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Prov i matematik
Funktionalanalys
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Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL!

Problems 1 — 8 should be attempted by all students. **Graduate students should also try to solve Problems 9 and 10**

1. Let $\mathcal{P}$ be the vector space of all polynomials of one real variable and with real coefficients. If $p \in \mathcal{P}$ has degree d, then it has the form

$$p(t) = c_0 + c_1 t + \ldots + c_d t^d$$

and we define

$$\|p\| = \sqrt{\sum_{j=0}^{d} c_j^2}.$$

Show that this is a norm. Is there an inner product such that $\|p\|^2 = \langle p, p \rangle$? Is this space complete?

2. Let X be a vector space and let $g \in X^*$ be such that $\mathcal{N}(g) \neq X$. Prove that if a functional $f \in X^*$ has the same null-space as g, then $g = \lambda f$ for some number $\lambda \neq 0$.

3. Let (u_n) and (v_n) be two orthonormal bases in a Hilbert space H and let (λ_n) be a bounded sequence of complex numbers. Define

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle v_n, \qquad x \in H.$$

Prove that this formula defines a bounded linear operator $T : H \longrightarrow H$. Find the norm of the operator T.

4. Let T be the linear operator defined in the previous problem. Determine T^* and show that $T^*T = TT^* = I$ if and only if $|\lambda_n| = 1$ for all n.

5. Let M be a subset of a normed space X. Show that $a \in X$ is an element of the closure of span(M) if and only if $f(a) = 0$ for every $f \in X'$ such that $M \subset \mathcal{N}(f)$.

6. Let H and let (P_n) be a sequence of orthogonal projections of H onto closed subspaces of H. Suppose that $P : H \longrightarrow H$ is a bounded linear operator such that $\|P_n - P\| \to 0$ as $n \to \infty$. Show that P is the orthogonal projection onto its range.

7. Let X, Y be Banach spaces and let $T : X \longrightarrow Y$ be a compact bijective operator. Show that X must be finite-dimensional.

8. Let $X = \mathcal{C}[0,1]$ be equipped with the usual norm $\|x\| = \sup\{|x(t)| : \ x \in [0,1]\}$. Consider the linear operator $T : X \longrightarrow X$ given by the formula

$$T(x) = y, \text{ where } y(t) = x(1-t) \text{ for any } x \in X.$$

Determine the eigenvalues and the eigenspaces of T. Find explicitly the resolvent operator and describe the spectrum of T.

Additional problems for graduate students:

9. Let H be a Hilbert space and let

$$E_1 \subset E_2 \subset E_2 \subset E_3 \subset \ldots \subset H$$

be a sequence of closed subspaces of H. Suppose that $x_1, x_2, x_3, \ldots$ is a bounded sequence in H such that x_n is the orthogonal projection of x_{n+1} onto E_n for every n. Show the following properties:

(a) $\|x_n\| \leq \|x_{n+1}\|$ for every n;

(b) x_n is the orthogonal projection of x_{n+k} onto E_n if $n = 1, 2, 3, \ldots$ and $k = 0, 1, 2, \ldots$;

(c) (x_n) is a Cauchy sequence;

(d) if $y = \lim_{n\to\infty} x_n$, then x_n is the orthogonal projection of y onto E_n.

10. Let (f_n) be a sequence of complex-valued continuous functions on an open interval $]a, b[$ such that for at each point $x \in]a, b[$ we have:

$$\sup_{n\in\mathbf{N}} |f_n(x)| < +\infty.$$

Show that if α, β are suitably chosen and $a < \alpha < \beta < b$, then

$$\sup_{n\in\mathbf{N}} \sup_{x\in[\alpha,\beta]} |f_n(x)| < +\infty.$$

GOOD LUCK!