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Prov i matematik
Funktionalanalys
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SOLUTIONS:

1. Let $\{\lambda_n\}_1^\infty$ be a dense sequence in K. Such a sequence exists because any subset of the separable space $\mathbb{C}$ is separable. Define $T : l^1 \to l^1$ by $Tx = (\lambda_1\xi_1,\ \lambda_2\xi_2,\ \lambda_3\xi_3,\ \dots)$, $x = (\xi_j)$. As all λ_n are eigenvalues ($Te_n = \lambda_n e_n$), $\overline{\{\lambda_n\}} = K$ and $\sigma(T)$ is closed we draw the conclusion that $K \subset \sigma(T)$.

If, on the other hand, $\lambda \notin K$ then there exists $\delta > 0$ such that $|\lambda - \mu| \geq \delta$ for all $\mu \in K$. It follows that $\|(T - \lambda I)^{-1}\| \leq \frac{1}{\delta}$, so $\lambda \in \rho(T)$ and consequently $\sigma(T) \subset K$.

2. From the definition we see that

$$Y = \{e_1 - e_2,\ e_3,\ e_2 + e_3 + e_4\}^\perp = \{e_1 - e_2,\ e_3,\ e_2 + e_4\}^\perp.$$

Consequently $Y^\perp = \text{span}\{e_1 - e_2,\ e_3,\ e_2 + e_4\}$. Using Gram-Schmidt we find an orthogonal basis $f_1 = e_1 - e_2$, $f_2 = e_3$, $f_3 = e_1 + e_2 + 2e_4$ in $Y^\perp$. The orthogonal projection P onto $Y^\perp$ is then given by

$$Px = \frac{\langle x, f_1\rangle}{\langle f_1, f_1\rangle} f_1 + \frac{\langle x, f_2\rangle}{\langle f_2, f_2\rangle} f_2 + \frac{\langle x, f_3\rangle}{\langle f_3, f_3\rangle} f_3 =$$
$$= \tfrac{1}{3}(2\xi_1 - \xi_2 + \xi_4,\ -\xi_1 + 2\xi_2 + \xi_4,\ 3\xi_3,\ \xi_1 + \xi_2 + 2\xi_4,\ 0,\ 0,\ \dots).$$

3. (i) Let $x = \alpha x_0 + y$. Then $f(y) = f(x) - \alpha f(x_0) = 0$ iff $\alpha = f(x)/f(x_0)$, so $y \in \mathcal{N}(f)$ iff $\alpha = f(x)/f(x_0)$. Existence and uniqueness of the representation follows.
(ii) If $f = \lambda g$, $\lambda \neq 0$ it is clear that $f(x) = 0$ iff $g(x) = 0$ so $\mathcal{N}(f) = \mathcal{N}(g)$. Suppose conversely that $\mathcal{N}(f) = \mathcal{N}(g)$. If $\mathcal{N}(f) = X = \mathcal{N}(g)$ then $f = 0 = g = 1 \cdot g$. If $\mathcal{N}(g) \neq X$ take $x_0 \in X \setminus \mathcal{N}(g)$. Any $x \in X$ then has the representation $x = (g(x)/g(x_0))x_0 + y$, where $y \in \mathcal{N}(g) = \mathcal{N}(f)$. $\Rightarrow f(x) = (g(x)/g(x_0))f(x_0) + f(y) = (f(x_0)/g(x_0))g(x)$ i e $f = \lambda g$, where $\lambda = f(x_0)/g(x_0)$.

4. If $f = 0$ then $z = 0$ works. If $f \neq 0$ then $\mathcal{N}(f)$ is a proper closed subspace of X, which implies that we can find $z_0 \perp \mathcal{N}(f)$ with $\|z_0\| = 1$. By the conclusions of the previous problem $\mathcal{N}(f)^\perp$ is one-dimensional and hence spanned by z_0, which means that $z_0^\perp = \mathcal{N}(f)$. Consequently the linear form $g(x) = \langle x, z_0\rangle$, $x \in X$, fulfils $\mathcal{N}(g) = \mathcal{N}(f)$. By the previous problem there exists $\lambda \neq 0$ such that $f = \lambda g$ i e $f(x) = \lambda\langle x, z_0\rangle = \langle x, \bar{\lambda} z_0\rangle = \langle x, z\rangle$, where $z = \bar{\lambda} z_0$. For proof of uniqueness see Kreyszig.

5. For $f \in (l^1)'$ given by

$$f(\xi_1, \xi_2, \dots) = \sum_{n=1}^{\infty} (-1)^n \xi_n,$$

we have $f(e_n) = (-1)^n$, so $(f(e_n))$ has no limit. Hence (e_n) has no weak limit. On the other hand, for $x = (\xi_j) \in c_0$ we have $e_n(x) = \xi_n \to 0 = 0(x)$, so e_n tends to 0 in the weak*-sense.

