UPPSALA UNIVERSITET
Matematiska institutionen
M. Klimek

Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
1998-06-02

SOLUTIONS:

1. To show completeness it is enough to notice that $x = (\xi_1, \xi_2, \xi_3, \dots) \in E$ if and only if $x^{(1)} = (\xi_1, \xi_3, \xi_5, \dots) \in l^1$ and $x^{(2)} = (\xi_2, \xi_4, \xi_6, \dots) \in l^2$. Moreover (x_n) is a Cauchy sequence in E if and only if the sequences $(x_n^{(1)})$ and $(x_n^{(2)})$ are Cauchy in l^1 and l^2 respectively. Since both l^1 and l^2 are complete, the required result follows. To check the second statement, it suffices to show that the parallelogram identity is not satisfied. For instance, if $x = (1, 0, 0, \dots)$ and $y = (0, 1, 0, 0, \dots)$, then

$$\|x + y\|^2 + \|x - y\|^2 = 8 \neq 4 = 2(\|x\|^2 + \|y\|^2).$$

2. The operator $(I - P)$ is the orthogonal projection onto $\mathcal{N}(T)$. So for any $x \in H$ we have $Tx = T((I - P)x + Px) = TPx$. S is obviously surjective; it is also injective because $Sx = 0$ if and only if Tx and $x \perp \mathcal{N}(T)$. By the Open Mapping Theorem S is open, which means that S^{-1} is continuous.

3. Take $x \in \mathcal{C}[-1, 1]$. Since x is continuous at 0, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$x(0) - \epsilon < x(t) < x(0) + \epsilon \qquad \text{if } |t| < \delta.$$

Consequently

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon \qquad \text{if } \frac{1}{n} < \delta.$$

In other words $\lim_{n\to\infty} f_n(x) = f(x)$.

4. We have: $\|x_n\| = 1$, $f_n(x_n) = 1/2$, $f(x_n) = 0$ and $1/2 = \|f_n(x_n) - f(x_n)\| \leq \|f_n - f\|$.

5. If $\xi = (\xi_k) \in l^\infty$, then we have:

$$\|S(x)\| = \sup_{j\geq 1} \left| \sum_{k=1}^{\infty} \alpha_{jk}\xi_k \right| \leq \sup_{j\geq 1} \left(\sum_{k=1}^{\infty} |\alpha_{jk}| \right) \left(\sup_{k\geq 1} |\xi_k| \right) \leq A\|\xi\|.$$

6. It is enough to notice that $P_Y(x)$ is characterized by the fact that $x - P_Y(x) \perp y_1, \dots, y_n$.

7. If Z is a closed subspace of a Hilbert space H and $f \in Z'$, then $f(x) = \langle x, z \rangle$ for some $z \in Z$ and for all $x \in H$ (by the Riesz theorem) and $\|f\| = \|z\|$. But f is well-defined for all $x \in H$ and has the same norm $\|z\|$ when regarded as an element of H'. The second part is also a consequence of the Riesz theorem:

$$\sup_{f\in H'\setminus\{0\}} \frac{|f(x)|}{\|f\|} = \sup_{y\in H\setminus\{0\}} \frac{|\langle x,y\rangle|}{\|y\|} = \sup_{\|z\|=1} |\langle x,z\rangle| = \|x\|.$$

(By the Cauchy-Schwarz estimate $|\langle x,z\rangle| \le \|x\|$ if $\|z\| = 1$, but we reach equality if either $x = 0$ or $z = x/\|x\|$.)

8. Define

$$T(x) = \sum_{n=1}^{\infty} \frac{\langle x, x_n\rangle}{n} x_n.$$

Then $\|Tx\|^2 \le \|x\|^2$ by Bessel's inequality, T is Hermitian because $\langle Tx, x\rangle$ is real for all x and T is compact being the limit of

$$T_m(x) = \sum_{n=1}^{m} \frac{\langle x, x_n\rangle}{n} x_n.$$

9. Since $S(B(0,1))$ is open and contains 0, there exists $\delta > 0$ such that $B(0,\delta) \subset S(B(0,1))$. If $y \in Y \setminus \{0\}$, then

$$\frac{\delta y}{2\|y\|} \in B(0,\delta).$$

Hence there exists $x \in B(0,1)$ such that

$$S(x) = \frac{\delta y}{2\|y\|}.$$

Thus

$$S\left(\frac{2\|y\|x}{\delta}\right) = y.$$

Since $S(0) = 0$, the above shows surjectivity of S. To show openness, it suffices to check that

$$\forall a \in X\ \forall r > 0\ :\ S\left(B(a,r)\right) \supset B\left(f(a), r\delta\right),$$

where δ is as above.

10. We have to show that the graph of T is closed. Suppose that $y_j \to y$ in H and $Ty_j \to z$ in l^2 as $j \to \infty$. It is enough to prove that $Ty = z$. Since (Ty_j) is a Cauchy sequence (and from the definition of the l^2 norm)

$$\forall \epsilon > 0\, \exists N\, \forall n \ge 1\, \forall i,j \ge N\ :\ \sum_{k=1}^{n} |\langle y_i, x_k\rangle - \langle y_j, x_k\rangle|^2 < \epsilon^2.$$

By letting first j and then n go to infinity, we conclude that

$$\|Ty_i - T_y\|_{l^2} < \epsilon.$$

So $Ty_i \to T_y$ in l^2 and hence $Ty = z$. By the Closed Graph Theorem T is bounded.