Functional Analysis (2006)

Homework assignment 1

All students should solve the following problems:

1. (Part of Problem 6, §1.4.) Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d), and let $a_n = d(x_n, y_n)$. Show that the sequence (a_n) converges.

2. Let $a < b$ and let $C[a, b]$ be the metric space of real valued continuous functions from $[a, b]$ to $\mathbb{R}$, with metric $d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|$ (as in §1.1-7 in the book). Let

$$D = \{x \in C[a, b] \mid x \text{ is increasing}\}.$$

(We say that $x \in C[a, b]$ is *increasing* if and only if $x(t_1) \leqq x(t_2)$ holds for all $t_1 < t_2$ in $[a, b]$.) Prove that D is closed but not open.

3. Let X be the vector space of all sequences of complex numbers with only finitely many nonzero terms. Consider the following two norms $|| \cdot ||_1$ and $|| \cdot ||_2$ on X:

$$||(\xi_j)||_1 := \sum_{j=1}^{\infty} |\xi_j|; \qquad ||(\xi_j)||_2 := \sqrt{\sum_{j=1}^{\infty} |\xi_j|^2}.$$

Prove that $|| \cdot ||_1$ and $|| \cdot ||_2$ are *not* equivalent.

4. Let $\tilde{B}(0; 1) = \{x \in \ell^1 \mid ||x|| \leqq 1\}$ be the closed unit ball in ℓ^1, and let M be the subset

$$M = \{(\xi_j) \in \tilde{B}(0; 1) \mid |\xi_j| \leqq j^{-1} \text{ for all } j = 1, 2, 3, ...\}.$$

Prove that M is not compact.

Students taking Functional Analysis as a 6 point course should also solve the following problems:

5. Let X be a normed space and let r be any number $r > 1$. Assume that it is possible to cover the open ball $B(0; r)$ by a finite number of translates of the open unit ball $B(0; 1)$. (By a *translate* of a subset $M \subset X$ we mean any set of the form $v + M := \{v + w \mid w \in M\}$ for some $v \in X$.) Prove that X is finite dimensional.

6. Let $t_1 = 0$, $t_2 = 1$ and let $t_3, t_4, \ldots$ be any pairwise distinct points in the open interval $(0,1)$ such that the set $\{t_1, t_2, t_3, t_4, \ldots\}$ is dense in $[0,1]$. Let $x_1 \in C[0,1]$ be the constant function $x_1(t) = 1$, and for $j \geqq 2$ let $x_j \in C[0,1]$ be the piecewise linear function which satisfies $x_j(t_1) = x_j(t_2) = \ldots = x_j(t_{j-1}) = 0$ and $x_j(t_j) = 1$ (and is linear at all points $t \notin \{t_1, t_2, \ldots, t_j\}$). Prove that $x_1, x_2, x_3, \ldots$ is a Schauder basis for $C[0,1]$!

Solutions should be handed in by Friday, February 10. (Either give the solutions to me directly or put them in my mailbox, third floor, House 3, Polacksbacken.)

Functional Analysis F3/F4/NVP
Solutions to homework assignment 1

1. We first prove that (a_n) is a Cauchy sequence on the real line (with respect to its usual metric "$|x - y|$"). Let $\varepsilon > 0$ be given. Then since (x_n) is Cauchy there is some integer N_1 such that $d(x_m, x_n) < \frac{\varepsilon}{2}$ for all $m, n > N_1$. Also, since (y_n) is Cauchy there is some integer N_2 such that $d(y_m, y_n) < \frac{\varepsilon}{2}$ for all $m, n > N_2$. Let $N = \max(N_1, N_2)$.

