Functional Analysis F3/F4/NVP (2005)
Homework assignment 3

All students should solve the following problems:

1. Section 4.8: Problem 8.

2. Section 4.9: Problem 4.

3. Let $T : \ell^2 \to \ell^2$ be the operator defined by

$$Tx = \big(\xi_1 + \xi_2, \xi_2, \xi_3 + \xi_4, \xi_4, \xi_5 + \xi_6, \xi_6, \xi_7 + \xi_8, ...\big), \qquad x = (\xi_j) \in \ell^2.$$

Determine the four sets $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$.

4. Prove the following (partial) converse to the spectral theorem for compact, self-adjoint operators: If H is a Hilbert space, $\{e_n\}$ is a total orthonormal sequence in H, and $\{\lambda_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} \lambda_n = 0$, then there exists a unique bounded linear operator $T : H \to H$ such that $T(e_n) = \lambda_n e_n$ for all n. This operator is compact and self-adjoint.

Students taking Functional Analysis as a 6 point course should also solve the following problems:

5. Section 8.2: Problem 8.

6. Let H be the Hilbert space $L^2(-\infty, \infty)$ and let $T : H \to H$ be the multiplication operator $Tx(t) = \sin(t) \cdot x(t)$. Verify that T is a bounded self-adjoint operator, and find its spectral family.

Solutions should be handed in by Thursday, March 3, 16.00.
(Either give the solutions to me directly or put them in my mailbox, third floor, House 3, Polacksbacken.)

Functional Analysis F3/F4/NVP
Comments to homework assignment 3 (to be updated!)

1.Many students claim that if (x_n) is a weak Cauchy sequence in X then there must exist some x in X such that (x_n) converges weakly to x. This is *not* true (even if X is a Banach space).

Example: Let $X = c_0$ (see p. 70, exercise 1) and let $a_1, a_2, a_3, ...$ be *any* bounded sequence of complex numbers. Then the following is a weak Cauchy sequence in $X = c_0$:

$$x_1 = (a_1, 0, 0, 0, ...),$$
$$x_2 = (a_1, a_2, 0, 0, 0, ...),$$
$$x_3 = (a_1, a_2, a_3, 0, 0, 0, ...),$$
$$x_4 = (a_1, a_2, a_3, a_4, 0, 0, 0, ...).$$

(Proof: one uses the fact given in exercise 8 on p. 126. In explicit terms this exercise tells us: For every $f \in (c_0)'$ there is some $(b_n) \in \ell^1$ such that $f((\xi_n)) = \sum_{n=1}^{\infty} b_n \xi_n$. It follows that if $x_1, x_2, x_3, ...$ is the sequence in c_0 given above then $\lim_{j\to\infty} f(x_j) = \sum_{n=1}^{\infty} b_n a_n$. Hence $x_1, x_2, x_3, ...$ is indeed weak Cauchy.)

However, for most choiches of numbers $a_1, a_2, a_3, ...$, the above sequence (x_n) does *not* converge weakly to an element $x \in c_0$.

Functional Analysis F3/F4/NVP

Solutions to homework assignment 3

1. Assume that (x_n) is a weak Cauchy sequence in X. Take an arbitrary $f \in X'$. Then $(f(x_n))$ is a Cauchy sequence in K, and hence it is a bounded sequence. Hence, for every $f \in X'$ there exists some constant c_f such that $|f(x_n)| \leq c_f$ for all n. Now we imitate the argument from p. 258 in the book (part (c)): For every $x \in X$ we define $g_x \in X''$ as usual by $g_x(f) = f(x)$ for all $f \in X'$ (cf. p. 239 and Lemma 4.6-1). We now have for all n,

$$|g_{x_n}(f)| = |f(x_n)| \leq c_f.$$

This tells us that the sequence (g_{x_n}) in $X'' = B(X', K)$ is *pointwise bounded*. We also know that X' is a Banach space by Theorem 2.10-4. Hence the Uniform Boundedness Theorem (Theorem 4.7-3) can be applied, and this gives that there exists a constant c such that $||g_{x_n}|| \leq c$ for all n. But $||x_n|| = ||g_{x_n}||$ by Lemma 4.6-1; hence we have proved that $||x_n|| \leq c$ for all n, i.e. the sequence (x_n) is bounded.

2. Let X be an arbitrary normed space, and let (f_n) be a sequence in X'.

Assume that (f_n) converges weakly to $f \in X'$. We wish to prove that (f_n) is weak* convergent to f. Let x be an arbitrary vector in X. Let $g_x \in X''$ be the image of x under the canonical mapping of X into X'' (cf. p. 239–240), i.e.

$$g_x(h) = h(x), \quad \forall h \in X'.$$

Since $g_x \in X''$ and (f_n) converges weakly to $f \in X'$, we have $\lim_{n\to\infty} g_x(f_n) = g_x(f)$. This means $\lim_{n\to\infty} f_n(x) = f(x)$. Since this holds for all $x \in X$, (f_n) is weak* convergent to f.