6. (a) $(\eta_n) = y = Tx$, $x = (\xi_j) \Leftrightarrow \eta_n = \xi_1 + \frac{1}{2}\xi_2 + \cdots + \frac{1}{n}\xi_n$, $n = 1,\ 2,\ 3,\ \ldots. \Rightarrow$

$$\begin{aligned}
|\eta_n| &\le |\xi_1| + \tfrac{1}{2}|\xi_2| + \cdots + \tfrac{1}{n}|\xi_n| \\
&\le \left(1 + \tfrac{1}{2^2} + \cdots + \tfrac{1}{n^2}\right)^{\frac{1}{2}} \left(|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{\infty} \tfrac{1}{k^2}\right)^{\frac{1}{2}} \|x\| \\
&= \frac{\pi}{\sqrt{6}}\|x\|
\end{aligned}$$

$\Rightarrow \|Tx\| = \|y\| \le \frac{\pi}{\sqrt{6}}\|x\|$. $\Rightarrow \|T\| \le \frac{\pi}{\sqrt{6}}$. For the converse inequality we let $x = (\frac{1}{j})$. Then $\|x\| = \frac{\pi}{\sqrt{6}}$ and $\|T\|\|x\| \ge \|Tx\| \ge \lim_{n\to\infty} \eta_n = \|x\|^2$. $\Rightarrow \|T\| \ge \|x\| = \frac{\pi}{\sqrt{6}}$. Hence $\|T\| = \frac{\pi}{\sqrt{6}}$.

(b) $y = Tx \Leftrightarrow \eta_k = \xi_1 + \frac{1}{2}\xi_2 + \cdots + \frac{1}{k}\xi_k = \eta_{k-1} + \frac{1}{k}\xi_k$. $\Leftrightarrow \xi_1 = \eta_1$ and $\xi_k = k(\eta_k - \eta_{k-1})$ $(k > 1)$. So T is invertible and $x = T^{-1}y$ iff $\xi_1 = \eta_1$ and $\xi_k = k(\eta_k - \eta_{k-1})$ $(k > 1)$.

(c) If $y \in c$ has the form $y = (\eta_1,\ \ldots,\ \eta_{n-1},\ L,\ L,\ L,\ \ldots)$ (we say that y *s*tabilizes) then for $x = T^{-1}y$ we have $\xi_k = k(\eta_k - \eta_{k-1}) = 0$, $k > n$, i e $x = (\xi_1,\ \ldots,\ \xi_n,\ 0,\ 0,\ 0,\ \ldots) \in l^2$. As the subspace of all stabilizing sequences is dense in c we conclude that $\mathcal{R}(T)$ is dense in c. It follows that if $\mathcal{R}(T)$ is closed then $\mathcal{R}(T) = c$, which implies that $T^{-1}y \in l^2$ for all $y \in c$, which is not true, as the following counterexample shows: Let $\eta_k = 1 + \frac{1}{2\sqrt{2}} + \cdots + \frac{1}{k\sqrt{k}}$. Then $y = (\eta_k) \in c$ but for $x = T^{-1}y$ we have $\xi_k = \frac{1}{\sqrt{k}}$, so $x \notin l^2$. This contradiction proves that $\mathcal{R}(T)$ cannot be closed.

7. (a) Let the sequence of operators $T_n : l^2 \longrightarrow l^2$, $n = 1,\ 2,\ 3,\ \ldots$, be defined by $(T_n x)_j = (Tx)_j$, if $1 \le j \le n$, and $(T_n x)_j = 0$, if $j > n$ ($x \in l^2$ is arbitrary). All the T_n's are operators of finite range and hence compact. For an arbitrary $x \in l^2$ we have the estimate $\|Tx - T_n x\| \le (|\alpha_{n+1}| + |\beta_n|)\|x\|$, which implies that $\|T - T_n\| \to 0$ as $n \to \infty$. But a uniform limit of a sequence of compact operators is compact. Hence T is compact.

(b) Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of T and that $x \neq 0$ is an eigenvector belonging to λ. Then we have

$$0 = Tx - \lambda x = ((\alpha_1 - \lambda)\xi_1, (\alpha_2 - \lambda)\xi_2 + \beta_1\xi_1,\ \ldots, (\alpha_n - \lambda)\xi_n + \beta_{n-1}\xi_{n-1},\ \ldots)$$

If λ coincides with none of the α_n's it is easy to see that $x = 0$ is the only solution. So it is neccessary that $\lambda = \alpha_n$, for some n. In that case ξ_n can be choosen arbitrary, $0 = \xi_1 = \cdots = \xi_{n-1}$ and for $k = 1, 2, 3, \ldots$ we have $0 = (\alpha_{n+k} - \lambda)\xi_{n+k} + \beta_{n+k-1}\xi_{n+k-1}$. If we choose $\xi_n = 1$ we get

$$\xi_{n+k} = \frac{\beta_{n+k-1}}{\lambda - \alpha_{n+k}} \frac{\beta_{n+k-2}}{\lambda - \alpha_{n+k-1}} \cdots \frac{\beta_n}{\lambda - \alpha_{n+1}}$$

$k = 1, 2, 3, \ldots$. Thus, for any n, $\lambda = \alpha_n$ is a simple eigenvalue.