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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL!

Problems 1 — 8 should be attempted by all students.
Graduate students should also try to solve Problems 9 and 10

1. Let E be the space of all sequences of complex numbers $x = (\xi_1, \xi_2, \xi_3, \ldots)$ such that:

$$\sum_{j=1}^{\infty} |\xi_{2j}| < \infty \text{ and } \sum_{k=0}^{\infty} |\xi_{2k+1}|^2 < \infty.$$

It can be shown that the formula

$$\|x\| = \sum_{j=1}^{\infty} |\xi_{2j}| + \left(\sum_{k=0}^{\infty} |\xi_{2k+1}|^2 \right)^{1/2}, \qquad x = (\xi_n) \in E,$$

defines a norm on E. Show that E is a Banach space with this norm and that there is no inner product such that $\langle x, x \rangle = \|x\|^2$ for all $x \in E$.

2. Let H_1, H_2 be Hilbert spaces and let $T : H_1 \longrightarrow H_2$ be a bounded linear operator whose range $\mathcal{R}(T)$ is closed in H_2. Let $\mathcal{N}(T)$ denote the null-space of T. Show that $T = T \circ P$ where P is the orthogonal projection onto the orthogonal complement of $\mathcal{N}(T)$. Define $S : \mathcal{N}(T)^{\perp} \longrightarrow \mathcal{R}(T)$ to be the restriction of T to $\mathcal{N}(T)^{\perp}$. Show that the operator S is bijective and has a bounded inverse.

3. Consider the space $\mathcal{C}[-1, 1]$ equipped with the norm

$$\|x\| = \sup_{t \in [-1,1]} |x(t)|.$$

Define $f_n, f \in (\mathcal{C}[-1, 1])'$ by the formulae:

$$f_n(x) = \frac{n}{2} \int_{-1/n}^{1/n} x(t) dt, \qquad x \in \mathcal{C}[-1, 1]$$

and

$$f(x) = x(0), \qquad x \in \mathcal{C}[-1,1].$$

Prove that $\|f_n\| = 1$ and that the sequence (f_n) is weak-$*$-convergent to f.

4. Re-examine Problem 3. Show that the convergence is not strong by looking at the sequence of functions:

$$x_n(t) = \min\{n|t|, 1\}, \qquad t \in [-1,1].$$

5. Suppose that α_{jk} are numbers such that

$$A = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mid \alpha_{jk} \mid < \infty.$$

Define

$$S(\xi) = (\sum_{k=1}^{\infty} \alpha_{1k}\xi_k, \sum_{k=1}^{\infty} \alpha_{2k}\xi_k, \sum_{k=1}^{\infty} \alpha_{3k}\xi_k, \ldots), \qquad \xi = (\xi_j) \in l^{\infty}.$$

Show that $S(\xi) \in l^{\infty}$ if $\xi \in l^{\infty}$. Show also that $S : l^{\infty} \to l^{\infty}$ is a bounded linear operator and that $\|S\| \leq A$.

6. Let Y be a finite dimensional subspace of a Hilbert space H. Let P_Y denote the orthogonal projection onto Y and $\{y_1, \ldots, y_n\}$ be a basis for Y. Let $x \in H$. Show that the system of linear equations

$$\begin{bmatrix} \langle y_1, y_1\rangle & \ldots & \langle y_n, y_1\rangle \\ \langle y_1, y_2\rangle & \ldots & \langle y_n, y_2\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1, y_n\rangle & \ldots & \langle y_n, y_n\rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \langle x, y_1\rangle \\ \langle x, y_2\rangle \\ \vdots \\ \langle x, y_3\rangle \end{bmatrix}$$

has a unique solution $(\alpha_1, \ldots, \alpha_n)$ and that

$$P_Y(x) = \sum_{j=1}^{n} \alpha_j y_j.$$

7. Explain why the Hahn-Banach theorem becomes trivial for closed subspaces of Hilbert spaces. In a Hilbert space H derive the formula

$$\|x\| = \sup_{f \in H' \setminus \{0\}} \frac{|f(x)|}{\|f\|}, \qquad x \in H,$$

without using the Hahn-Banach theorem.

8. Let $\{x_j\}$ be an orthonormal sequence in a complex Hilbert space H. Give an example of a Hermitian compact operator for which the numbers 1, 1/2, 1/3, 1/4, ... are eigenvalues with $x_1, x_2, \ldots$ as corresponding eigenvectors. Justify your answer.

Additional problems for graduate students:

9. Let X, Y be normed spaces. Assume that $S : X \longrightarrow Y$ is a linear operator which maps the open unit ball $B(0,1)$ in X onto an open subset of Y. Prove that then S is surjective and that S is an open mapping.

10. Let $\{x_n\}_{n\geq 1}$ be a sequence of vectors in a Hilbert space H such that

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq \infty, \qquad \text{for every } x \in H.$$

Let $T : H \longrightarrow l^2$ be the linear operator given by the formula

$$T(x) = (\langle x, x_n\rangle)_{n\geq 1}, \qquad \text{for } x \in H.$$

Using the Closed Graph Theorem or otherwise show that this is a bounded operator.

GOOD LUCK!