Now assume that X is reflexive. We wish to prove that weak* convergence in X' implies weak convergence in X'. Let (f_n) be an arbitrary sequence in X' which is weak* convergent to some $f \in X'$. Let g be an arbitrary element in X''. Then *since X is reflexive*, there is an element $x \in X$ such that $g_x = g$. Since (f_n) is weak* convergent to f we have $\lim_{n\to\infty} f_n(x) = f(x)$. This can also be written: $\lim_{n\to\infty} g_x(f_n) = g_x(f)$. But $g_x = g$, hence we have $\lim_{n\to\infty} g(f_n) = g(f)$. But $g \in X''$ was arbitrary; hence (f_n) is weakly convergent to f.

3. T is a bounded linear operator $\ell^2 \to \ell^2$, since, for all $(\xi_n) \in \ell^2$,

$$\begin{aligned} ||T((\xi_n))||^2 &= |\xi_1 + \xi_2|^2 + |\xi_2|^2 + |\xi_3 + \xi_4|^2 + |\xi_4|^2 + ... \\ &\leq (1+1) \cdot (|\xi_1|^2 + |\xi_2|^2) + |\xi_2|^2 + (1+1) \cdot (|\xi_3|^2 + |\xi_4|^2) + |\xi_4|^2 + ... \\ &\leq 3 \sum_{n=1}^{\infty} |\xi_n|^2 = 3||(\xi_n)||^2. \end{aligned}$$

(This also shows $||T|| \leq \sqrt{3}$.)

In the computation above we used Schwarz inequality on $\mathbb{C}^2$, which says the following:

$$(*) \qquad |a_1 b_1 + a_2 b_2|^2 \leq (|a_1|^2 + |a_2|^2) \cdot (|b_1|^2 + |b_2|^2), \qquad \forall a_1, a_2, b_1, b_2 \in \mathbb{C}.$$

(We used this with $a_1 = a_2 = 1$ and $b_1 = \xi_1$, $b_2 = \xi_2$, etc.)

To get started, we investigate when $T_\lambda = T - \lambda I$ is injective. Let us write $x = (\xi_n) \in \ell^2$ and $y = (T - \lambda I)x = (\eta_n) \in \ell^2$. Then

$$(**) \qquad y = (\eta_n) = ((1-\lambda)\xi_1 + \xi_2, (1-\lambda)\xi_2, (1-\lambda)\xi_3 + \xi_4, (1-\lambda)\xi_4, ...).$$

Now if $\lambda = 1$ then $Tx = 0$ for *every* vector $x = (\xi_n) \in \ell^2$ with $\xi_2 = \xi_4 = \xi_6 = ... = 0$, and there exist *many* such vectors which are $\neq 0$, e.g. $x = (1, 0, 0, 0, ...)$. This shows that if $\lambda = 1$ then $T - \lambda I$ is not injective; hence $1 \in \sigma_p(T)$.

From now on we assume $\lambda \neq 1$. Then for any $x = (\xi_n) \in \ell^2$, formula (**) implies:

$$(***) \qquad \begin{cases} \xi_1 = \frac{1}{1-\lambda}\eta_1 - \frac{1}{(1-\lambda)^2}\eta_2 \\ \xi_2 = \frac{1}{1-\lambda}\eta_2 \\ \xi_3 = \frac{1}{1-\lambda}\eta_3 - \frac{1}{(1-\lambda)^2}\eta_4 \\ \xi_4 = \frac{1}{1-\lambda}\eta_4 \\ \ldots \end{cases}$$

Now if some $x' = (\xi'_n) \in \ell^2$ would give $Tx' = Tx = (\eta_n)$ then each ξ'_n is given by the *same* formula as ξ_n in (***), i.e. $\xi'_n = \xi_n$ for all n and hence $x = x'$. This proves that $T - \lambda I$ *is injective* (under our present assumption $\lambda \neq 1$). From the above computations we also obtain explicit formulas for $(T - \lambda I)^{-1}$ and $\mathcal{D}((T - \lambda I)^{-1})$:

$$\mathcal{D}\left((T - \lambda I)^{-1}\right) = \left\{ (\eta_n) \in \ell^2 \mid \text{if } \xi_n \text{ is computed using (***) then } \sum_{n=1}^{\infty} |\xi_n|^2 < \infty \right\}$$

and

$$(T - \lambda I)^{-1}\Big((\eta_n)\Big) = (\xi_n), \qquad \forall (\eta_n) \in \mathcal{D}\left((T - \lambda I)^{-1}\right).$$

