UPPSALA UNIVERSITET
Matematiska institutionen
Bo Styf

Prov i matematik
Funktionalanalys
Kurs: 1MA283
2004-04-02

Skrivtid: 09.15-12.15

Tillåtna hjälpmedel: Manuella skrivdon och Kreyszigs bok *Introductory Functional Analysis with Applications.*

8. Let T be the linear operator on $L^2[-1,1]$ given by

$$y = Tx \qquad \text{if and only if} \qquad y(t) = \int_{-1}^{1} \left(e^{t-s} + e^{s-t}\right) x(s)\, ds$$

(a) Show that T is self-adjoint and compact. Find all eigenvalues and eigenvectors of T.

(b) Find the spectral decomposition of T, i.e. find orthogonal projections P_k and scalars λ_k such that $I = \sum_k P_k$, $T = \sum_k \lambda_k P_k$ and $P_j P_k = 0$ $(j \neq k)$.

(c) Find the spectral decomposition of $(\lambda I - T)^{-1}$ when λ is not an eigenvalue of T.

9. Let X be a separable Banach space. Prove, for example by using the diagonal method (see p. 408), the so-called Banach-Alaoglu theorem: Every bounded subset M of X' is relatively weak*-compact, i.e. every sequence in M contains a subsequence which is weak*-convergent to an element of X'.

10. A vector $x_0 \in X$, where X is a normed space, is said to be *g-orthogonal* (a temporary concept invented for this problem!) to a subspace Y if

$$\|x_0\| \leq \|x_0 - y\| \quad \text{for all} \quad y \in Y$$

(a) Prove that x_0 $(x_0 \neq 0)$ is g-orthogonal to Y if and only if there exists $f \in X'$ such that $\|f\| = 1$, $Y \subset \mathcal{N}_f$ and $f(x_0) = \|x_0\|$.

(b) Prove that if X is an inner product space then x_0 is g-orthogonal to Y if and only if x_0 is orthogonal to Y.

(c) Let $X = l^1$ and $Y = \mathcal{N}_g$, where $g \in l^\infty$ is given by $g = (1, 1, 1, \dots)$. Find all unit vectors which are g-orthogonal to Y.