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Prov i matematik
Funktionalanalys
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Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL!

Problems 1 — 8 should be attempted by all students. **Graduate students should also try to solve Problems 9 and 10**

1. Let $\mathcal{P}_4$ be the space of the polynomials of one complex variable of degree at most 4. Define

$$\|p\| = \sum_{j=0}^{4} |p(j)|, \qquad p \in \mathcal{P}_4.$$

Show that this is a norm on $\mathcal{P}_4$, but that there is no inner product such that $\langle p, p\rangle = \|p\|^2$ for all $p \in \mathcal{P}_4$. Is the space $\mathcal{P}_4$ complete?

2. Let $S : X \longrightarrow Y$ be a bijective linear operator between normed spaces X, Y. Show that the inverse S^{-1} is continuous if and only if

$$\inf_{\|x\|=1} \|Sx\| > 0.$$

3. Let X and Y be normed spaces and let $T \in B(X, Y)$. Show that if $x \in X$, $y \in Y$, and (x_n) is a sequence in X such that $x_n \xrightarrow{w} x$ and $Tx_n \longrightarrow y$, then $Tx = y$.

4. Show that if Y is a closed subspace of a normed space X and $a \in X \setminus Y$, then there exists $f \in X'$ such that $f(a) = 1$ and Y is contained in the null space of f.

5. Suppose that α_{jk} are numbers such that

$$A = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mid \alpha_{jk} \mid^2 < \infty.$$

Define

$$S(\xi_1, \xi_2, \xi_3, \xi_4, \ldots) = (\sum_{k=1}^{\infty} \alpha_{1k}\xi_k, \sum_{k=1}^{\infty} \alpha_{2k}\xi_k, \sum_{k=1}^{\infty} \alpha_{3k}\xi_k, \ldots), \qquad \xi = (\xi_j) \in l^2.$$

Show that $S : l^2 \to l^2$ is a bounded linear operator and that $\|S\| \leq \sqrt{A}$.

6. Let Y be a closed subspace of a Hilbert space H and let P_Y denote the orthogonal projection onto Y. Show that

$$Y = \{x \in H : \|P_Y(x)\| = \|x\|\}.$$

7. Let H be a Hilbert space and let $T : H \longrightarrow H$ be a bounded linear operator. Show that $\|T\| = |\lambda|$ for some eigenvalue λ if and only if $|\langle Tx, x\rangle| = \|T\|$ for some vector $x \in H$ such that $\|x\| = 1$.

8. Assume that the space $\mathcal{C}[0, \pi]$ is equipped with the norm

$$\|x\| = \sup_{t \in [0,\pi]} |x(t)|.$$

Consider the linear operator

$$K : \mathcal{C}[0, \pi] \longrightarrow \mathcal{C}[0, \pi]$$

given by the formula

$$(Kx)(s) = \int_0^\pi (\sin s + \cos t) x(t) dt, \qquad x \in \mathcal{C}[0, \pi].$$

Is this operator compact? Find the ranges of K and KK. Show that K has only one non-zero eigenvalue.

Additional problems for graduate students:

9. Let $T : H \longrightarrow H$ be a bounded linear operator on a Hilbert space H. We define the *approximative spectrum* $\sigma_a(T)$ of T by the formula

$$\sigma_a(T) = \left\{ \lambda \in \mathbf{C} : \inf_{\|x\|=1} \|Tx - \lambda x\| = 0 \right\}.$$

Show that

$$\sigma_p(T) \cup \sigma_c(T) \subset \sigma_a(T) \subset \sigma(T).$$

Hint: See Problem 2.

10. Let X be a separable Banach space. Prove the so-called Banach-Alaoglu Theorem: Every bounded subset M of X' is relatively weak* compact, that is every sequence of elements of M contains a subsequence which is weak* convergent to an element of X'.

GOOD LUCK!