To decide whether λ belongs to $\sigma_c(T)$ or $\sigma_r(T)$ or $\rho(T)$ we must determine whether $\mathcal{D}\left((T-\lambda I)^{-1}\right)$ is dense in ℓ^2 and whether $(T-\lambda I)^{-1}$ is bounded. In the present problem it turns out to be easiest to consider the first question first. This means trying to bound $||(\xi_n)||$ in terms of $||(\eta_n)||$ in (***). For every $(\eta_n) \in \ell^2$ we have, if (ξ_n) is computed using (***):

$$\sum_{n=1}^{\infty} |\xi_n|^2 = \left|\tfrac{1}{1-\lambda}\eta_1 - \tfrac{1}{(1-\lambda)^2}\eta_2\right|^2 + \left|\tfrac{1}{1-\lambda}\eta_2\right|^2 + \left|\tfrac{1}{1-\lambda}\eta_3 - \tfrac{1}{(1-\lambda)^2}\eta_4\right|^4 + \left|\tfrac{1}{1-\lambda}\eta_4\right|^2 + ...$$

$$\leq \left(\left|\tfrac{1}{1-\lambda}\right|^2 + \left|\tfrac{1}{(1-\lambda)^2}\right|^2\right) \cdot (|\eta_1|^2 + |\eta_2|^2) + \left|\tfrac{1}{1-\lambda}\right|^2 \cdot |\eta_2|^2$$

$$+ \left(\left|\tfrac{1}{1-\lambda}\right|^2 + \left|\tfrac{1}{(1-\lambda)^2}\right|^2\right) \cdot (|\eta_3|^2 + |\eta_4|^2) + \left|\tfrac{1}{1-\lambda}\right|^2 \cdot |\eta_4|^2 + \cdots$$

Here we used again the inequality (*) above. It follows from the above that if we let C_λ be the constant

$$C_\lambda = \left|\tfrac{1}{1-\lambda}\right|^2 + \left|\tfrac{1}{(1-\lambda)^2}\right|^2 + \left|\tfrac{1}{1-\lambda}\right|^2,$$

then

$$\sum_{n=1}^{\infty} |\xi_n|^2 \leq C_\lambda \cdot \sum_{n=1}^{\infty} |\eta_n|^2.$$

In particular, for *every* $(\eta_n) \in \ell^2$ we have $\sum_{n=1}^{\infty} |\xi_n|^2 < \infty$, i.e. $(\xi_n) \in \ell^2$. In view of our earlier formular for $\mathcal{D}\left((T-\lambda I)^{-1}\right)$, this proves that

$$\mathcal{D}\left((T-\lambda I)^{-1}\right) = \ell^2,$$

i.e. $\mathcal{D}\left((T-\lambda I)^{-1}\right)$ is dense in ℓ^2 in the trivial way that it is *equal* to ℓ^2. The above computation also proves that

$$||(T-\lambda I)^{-1}((\eta_n))|| = \sqrt{\sum_{n=1}^{\infty} |\xi_n|^2} \leq \sqrt{C_\lambda} \cdot ||(\eta_n)||$$

for all $(\eta_n) \in \ell^2$, i.e. the operator $(T-\lambda I)^{-1}$ is bounded, with norm $\leq \sqrt{C_\lambda}$. Together, this implies that $\lambda \in \rho(T)$. This is true for all $\lambda \neq 1$.

Hence we have proved:

$$\sigma_p(T) = \{1\}; \quad \sigma_c(T) = \emptyset; \quad \sigma_r(T) = \emptyset; \quad \rho(T) = \mathbb{C} \setminus \{1\}.$$

4. Let $\{e_n\}$ be a total sequence in H, and $\{\lambda_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} \lambda_n = 0$. By the main Theorem 3.5-2 (in the version I proved in class), we know that

$$H = \{\sum_{n=1}^{\infty} \alpha_n e_n \mid \alpha_n \in K, \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty\}$$

Now if T is a bounded linear operator $T : H \to H$ with $T(e_n) = \lambda_n e_n$ for all n, we have for all sequences (α_n) in K with $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$:

$$\begin{aligned} T\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) &= T\left(\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n e_n\right) = \lim_{N\to\infty} T\left(\sum_{n=1}^{N} \alpha_n e_n\right) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n \lambda_n e_n = \sum_{n=1}^{\infty} \alpha_n \lambda_n e_n. \end{aligned}$$

