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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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Skrivtid: 8-13.

Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

LYCKA TILL!

1. Let $c = \{x = (\xi_j) \in l^\infty : \lim \xi_j \text{ exists (and is finite)}\}$. In Kreyszig's book it is shown that c is a closed subspace of l^∞ and hence a Banach space. Define $f : c \to \mathbb{C}$ by $f(x) = \lim \xi_j$, $x = (\xi_j) \in c$. Show that f is a continuous linear functional on c and calculate it's norm $\|f\|$.

2. Use the linear functional f defined in the previous problem to prove that

$$c_0 = \{x = (\xi_j) : \lim \xi_j = 0\}$$

(normed as a subspace of l^∞) is a Banach space.

3. Define a linear functional $g : C[-1,1] \to \mathbb{C}$ by

$$g(x) = \int_{-1}^{1} x(t) \sin \pi t \, dt$$

($C[-1,1]$ is equipped with it's usual norm $\|x\| = \max_{-1 \le t \le 1} |x(t)|$). Show that g is continuous with norm $\|g\| = 4/\pi$.

4. Let X, Y be two Banach spaces and let $0 \neq T \in B(X,Y)$. A vector $x \in X$ is said to be maximal for T if $x \neq 0$ and $\|Tx\| = \|T\| \, \|x\|$. Show that it is impossible for the linear functional g defined in the previous problem to have a maximal vector.

5. Let (λ_n) be a sequence of non-zero scalars and let

$$\mathcal{D} = \{x = (\xi_j) \in l^2 : \sum_{j=1}^{\infty} |\lambda_j|^2 |\xi_j|^2 < \infty\}$$

Show that $\mathcal{D}$ is a dense subspace of l^2. Also show that the linear operator $T : \mathcal{D} \to l^2$ defined by $Tx = (\lambda_j \xi_j)$, $x = (\xi_j) \in \mathcal{D} = \mathcal{D}(T)$ is invertible with range $\mathcal{R}(T)$ dense in l^2.

6. Let g be the linear functional defined in problem 3 and let $z(t) = \sin \pi t$. Find the distance from z to the nullspace $\mathcal{N}_g$ of g, i.e. calculate

$$D(z, \mathcal{N}_g) = \inf\{\|z - y\| : y \in \mathcal{N}_g\}$$

7. Let T be the operator defined in problem 5. Show that T and T^{-1} are closed (but not neccessarily continuous) linear operators. Also calculate the norms $\|T\|$ and $\|T^{-1}\|$.

$\boxed{\textbf{You 6-pointers: Don't forget to contact me for further examination!}}$

GOOD LUCK!