Now let m, n be any integers with $m, n > N$. Then both $m, n > N_1$ and $m, n > N_2$, and hence $d(x_m, x_n) < \frac{\varepsilon}{2}$ and $d(y_m, y_n) < \frac{\varepsilon}{2}$. Hence by the generalized triangle inequality, see p.4(1):

$$d(x_n, y_n) \leqq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < \frac{\varepsilon}{2} + d(x_m, y_m) + \frac{\varepsilon}{2} = d(x_m, y_m) + \varepsilon$$

and also

$$d(x_m, y_m) \leqq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) < \frac{\varepsilon}{2} + d(x_n, y_n) + \frac{\varepsilon}{2} = d(x_n, y_n) + \varepsilon.$$

In other words we have proved

$$a_n < a_m + \varepsilon \quad \text{and} \quad a_m < a_n + \varepsilon.$$

Together these two inequalities imply $-\varepsilon < a_n - a_m < \varepsilon$, i.e.

$$|a_n - a_m| < \varepsilon.$$

In conclusion, we have proved that for all $m, n > N$ we have $|a_n - a_m| < \varepsilon$.

The above argument works for any $\varepsilon > 0$; hence for any $\varepsilon > 0$ there exists an integer N such that $m, n > N$ implies $|a_n - a_m| < \varepsilon$. Hence (a_n) is a Cauchy sequence of real numbers! Hence by Theorem 1.4-4, the sequence (a_n) is convergent, Q.E.D.

2. We first prove that D *is closed,* i.e. (by def 1.3-2) that D^C is open. Let $x \in D^C$. Then x is *not* increasing, i.e. there exist some numbers $t_1 < t_2$ (with $t_1, t_2 \in [a, b]$) such that $x(t_1) > x(t_2)$. Let $r = \frac{x(t_1) - x(t_2)}{3}$. (Of course, $r > 0$.) We then claim that D^C contains the ball $B(x; r)$, i.e. $B(x; r) \subset D^C$. To prove this, let y be an arbitrary element in $B(x; r)$. Then $d(x, y) < r$, and in particular $|x(t_1) - y(t_1)| < r$ and $|x(t_2) - y(t_2)| < r$. It follows that $y(t_1) > x(t_1) - r$ and $y(t_2) < x(t_2) + r$. But by our definition of r we have $x(t_1) = x(t_2) + 3r$. Using all these facts we obtain:

$$y(t_1) > x(t_1) - r = x(t_2) + 2r > x(t_2) + r > y(t_2).$$

But remember here $t_1 < t_2$; hence y is *not* an increasing function. Hence $y \in D^C$. This is true for every $y \in B(x;r)$; hence we have proved $B(x;r) \subset D^C$. But $x \in D^C$ was arbitrary; hence for every $x \in D^C$ there is some $r > 0$ such that $B(x;r) \subset D^C$. This proves that D^C is open. Hence D is closed, Q.E.D.

Next we prove that D *is not open.* Let us choose x as the constant function $x(t) = 0$ for all $t \in [a,b]$. Clearly x is an increasing continuous function, i.e. $x \in D$. Let $r > 0$ be arbitrary and consider the ball $B(x;r)$. Clearly there is a continuous function $y \in B(x;r)$ which is not increasing, for example we may take $y(t) = \frac{r}{2} \cdot \frac{b-x}{b-a}$. (This is the linear function with $y(a) = \frac{r}{2}$, $y(b) = 0$.) Hence, we have found a function $y \in B(x;r)$ with $y \notin D$. It follows that $B(x;r)$ is not contained in D. The above argument works for each $r > 0$, hence D does not contain *any* ball about the point $x \in D$. Hence D is not open, Q.E.D.

3. Assume that $||\cdot||_1$ and $||\cdot||_2$ *are* equivalent (this will be shown to lead to a contradiction). Then there are some numbers $a, b > 0$ such that $a||x||_1 \leqq ||x||_2 \leqq b||x||_1$ for all $x \in X$.

We then let n be any integer which is greater than a^{-2}, and let $x \in X$ be the sequence whose first n entries equal n^{-1} and all the other entries equal 0. In other words, $x = (\xi_1, \xi_2, \xi_3, \cdots)$ where $\xi_j = n^{-1}$ for $j = 1, 2, \cdots, n$ and $\xi_j = 0$ for all $j > n$. We now compute:

$$||x||_1 = \sum_{j=1}^{n} n^{-1} = 1$$

and

$$||x||_2 = \sqrt{\sum_{j=1}^{n} n^{-2}} = \sqrt{n^{-1}} = n^{-\frac{1}{2}}.$$

Hence since we are assuming $a||x||_1 \leqq ||x||_2$ it follows that $a \leqq n^{-\frac{1}{2}}$, i.e. $n \leqq a^{-2}$. This contradicts our original choice of n, where we took n so that $n > a^{-2}$.

