alpha x+\alpha y\mid y\in Y\}\\&=\{\alpha x+y\mid y\in Y\}\\&=\alpha x+Y,\end{aligned}$$

where in the next to last step we used $\{\alpha y\mid y\in Y\}=Y$ which is true since $\alpha\neq 0$ and Y is a vector space. The above calculation shows that *if* $\alpha\neq 0$ then the operation "multiplication by α" on cosets as defined in the problem is well-defined, and is a special case of "$\tilde{\cdot}$-multiplication by α".

Finally, to cover the case $\alpha=0$, note that the operation "multiplication by 0" as defined in the problem is well-defined, since $0\cdot x+Y=Y$ for *all* $x\in X$. (But note that $0\,\tilde{\cdot}\,Y=\{0\}$ and hence $0\,\tilde{\cdot}\,Y\neq 0+Y$ if $Y\neq\{0\}$, i.e. the two operations are in general *not* the same in the special case $\alpha=0$!)

Now that we have proved that the two operations are well-defined, it is very easy to prove that these operations satisfy all the vector space laws; these are direct consequences of the corresponding laws for the vector space X. In precise terms, for all $x, y, z \in X$ and all $\alpha, \beta \in K$ we have (see p.50–51, and use the given definitions of the two operations in X/Y):

(V1) $(x+Y)+(y+Y) = (x+y)+Y = (y+x)+Y = (y+Y)+(x+Y)$.

(V2) $(x+Y)+((y+Y)+(z+Y)) = (x+(y+z))+Y = ((x+y)+z)+Y = ((x+Y)+(y+Y))+(z+Y)$.

(V3) Set $0_{X/Y} := 0+Y$; then $(x+Y)+0_{X/Y} = (x+0)+Y = x+Y$.

(V4) The coset $(-x)+Y$ satisfies $(x+Y)+((-x)+Y) = 0+Y = 0_{X/Y}$.

(V5) $\alpha(\beta(x+Y)) = (\alpha(\beta x))+Y = ((\alpha\beta)x)+Y = (\alpha\beta)(x+Y)$.

(V6) $1(x+Y) = (1x)+Y = x+Y$.

(V7) $\alpha((x+Y)+(y+Y)) = \alpha((x+y)+Y) = (\alpha(x+y))+Y = (\alpha x+Y)+(\alpha y+Y) = \alpha(x+Y)+\alpha(y+Y)$.

(V8) $(\alpha+\beta)(x+Y) = (\alpha+\beta)x+Y = (\alpha x+Y)+(\beta x+Y) = \alpha(x+Y)+\beta(x+Y)$.

This proves that the cosets indeed constitute the elements of a vector space under the given operations. This completes the solution of problem 14 on p.57.

We now solve problem 14 on p.71. It is clear that $||\cdot||_0$ is a well-defined function $X/Y \to [0,\infty)$ by the definition in the problem, since each $\hat{x} \in X/Y$ is a nonempty set, and $||x|| \in [0,\infty)$ for all $x \in \hat{x}$. Hence the law (N1) is satisfied. We now prove that the three other norm laws are satisfied as well:

(N2) If $\hat{x} = 0_{X/Y}$ then $0 \in \hat{x}$ and hence $||\hat{x}||_0 = ||0|| = 0$. Conversely, assume $\hat{x} \in X/Y$ and $||\hat{x}||_0 = 0$. Take $x_0 \in \hat{x}$; then $\hat{x} = x_0 + Y = \{x_0 + y \mid y \in Y\}$. Now $\inf_{x\in\hat{x}} ||x|| = 0$, i.e. $\inf_{y\in Y} ||x_0+y|| = 0$. Hence there is a sequence $y_1, y_2, y_3, \ldots$ in Y such that $\lim_{n\to\infty} ||x_0+y_n|| = 0$, and thus $y_n \to -x_0$, by the definition of converging sequence (Def. 1.4-1). Since Y is closed, this implies $-x_0 \in Y$ (by Theorem 1.4-6(b)). Hence $x_0 \in Y$ and $x_0 + Y = Y = 0_{X/Y}$, i.e. $\hat{x} = 0_{X/Y}$.

(N3) Let $\hat{x} \in X/Y$ and $\alpha \in K$. Take $x_0 \in \hat{x}$, so that $\hat{x} = x_0 + Y$. If $\alpha \neq 0$ then

$$\alpha\hat{x} = \alpha x_0 + Y = \{\alpha x_0 + y \mid y \in Y\} = \{\alpha(x_0+y) \mid y \in Y\}$$

(using $y \in Y \iff \alpha^{-1}y \in Y$), and thus

$$||\alpha\hat{x}||_0 = \inf_{y\in Y} ||\alpha(x_0+y)|| = |\alpha| \inf_{y\in Y} ||x_0+y|| = |\alpha| \cdot ||\hat{x}||_0.$$

On the other hand, if $\alpha = 0$ then $||\alpha\hat{x}||_0 = ||0_{X/Y}|| = 0 = |\alpha| \cdot ||\hat{x}||_0$, i.e. (N3) holds in all cases.

