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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL!

Problems 1 — 8 should be attempted by all students. **Graduate students should also try to solve Problems 9 and 10**

1. Let Y be a subspace of an inner product space X and let $x_1, x_2, \dots, x_n \in Y$. Show that if

$$Y = \operatorname{span}\{x_1, x_2, \dots, x_n\},$$

then

$$Y^\perp = \left\{ x \in X \,:\, \sum_{j=1}^{n} \langle x, x_j \rangle x_j = 0 \right\}.$$

2. Let

$$c = \{x = (\xi_j) \in l^\infty \,:\, \lim_{j \to \infty} \xi_j \text{ exists (and is finite)}\}.$$

We say that $x = (\xi_j) \in l^\infty$ stabilizes if there exist N such that for all $j \geq N$

$$\xi_N = \xi_j.$$

Let $M = \{x \in c \,:\, x \text{ stabilizes}\}$. Show that c is the closure of M.

3. Let H be a real Hilbert space. Let $u, v \in H$ be such that u is not orthogonal to v. Define $T : H \longrightarrow H$ by the condition that $y = Tx$ if and only if y is the only point on the line

$$L_x = \{x + tv \,:\, t \in \mathbf{R}\}$$

which is orthogonal to u. Show that T is a bounded linear operator and find its range. *Hint:* Given x, calculate y in terms of x, u and v.

4. Show that if T is the operator from Problem 3 and u, v are linearly dependent, then T is self-adjoint.

5. Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator between Hilbert spaces H_1 and H_2. Let $(e_j)_{j\geq 1}$ be an orthonormal basis for H_2. Show that there exists a sequence of vectors $(f_j)_{j\geq 1}$ in H_1 such that

$$Tx = \sum_{j\geq 1} \langle x, f_j \rangle e_j, \qquad x \in H_1.$$

6. Let α_{jk} be numbers (for all $j, k \geq 1$) such that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_{jk}|^2 < \infty.$$

Define $T : l^2 \longrightarrow l^2$ as follows. If $x = (\xi_j)$, $y = (\eta_j)$ and $Tx = y$, then

$$\eta_j = \sum_{k=1}^{\infty} \alpha_{jk} \xi_k, \qquad j \geq 1.$$

Show that T is a compact operator.

7. Let H be a complex Hilbert space. Let $P : H \longrightarrow H$ be an orthogonal projection and let $S : H \longrightarrow H$ be a unitary operator. Show that the operator $Q = S^{-1}PS$ is an orthogonal projection.

8. Recall that if X, Y are normed spaces and $T : X \longrightarrow Y$ is a linear operator then the *graph* of T is the set $\mathcal{G}(T) = \{(x, y) \in X \times Y \ : \ y = Tx\}$. We will treat $X \times Y$ as a normed space with the norm $\|(x, y)\| = \|x\| + \|y\|$. Let $T_1, T_2 : X \longrightarrow Y$ be two linear operators. Show that if $\mathcal{G}(T_1)$ is closed in $X \times Y$ and T_2 is bounded, then $\mathcal{G}(T_1 + T_2)$ is closed in $X \times Y$.

Additional problems for graduate students:

9. Let X be an infinite dimensional normed space and let $T : X \longrightarrow X$ be a compact operator. Show that $0 \in \sigma(T)$.

10. Let c be the subspace of l^∞ described in Problem 2. For every $n \geq 1$ define the functional $f_n : c \longrightarrow \mathbf{C}$ by the formula

$$f_n(x) = \sum_{k=1}^{\infty} \frac{\left(1 - \frac{1}{n}\right)^{k-1}}{n} \xi_k, \qquad x = (\xi_k) \in c.$$

Define also $f : c \longrightarrow \mathbf{C}$ by

$$f(x) = \lim_{k\to\infty} \xi_k, \qquad x = (\xi_k) \in c.$$

Show that $\|f_n\| = \|f\| = 1$, for all n, and that the sequence (f_n) is w^*-convergent to f.

GOOD LUCK!