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Prov i matematik
Funktionalanalys
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SOLUTIONS:

1. To get a counterexample we can take $x = (1, i)$ and $y = (1, -i)$ in $\mathbf{C}^2$.

2. $\|Tx\| \leq (b-a)\|x\|$ and we get equality when $x \equiv 1$. Obviously $T^{-1}x = x'$ because if $x \in Y$, then

$$\int_a^t x'(s)ds = x(t).$$

The operator T^{-1} is not bounded because, for example, if n is big enough and $x_n(t) = \sin(n(t-a))$, then $\|x_n\| = 1$ and $\|x_n'\| = n \to \infty$ as $n \to \infty$.

3. Let $Ta_n \to c \in Y$, where $(a_n) \subset K$. Since K is compact (a_n) has a subsequence, say (b_n) which is convergent to a limit $b \in K$. Since the graph $\mathcal{G}$ of T is closed $(b_n, Tb_n) \to (b, c) \in \mathcal{G}$ and thus $Tb = c$. So $c \in T(K)$.

4. "$\Rightarrow$" If $y = Tx$ where $x \in M$, then $y = TPx = PTx \in M$. So $T(M) \subset M$ and similarly $T(M^\perp) \subset M^\perp$.
"$\Leftarrow$" Let $x \in H$. Then $PTx = PT(Px+Qx) = PTPx+PTQx = PTPx$ because $TQx \in M^\perp$. On the other hand $TPx = PTPx+QTPx = PTPx$ because $TPx \in M$. The equality $QT = TQ$ can be shown similarly.

5. The duals of l^2 and l^1 are l^2 and l^∞ respectively. For weak convergence in l^2 see Kreyszig p.259. If $f \in (l^1)'$ is given by

$$f(\xi_1, \xi_2, \ldots) = \sum_{n=1}^{\infty} (-1)^n \xi_n,$$

then $f(e_n)$ oscillates between just two values 1 and -1.

6. If T_k is the operator given by the k-th partial sum in the definition of T, then T_k is bounded with a finite dimensional range and hence it is compact. Moreover,

$$\|Tx - T_kx\|^2 = \sum_{n=k+1}^{\infty} |\lambda_n|^2 |\langle x, e_n \rangle|^2 \leq \sup_{n>k}\{|\lambda_n|^2\}\|x\|^2,$$

and hence $\|T - T_k\| \to 0$ as $k \to \infty$.

7. There exists n_0 such that $\lambda_n \geq 1/2$ for all $n \geq n_0$. The sequence $(e_n)_{n \geq n_0}$ is bounded, but if $n_0 \leq n < m$, then

$$\|Te_m - Te_n\|^2 = \|\lambda_m e_m - \lambda_n e_n\|^2 = |\lambda_m|^2 + |\lambda_n|^2 \geq 1/2.$$

Consequently (Te_n) has no convergent subsequence.

8. We have

$$(T-\lambda I)\left(-\sum_{n=0}^{N-1}\frac{T^n}{\lambda^{n+1}}\right)=-\sum_{n=1}^{N-1}\frac{T^n}{\lambda^n}+\sum_{n=0}^{N-1}\frac{T^n}{\lambda^n}=I.$$

There exists $x \in X$ such that $y = T^{N-1}x \neq 0$. Hence $Ty = 0$ and so y is an eigenvector. Consequently $\rho(T) = \mathbf{C} \setminus \{0\}$ and $\sigma(T) = \sigma_p(T) = \{0\}$.

9. Done in class.

10. Done in class.