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Tillåtna hjälpmedel: Manuella skrivdon

LYCKA TILL!

1. Show that if p is a positive integer, then the vector space l^p (with the usual norm) is a Banach space. Give an example of a normed space which is not complete.

2. Prove that for every normed space X, the dual space X' is complete. Suppose that X is the vector space of all polynomials (of one real variable and with real coefficients) with the norm

$$\|x\| = \sup_{0 \le t \le 1} |x(t)|, \qquad x \in X.$$

Give an example of a linear functional $f : X \longrightarrow \mathbf{R}$ which is not continuous.

3. Let X be a normed space and let Y be a finite-dimensional subspace of X. Let $x \in X$. Show that there exists $y \in Y$ such that $\|x - y\| \le \|x - z\|$ for all $z \in Y$. **Hint:** Observe that if $x \neq 0$ and such y can be found, then y must belong to the closed ball with centre at x and radius $\|x\|$.

4. Let X and Y be normed spaces and let $T : X \longrightarrow Y$ be a bounded linear operator linear operator. The norm of T can be defined in three ways:
- as $\sup\{\|Tx\| : \|x\| = 1\}$,
- or as $\sup\{\|Tx\| : \|x\| \le 1\}$,
- or as $\inf\{M \ge 0 :$ such that $\|Tx\| \le M\|x\|$ for all $x \in X\}$.

Show that all three definitions give the same number.

5. Assume that $x = (\xi_j)_{j \ge 1} \in l^2$, $\|x\| = 1$ and

$$x_n = (\underbrace{0, \ldots, 0}_{n-\text{times}}, \xi_1, \xi_2, \xi_3, \ldots), \qquad n = 1, 2, 3, \ldots.$$

Show that $\|x_n\| = 1$ for all n and that $(x_n)_{n\geq 1}$ converges weakly to $0 \in l^2$.

6. Let X be a normed space and let M be a positive number. Suppose that $x_0 \in X$ has the property that $|f(x_0)| \leq M$ for all $f \in X'$ such that $\|f\| = 1$. Show that $\|x_0\| \leq M$.

7. Suppose that H is a Hilbert space and that $P : H \longrightarrow H$ is a self-adjoint linear operator such that $PP = P$. Show that the range $\mathcal{R}(P)$ of P is closed and that P is the orthogonal projection onto $\mathcal{R}(P)$.
Hint: First show that $Px \perp (x - Px)$ for every $x \in H$.

8. Let $(c_j)_{j\geq 0}$ be a sequence of positive numbers such that $c_1 > c_2 > c_3 > \ldots$ and $\lim_{j\to\infty} c_j = 0$. Let $T : l^2 \longrightarrow l^2$ be the linear operator defined by

$$T((\xi_1, \xi_2, \xi_3, \ldots)) = (c_1\xi_1, c_2\xi_2, c_3\xi_3, \ldots),$$

where $(\xi_1, \xi_2, \xi_3, \ldots) \in l^2$. Show that T is a compact operator. Describe the spectrum of this operator.

GOOD LUCK!