(N4) Let $\hat{x}, \hat{w} \in X/Y$. Take $x_0 \in \hat{x}$ and $w_0 \in \hat{w}$, so that $\hat{x} = x_0 + Y$ and $\hat{w} = w_0 + Y$. Then if $x \in \hat{x}$ and $w \in \hat{w}$, we have $x = x_0 + y_1$ and $w = w_0 + y_2$ for some $y_1, y_2 \in Y$, and thus $x + w = (x_0+w_0)+(y_1+y_2) \in$

$(x_0 + w_0) + Y = \hat{x} + \hat{w}$. Hence, for all $x \in \hat{x}$, $w \in \hat{w}$,

$$||\hat{x} + \hat{w}||_0 = \inf_{u \in \hat{x}+\hat{w}} ||u|| \leq ||x + w||.$$

By the triangle inequality, this implies

$$||\hat{x} + \hat{w}||_0 \leq ||x|| + ||w||.$$

Since this is true for all $x \in \hat{x}$ and all $w \in \hat{w}$, we have

$$||\hat{x} + \hat{w}||_0 \leq \inf_{x \in \hat{x}} ||x|| + \inf_{w \in \hat{w}} ||w|| = ||\hat{x}||_0 + ||\hat{w}||_0.$$

6. Let us assume that there does *not* exist such numbers $\gamma_1, \gamma_2, ...$ In other words, we assume that there is some k such that there does *not* exist a number γ_k such that $|\xi_k| \leqq \gamma_k$ holds for all $(\xi_1, \xi_2, \xi_3, ...) \in M$. In other words, we assume that for every number $c > 0$ there is some element $(\xi_1, \xi_2, \xi_3, ...) \in M$ such that $|\xi_k| > c$. In symbols:

$$(*) \qquad \forall c > 0 : \exists (\xi_1, \xi_2, \xi_3, ...) \in M : |\xi_k| > c.$$

Taking $c = 1$ in (*) we see that there is a sequence $x_1 = (\xi_{1,1}, \xi_{2,1}, \xi_{3,1}, ...) \in M$ such that $|\xi_{k,1}| > 1$. Next we take $c = |\xi_{k,1}| + 1$ in (*) and hence we see that there is a sequence $x_2 = (\xi_{1,2}, \xi_{2,2}, \xi_{3,2}, ...) \in M$ such that $|\xi_{k,2}| > |\xi_{k,1}| + 1$. This is repeated recursively; i.e., when $x_n = (\xi_{1,n}, \xi_{2,n}, \xi_{3,n}, ...) \in M$ has been chosen, we apply (*) with $c = |\xi_{k,n}| + 1$ in (*) and hence we see that there is a sequence $x_{n+1} = (\xi_{1,n+1}, \xi_{2,n+1}, \xi_{3,n+1}, ...) \in M$ such that $|\xi_{k,n+1}| > |\xi_{k,n}| + 1$.

In this way we obtain an infinite sequence $x_1, x_2, x_3, ...$ in M such that $x_n = (\xi_{1,n}, \xi_{2,n}, \xi_{3,n}, ...)$ with $|\xi_{k,n+1}| > |\xi_{k,n}| + 1$ for all n. Using the last inequality repeatedly we see that $|\xi_{k,m}| > |\xi_{k,n}| + (m - n)$ for all $m > n \geq 1$. Hence, for all $m > n \geq 1$,

$$|\xi_{k,m} - \xi_{k,n}| \geq |\xi_{k,m}| - |\xi_{k,n}| > m - n \geq 1.$$

It follows that, for all $m > n \geq 1$,

$$d(x_m, x_n) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_{j,m} - \xi_{j,n}|}{1 + |\xi_{j,m} - \xi_{j,n}|} \geq \frac{1}{2^k} \frac{|\xi_{k,m} - \xi_{k,n}|}{1 + |\xi_{k,m} - \xi_{k,n}|}$$

$$> \frac{1}{2^k} \cdot \frac{1}{2} = 2^{-k-1}.$$

(In the next to last step we used the fact that for $r := |\xi_{k,m} - \xi_{k,n}| > 1$ we have $\frac{r}{1+r} = 1 - \frac{1}{1+r} > 1 - \frac{1}{2} = \frac{1}{2}$.)

Now we may continue as in the solution of Problem 4: Let $x_{j_1}, x_{j_2}, x_{j_3}, ...$ be an arbitrary subsequence of the sequence $x_1, x_2, x_3, ...$ (here $1 \leq j_1 < j_2 < j_3 < ...$). Then the above inequality implies that $d(x_{j_n}, x_{j_m}) >$

2^{-k-1} for all $n \neq m$. Hence the sequence $x_{j_1}, x_{j_2}, x_{j_3}, ...$ is *not* Cauchy. Hence by Theorem 1.4-5, $x_{j_1}, x_{j_2}, x_{j_3}, ...$ is *not* a convergent sequence.

We have proved that the sequence $x_1, x_2, x_3, ...$ in M does not have any convergent subsequence. Hence, by Def 2.5-1, M is *not* compact.

Extra information: The converse mentioned in problem 4, p. 82: This is actually *false!* Example: Let

$$M = \{(\xi_n) \in s \mid (\xi_1, \xi_2, ...) \neq 0 \text{ and } \forall n : |\xi_n| \leq 1\}.$$

Then M is an infinite set and M satisfies the criterion in the problem (with $\gamma_1 = \gamma_2 = ... = 1$). However, consider the sequence $x_n = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, ...)$, $n = 1, 2, 3, ...$ in M. This sequence converges to $(0, 0, 0, ...)$ in s, and hence every subsequence also converges to $(0, 0, 0, ...)$. However, $(0, 0, 0, ...) \notin M$; hence no subsequence of $x_1, x_2, x_3, ...$ converges to an element in M. Hence M is *not* compact.

However, the condition in the problem *does* imply that $\overline{M}$ *is compact*. This is a very pleasant exercise to prove!