The last sum converges since $\sum_{n=1}^{\infty} |\alpha_n \lambda_n|^2 \le \left(\sup_n |\lambda_n|\right) \cdot \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ (here $\sup_n |\lambda_n| < \infty$ since $\lim_{n\to\infty} \lambda_n = 0$). This proves *uniqueness*, i.e. that there is only one possible bounded linear operator $T : H \to H$ with $T(e_n) = \lambda_n e_n$ for all n. On the other hand, the formula $T\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \sum_{n=1}^{\infty} \alpha_n \lambda_n e_n$ truly defines a bounded linear operator $T : H \to H$. (Proof: Linearity is checked by a straightforward computation. Furthermore, we have $||T\left(\sum_{n=1}^{\infty} \alpha_n e_n\right)|| = ||\sum_{n=1}^{\infty} \alpha_n \lambda_n e_n|| = \sqrt{\sum_{n=1}^{\infty} |\alpha_n \lambda_n|^2} \le (\sup_n |\lambda_n|) \cdot \sqrt{\sum_{n=1}^{\infty} |\alpha_n|^2} = (\sup_n |\lambda_n|) \cdot ||\sum_{n=1}^{\infty} \alpha_n e_n||$. Hence T is bounded with norm $||T|| \le \sup_n |\lambda_n|$.)

We now prove that T *is self-adjoint*. Let x, y be two arbitrary vectors in H. Then we have $x = \sum_{n=1}^{\infty} \alpha_n e_n$ and $y = \sum_{n=1}^{\infty} \beta_n e_n$ for some $\alpha_n, \beta_n \in K$ with $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$, $\sum_{n=1}^{\infty} |\beta_n|^2 < \infty$. Now, by a generalization of the Parseval relation which I have mentioned in class (compare p.175, exercise 4),

$$\langle Tx, y\rangle = \langle \sum_{n=1}^{\infty} \lambda_n \alpha_n e_n, \sum_{n=1}^{\infty} \beta_n e_n \rangle = \sum_{n=1}^{\infty} \lambda_n \alpha_n \overline{\beta_n}$$

and

$$\langle x, Ty\rangle = \langle \sum_{n=1}^{\infty} \alpha_n e_n, \sum_{n=1}^{\infty} \lambda_n \beta_n e_n \rangle = \sum_{n=1}^{\infty} \alpha_n \overline{\lambda_n \beta_n}$$

But all numbers λ_n are assumed to be real; hence $\langle Tx, y\rangle = \langle x, Ty\rangle$. Since x, y were arbitrary this proves that T is self-adjoint.

Finally, we now prove that T *is compact*, by imitating the argument in Example 8.1-6, p.409 in the book. For each $m \in \mathbb{Z}^+$ we define the operator $T_m : H \to H$ by

$$T_m \left(\sum_{n=1}^{\infty} \alpha_n e_n \right) = \sum_{n=1}^{m} \lambda_n \alpha_n e_n.$$

This operator is clearly linear, and it is bounded since

$$\begin{aligned} \left\| T_m \left(\sum_{n=1}^{\infty} \alpha_n e_n \right) \right\|^2 &= \sum_{n=1}^{m} |\lambda_n \alpha_n|^2 \leq \left(\sup_{1 \leq n \leq m} |\lambda_n|^2 \right) \cdot \sum_{n=1}^{m} |\alpha_n|^2 \\ &\leq \left(\sup_{1 \leq n \leq m} |\lambda_n|^2 \right) \cdot \left\| \sum_{n=1}^{\infty} \alpha_n e_n \right\|^2, \end{aligned}$$

for all vectors $x = \sum_{n=1}^{\infty} \alpha_n e_n \in H$. (This shows that $||T_m|| \leq \sqrt{\sup_{1 \leq n \leq m} |\lambda_n|^2} = \sup_{1 \leq n \leq m} |\lambda_n|$.) Furthermore the range $\mathcal{R}(T_m) \subset \mathrm{Span}\{e_1, e_2, ..., e_m\}$ is finite dimensional, hence by Theorem 8.1-4(a), T_m is compact.

For each m and each $x = \sum_{n=1}^{\infty} \alpha_n e_n \in H$ we have

$$\begin{aligned} \left\| (T - T_m) \left(\sum_{n=1}^{\infty} \alpha_n e_n \right) \right\|^2 &= \sum_{n=m+1}^{\infty} |\lambda_n \alpha_n|^2 \leq \left(\sup_{n \geq m+1} |\lambda_n|^2 \right) \cdot \sum_{n=m+1}^{\infty} |\alpha_n|^2 \\ &\leq \left(\sup_{n \geq m+1} |\lambda_n|^2 \right) \cdot \left\| \sum_{n=1}^{\infty} \alpha_n e_n \right\|^2, \end{aligned}$$

hence

$$\|T - T_m\| \leq \sqrt{\sup_{n \geq m+1} |\lambda_n|^2} = \sup_{n \geq m+1} |\lambda_n|.$$

This number tends to 0 as $m \to \infty$, since $\lim_{n \to \infty} \lambda_n = 0$. Hence

$$\|T - T_m\| \to 0 \qquad \text{as } m \to \infty.$$

Hence by Theorem 8.1-5, the operator T is compact.