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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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SOLUTIONS:

1. Every step function f can be identified with an element of l^2, for example with

$$s_f = (f(0), f(1), f(-1), f(2), f(-2), f(3), f(-3), \ldots).$$

and the operator $f \mapsto s_f$ is an isometry of normed spaces.

2. Let $\mathbf{a} = (\alpha, \alpha^2, \alpha^3, \ldots)$ and $\mathbf{b} = (\beta, \beta^2, \beta^3, \ldots)$. Then $L(x) = \langle \mathbf{b}, x\rangle \mathbf{a}$ for all $x \in l^2$ which yields the result easily.

3. It is enough to notice that

$$\sqrt{2}T((x_n)) = (x_1 - x_2, x_1 + x_2, x_3 - x_4, x_3 + x_4, \ldots)$$

and

$$\|(x_1 - x_2, x_1 + x_2, x_3 - x_4, x_3 + x_4, \ldots)\|^2 = 2\|x\|^2.$$

4. We have

$$\langle S(x), y\rangle = \left\langle \sum_{j=1}^{n} \langle x, x_j\rangle x_j, y \right\rangle = \sum_{j=1}^{n} \langle x, x_j\rangle \langle x_j, y\rangle = \left\langle x, \sum_{j=1}^{n} \langle y, x_j\rangle, x_j \right\rangle = \langle x, S(t)\rangle.$$

In particular if $x = y \in K$ and $S(x) = 0$, then $x \perp K$ and so $x = 0$.

5. It is enough to show that for each $x \in H$ and for all x_n

$$\langle x - P_K(x), x_n\rangle = 0.$$

We have

$$\langle P_K(x), x_n\rangle = \left\langle \sum_{j=1}^{k} \langle x, S^{-1}x_j\rangle x_j, x_n \right\rangle = \sum_{j=1}^{k} \langle x, S^{-1}x_j\rangle \langle x_j, x_n\rangle =$$

$$= \langle Sx, S^{-1}x_n\rangle = \langle x, SS^{-1}x_n\rangle = \langle x, x_n\rangle.$$

6. Suppose that (f_n) is weakly convergent in X' to $f \in X$. This means that $g(f_n) \to g(f)$ for any $g \in X''$. Take $x \in X$ and define $g \in X''$ by the formula $g(h) = h(x)$ for all $h \in X'$. Then $f_n(x) = g(f_n) \to g(f) = f(x)$. Since x was arbitrarily chosen we can conclude that (f_n) is weak* convergent to f.

7. The range consist of all cubic polynomials q such that $q(0) = 0$. The operator is not compact (and not even bounded). For example, if $p(t) = t^n$, then $\|p\| = 1$, but $\|F(p_n)\| = 1 + 2^n$.

8. Let $z = T_\lambda(x)$. Then

$$\begin{aligned} z(t) &= (a-\lambda)x(t) + bx(1-t), \\ z(1-t) &= bx(t) + (a-\lambda)x(1-t). \end{aligned}$$

Therefore

$$x(t) = \frac{(a-\lambda)z(t) - bz(1-t)}{(a-\lambda)^2 - b^2}$$

provided that $|b| \neq |a-\lambda|$. Hence there are exactly two eigenvalues $a-b$ and $a+b$. Moreover, there are no other elements in the spectrum.