Funktionalanalys F3 och F4

Solutions of exam problems from 96-12-09

1. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in l^2. Then $x_n = (\xi_j^{(n)})_{j\geq 1}$ and since $|\xi_j^{(n)} - \xi_j^{(m)}| \leq \|x_n - x_m\|$, the numerical sequence $(\xi_j^{(n)})_{n\geq 1}$ is Cauchy for any fixed j. Let $\xi_j = \lim_{n\to\infty} \xi_j^{(n)}$. We have to show that $x = (\xi_j)_{j\geq 1} \in l^2$. Since $(x_n)_{n\geq 1}$ is a Cauchy sequence, there exists a positive integer N such that for all $n \geq N$, we have $\|x_n - x_N\| \leq 1$. By the triangle inequality

$$\sum_{j=1}^{k} |\xi_j^{(n)}| \leq \|x_n - x_N\| + \|x_N\| \leq 1 + \|x_N\|$$

for all k and $n \geq N$. Hence $\|x\| \leq 1 + \|x_N\| < \infty$.

The space of all polynomials with the sup-norm or $\mathcal{C}[a,b]$ with the L^2-norm are not complete.

2. Let (f_n) be a Cauchy sequence in X'. If $x \in X$, then $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|\,\|x\|$ and hence $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in $\mathbf{C}$. Define $f(x)$ as the limit of this sequence. Clearly $x \mapsto f(x)$ is a linear functional. We have to show that it is bounded. Since (f_n) is Cauchy, there exists N such that $\|f_n - f_N\| \leq 1$ for all $n \geq N$. Therefore $|f_n(x)| \leq \|f_n\|\,\|x\| \leq (\|f_n - f_N\| + \|f_N\|)\|x\| \leq (1 + \|f_N\|)\|x\|$ for all $n \geq N$ and all $x \in X$. So $|f(x)| \leq (1 + \|f_N\|)\|x\|$.

Take e.g. $f : X \longrightarrow \mathbf{R}$ given by the formula $f(x) = x(2)$. If $x_n(t) = t^n$, then $\|x_n\| = 1$ but $f(x_n) = 2^n \to \infty$.

3. If $x = 0$, take $y = 0$. If $x \neq 0$, then $K = \tilde{B}(x, \|x\|) \cap Y$ is compact because Y is finite dimensional. Moreover, if $z \in Y \setminus K$, then $\|x - z\| \geq \|x - 0\|$ and $0 \in K$. So if the required y exists it must be an element of K. The existence of such y follows from the fact that the continuous function $z \mapsto \|z - x\|$ (regarded as a function from K to $\mathbf{R}$) attains a minimum at a point in K.

4. Let α, β and γ denote these three numbers. The inequality $\alpha \leq \beta$ is trivial. On the other hand, if $0 < \|x\| \leq 1$, then $\|T(x)\| = \|x\|\,\|T(x/\|x\|)\| \leq \|x\|\alpha$. So $\beta \leq \alpha$.

Note that $\|T(x)\| \leq M\|x\|$ for some M and all x, if and only if $\|T(x/\|x\|)\| \leq M$ for all $x \neq 0$. The last condition is equivalent to saying that $\|T(x)\| \leq M$ for all x of norm 1. Hence $\alpha = \gamma$.

5. The first statement follows from the definition of the norm in l^2. Because of the Riesz representation theorem it suffices to show that for every y we have $|\langle x_n, y\rangle| \to 0$ as $n \to \infty$. If $y = (\eta_j)$, then $|\langle x_n, y\rangle| \leq \|x_n\| \, \|(0, \ldots, 0, \eta_{n+1}, \eta_{n+2}, \ldots)\|$. Since $\|x_n\| = 1$ and $\|(0, \ldots, 0, \eta_{n+1}, \eta_{n+2}, \ldots)\| \to 0$ as $n \to \infty$, the required property follows.

6. The property follows directly from the formula

$$\|x\| = \sup_{f \in X' \setminus \{0\}} \frac{|f(x)|}{\|f\|},$$

or from the Hahn-Banach theorem.

7. We have $\langle Px, x - Px\rangle = \langle Px, x\rangle - \langle Px, Px\rangle = \langle Px, x\rangle - \langle P^*Px, x\rangle = \langle Px, x\rangle - \langle Px, x\rangle = 0$. So $H = \mathcal{R}(P) \oplus \mathcal{R}(I - P)$ and $\mathcal{R}(P) \perp \mathcal{R}(I - P)$. Note that since $PP = P$, we have $Py = y$ if $y \in \mathcal{R}(P)$. So if (y_n) is a sequence in $\mathcal{R}(P)$ convergent to y, then $y = \lim y_n = \lim Py_n = Py$ because P is continuous, and therefore y is in the range of P.

8. The first part is virtually identical to Example 8.1-6 on page 409 in the textbook (with c_j replacing $1/j$). The numbers c_j are the eigenvalues (and constitute the point spectrum), 0 is the only element of the continuous spectrum and the residual spectrum is empty.