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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
2001-03-02

Skrivtid: 9–14.

Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL! — GOOD LUCK!

Problems 1–8 should be solved by all students. Problems 9 and 10 are only for those who take Functional Analysis as a 6 point course.

1. Let

$$c_{00} = \{x = (\xi_j) \in l^2 \,:\, \text{at most finitely many } \xi_j \neq 0\}.$$

Show that c_{00} is not complete.

2. Let X be a real inner product space. For any $z \in X$, let $f_z \in X'$ be defined by the formula $f_z(x) = \langle x, z\rangle$ for all $x \in X$. Show that the operator $T : X \longrightarrow X'$ given by the formula $T(z) = f_z$ for $z \in X$ is a linear isometry. Why do we have to assume that X is a real vector space?

3. Let H_1 and H_2 be Hilbert spaces and let $A \in B(H_1, H_2)$. Show that

$$\mathcal{R}(A)^\perp = \mathcal{N}(A^*) \quad \text{and} \quad \overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^\perp.$$

4. Let u, v be two linearly independent vectors in a Hilbert space H. Define

$$Px = \langle x, u\rangle u + \langle x, v\rangle v$$

for all $x \in H$. Show that if P is an orthogonal projection, then $\|u\| = \|v\| = 1$ and u is orthogonal to v.

5. Let X be a normed space and let $S \subset X$ be "weakly bounded" in the sense that $f(S)$ is bounded for each functional $f \in X'$. Use the Banach-Steinhaus theorem to show that S is a bounded set.

6. Suppose that a vector space X has two norms $\| \ \|_1$ and $\| \ \|_2$, such that there is a constant $M > 0$ for which $\|x\|_2 \leq M\|x\|_1$ for all $x \in X$. Suppose that X is complete with respect to both norms. Use the Open Mapping Theorem to show that there exist a constant $m > 0$ such that $m\|x\|_1 \leq \|x\|_2$ for all $x \in X$.

7. Let (e_n) be an orthonormal basis for a Hilbert space H. Define the operator T by the formula:

$$T(x) = \sum_{n=1}^{\infty} \frac{\langle x, e_{n+1} \rangle}{n+1} e_n, \qquad x \in H.$$

Show that T is compact and find T^*.

8. Let $K : L^2[0,1] \longrightarrow L^2[0,1]$ be given by the formula

$$y = Kx \text{ if and only if } y(t) = \int_0^1 tsx(s)\, ds$$

for $x \in L^2[0,1]$. Show that $\mathcal{R}(K)$ is one-dimensional. Deduce from this that K has only one non-zero eigenvalue and find it.

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9. Let (e_n) be an orthonormal basis for a Hilbert space H. Define the *left-shift operator* by the formula:

$$L(x) = \sum_{n=1}^{\infty} \langle x, e_{n+1} \rangle e_n, \qquad x \in H.$$

Find $\|L\|$. Find the spectrum of L.

10. Let $A : c_0 \longrightarrow l^\infty$ be a bounded linear operator. Prove that there exists an infinite matrix $(\alpha_{ij})_{i,j \geq 1}$ of numbers, such that $y = Ax$ if and only if

$$\eta_i = \sum_{j=1}^{\infty} \alpha_{ij} \xi_j,$$

where $x = (\xi_j) \in c_0$ and $y = (\eta_i) \in l^\infty$. Show that

$$\|A\| = \sup_{i \geq 1} \sum_{j=1}^{\infty} |\alpha_{ij}|.$$