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Prov i matematik
Funktionalanalys
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SOLUTIONS:

1. The norm axioms are easy to verify. In particular, if $\|p\| = 0$, then p has five roots (at 0,1,2,3,4) and hence is identically zero. To check the second statement, it suffices to show that the parallelogram identity is not satisfied. Consider, for instance, the polynomials $p_1(z) = 1$ and $p_2(z) = z$. Then

$$\|p_1 + p_2\|^2 + \|p_1 - p_2\|^2 = 225 + 49 \neq 2(25 + 100) = 2(\|p_1\|^2 + \|p_2\|^2).$$

The space is complete, because it is finite dimensional.

2. Let

$$\alpha = \inf_{\|x\|=1} \|Sx\| > 0.$$

Then for $x \neq 0$

$$\|S(x)\| = \|x\| \left\| S\left(\frac{x}{\|x\|}\right) \right\| \geq \alpha \|x\|.$$

Hence $\|S(x)\| \geq \alpha\|x\|$ for all x. Consequently, if $y = S(x)$,

$$\|S^{-1}(y)\| = \|x\| \leq \frac{1}{\alpha}\|S(x)\| = \alpha\|y\|.$$

3. Since $x_n \xrightarrow{w} x$, we have $f(T(x_n)) \longrightarrow f(T(x))$ for any $f \in Y'$. This means that $T(x_n) \xrightarrow{w} T(x)$. But since $(T(x_n))$ is also strongly convergent, the strong and weak limits must coincide.

4. Since Y is closed, $\delta = \text{dist}(a, Y) > 0$. Let $Z = Y + \text{Span}(a)$. Define $g : Z \longrightarrow \mathbf{K}$ by the formula $g(y + \lambda a) = \lambda$ for $y \in Y$ and $\lambda \in \mathbf{K}$. Clearly g is linear and $g(a) = 1$. Moreover g is bounded because

$$|g(y + \lambda a)| = |\lambda| \leq \frac{|\lambda|}{\delta} \left\| a - \left(\frac{-y}{\lambda}\right) \right\| = \frac{1}{\delta}\|y + \lambda a\|, \qquad \lambda \neq 0.$$

So the required statement follows directly from the Hahn-Banach theorem.

5. If $x = (\xi_k) \in l^2$, then — in view of the Cauchy-Schwarz inequality — we have:

$$\|S(x)\|^2 = \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} \alpha_{jk}\xi_k \right|^2 \leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |\alpha_{jk}|^2 \right) \left(\sum_{k=1}^{\infty} |\xi_k|^2 \right) = A\|x\|^2.$$

6. If $x \in Y$, then $P_Y(x) = x$ and hence $\|P_Y(x)\| = \|x\|$. Conversely, suppose that $\|P_Y(x)\| = \|x\|$. By the Pythgorean theorem

$$\|x\|^2 = \|x - P_Y(x)\|^2 + \|P_Y(x)\|^2,$$

and so $P_Y(x) = x$.

7. If $\|T\| = |\lambda|$ for some eigenvalue λ, then take an eigenvector x corresponding to λ. We have

$$|\langle Tx, x\rangle| = |\langle \lambda x, x\rangle| = |\lambda|\|x\|^2.$$

Therefore

$$\left|\left\langle T\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle\right| = |\lambda| = \|T\|.$$

Now, let us assume that $|\langle Tx, x\rangle| = \|T\|$ for some vector $x \in H$ such that $\|x\| = 1$. Then

$$\|T\| = |\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|,$$

by the Schwarz inequality and the definition of the operator norm. So $|\langle Tx, x\rangle| = \|Tx\|\|x\|$, which implies that $Tx = \lambda x$ for some scalar λ.

8. It follows from the definition of K, that $\|Kx\| \leq 2\pi\|x\|$ and thus K is a bounded operator. Moreover $(Kx)(s) = a\sin(s) + b$ (for some numbers a, b) and hence $\mathcal{R}(K) \subset \text{Span}(\sin, 1)$. Therefore K is compact. Since $(K(a\sin t + b))(s) = (2a + b\pi)\sin s$, the range of KK is one dimensional and consists of multiples if the sine function. In particular, this implies that K can have only one non-zero eigenvalue and that — if this is the case — the sine function must be an eigenvector. Indeed, $(K(\sin t))(s) = 2\sin s$. Since $K(\cos)$ is a constant function, the range of K is equal to the span of sin and the constant function 1.

9. If $\lambda \notin \sigma(T)$, then $(T - \lambda I)^{-1}$ is well-defined on a dense subset of H and is bounded. But then, in view of Problem 2, $\lambda \notin \sigma_a(T)$. Hence $\sigma_a(T) \subset \sigma(T)$. It is obvious that $\sigma_p(T) \subset \sigma_a(T)$. If $\lambda \in \sigma_c(T)$, then $(T - \lambda I)^{-1}$ is well-defined on a dense subset of H but is not bounded. In view of Problem 2, $\lambda \in \sigma_a(T)$.

10. Let $(a_j)_{j\geq 1}$ be a sequence of points in X which, as a set, is dense in the closed unit ball in X. Let $(x_i)_{i\geq 1}$ be a bounded sequence in the dual space X'. It suffices to show that a subsequence of (x_i) is convergent at all points of the sequence (a_j). Since the sequence of numbers $(x_i(a_j))_{i\geq 1}$ is bounded, it has a convergent subsequence for any fixed value of j. As a consequence, the required property follows from the standard diagonal selection process.