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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
2002-03-01

Skrivtid: 9-14.

Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL!

1. Let K be an arbitrary, non-empty and compact set in $\mathbb{C}$. Give an example of a continuous linear operator T such that $\sigma(T) = K$.

2. Let $Y = \{x = (\xi_j) \in l^2 \mid \xi_1 - \xi_2 = 0,\ \xi_3 = 0,\ \xi_2 + \xi_3 + \xi_4 = 0\}$ and let P be the orthogonal projection onto $Y^{\perp}$. Determine P.

3. Let X be a normed space and let $f, g \in X'$. First show that if $\mathcal{N}(f) \neq X$ and $x_0 \notin \mathcal{N}(f)$ then every vector $x \in X$ has a unique representation $x = \alpha x_0 + y$, where α is a scalar and $y \in \mathcal{N}(f)$. Then show that $\mathcal{N}(f) = \mathcal{N}(g)$ if and only if $f = \lambda g$ for some scalar $\lambda \neq 0$.

4. Use the result of the previous problem to prove the Riesz' theorem: If H is a Hilbert space and $f \in H'$ then there exists a unique $z \in H$ such that $\|f\| = \|z\|$ and $f(x) = \langle x, z\rangle$, $x \in H$. (Please don't give the proof in the book!)

5. Consider l^1 as the dual of c_0. Show that the sequence of vectors

$$e_n = (0, \ldots, 0, 1, 0, 0, \ldots), \qquad n = 1, 2, \ldots,$$

where 1 is placed on the n-th position, is not weakly convergent in l^1, but converges in the weak*-sense.

6. Let c be the Banach space of all sequences $y = (\eta_j)_1^\infty$ such that $\lim_{j\to\infty} \eta_j$ exists. The norm on c is given by $\|y\| = \sup_j |\eta_j|$. Consider the operator $T : l^2 \longrightarrow c$ defined by

$$Tx = (\xi_1,\ \xi_1 + \tfrac{1}{2}\xi_2,\ \xi_1 + \tfrac{1}{2}\xi_2 + \tfrac{1}{3}\xi_3, \ldots), \qquad x = (\xi_j) \in l^2.$$

(a) Show that T is bounded and find it's norm.

(b) Show that T is invertible and determine T^{-1}.

(c) Show that $\mathcal{R}(T)$ is dense but not closed in c.

(*Hint:* It might be useful to know that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$)

7. Let (α_n) and (β_n) be complex sequences such that $|\alpha_{n-1}| > |\alpha_n| \to 0$ and $|\beta_{n-1}| > |\beta_n| \to 0$. $T : l^2 \longrightarrow l^2$ is defined by

$$Tx = (\alpha_1\xi_1,\ \alpha_2\xi_2 + \beta_1\xi_1,\ \alpha_3\xi_3 + \beta_2\xi_2,\ \ldots), \qquad x = (\xi_j) \in l^2.$$

(a) Show that T is compact.

(b) Find all eigenvalues and eigenvectors of T.

$\boxed{\textbf{You 6-pointers: Don't forget to contact me for further examination!}}$

GOOD LUCK!