Hence we have seen that the assumption that $||\cdot||_1$ and $||\cdot||_2$ are equivalent leads to a contradiction. Hence $||\cdot||_1$ and $||\cdot||_2$ are *not* equivalent.

Remark: However, the *other* inequality, $||x||_2 \leqq b||x||_1$ is actually *true*, with constant $b = 1$! Proof: For every $x = (\xi_j) \in X$ we have

$$||(\xi_j)||_2 = \sqrt{\sum_{j=1}^{\infty} |\xi_j|^2} \leqq \sqrt{\left(\sum_{j=1}^{\infty} |\xi_j|\right)^2} = \sum_{j=1}^{\infty} |\xi_j| = ||(\xi_j)||_1.$$

4. For each $n = 1, 2, 3, \cdots$ we define x_n as the sequence $x_n = (2^{-n}, 2^{-n}, \cdots, 2^{-n}, 0, 0, \cdots)$, where the entries 2^{-n} start at position 1 and end at position 2^n. In other words:

$$\begin{aligned} &x_1 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, \cdots);\\ &x_2 = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, 0, 0, \cdots);\\ &x_3 = (\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, 0, 0, 0, \cdots);\\ &\cdots \end{aligned}$$

The ℓ^1 norm of x_n is $||x_n|| = 2^n \cdot 2^{-n} = 1$, hence $x_n \in \tilde{B}(0; 1)$. We also see that $x_n \in M$, for if we write $x_n = (\xi_j^{(n)})$ then we have for all $j \leqq 2^n$: $|\xi_j^{(n)}| = |2^{-n}| = 2^{-n} \leqq j^{-1}$, and for all $j > 2^n$: $|\xi_j^{(n)}| = 0 \leqq j^{-1}$. Hence $(x_1, x_2, x_3, \cdots)$ is a sequence of points in M.

However, the distance between any two points in the sequence $(x_1, x_2, x_3, \cdots)$ is $\geqq 1$. [Proof: For any $1 \leqq n < m$ we have

$$\begin{aligned} ||x_n - x_m|| &= \sum_{j=1}^{2^n} |2^{-n} - 2^{-m}| + \sum_{j=2^n+1}^{2^m} |0 - 2^{-m}| + \sum_{j=2^m+1}^{\infty} |0 - 0|\\ &= 2^n(2^{-n} - 2^{-m}) + (2^m - 2^n)2^{-m} + 0\\ &= 1 - 2^{n-m} + 1 - 2^{n-m} = 2(1 - 2^{n-m}) \geqq 2 \cdot (1 - \tfrac{1}{2}) = 1, \end{aligned}$$

since $2^{n-m} \leqq \frac{1}{2}$ because $n < m$.]

Since any two points in the sequence $(x_1, x_2, x_3, \cdots)$ have distance $\geqq 1$, it follows that no subsequence of $(x_1, x_2, x_3, \cdots)$ can be Cauchy; hence our sequence does not contain any convergent subsequence! Hence we have seen that there is a sequence in M which does not have any convergent subsequence; this means that M *is not compact*.

5. Assume that $B(0; r)$ is covered by the translates

$$v_1 + B(0; 1),\ v_2 + B(0; 1),\ \cdots, v_n + B(0; 1),$$

for some vectors $v_1, \cdots, v_n \in X$. Let $Y = \text{Span}\{v_1, \cdots, v_n\}$. Note that Y is a *closed* subset of X, by Theorem 2.4-3 on p. 74.

Let us assume that Y is a *proper* subset of X, i.e. $Y \neq X$. Let θ and r_1 be any numbers with $1 < \theta^{-1} < r_1 < r$. Then by Riesz' Lemma (2.5-4 on p. 78), applied with $Z = X$, there is a vector $x \in X$ with $||x|| = 1$ and $||x - y|| \geqq \theta$ for all $y \in Y$. Multiplying by r_1 we then obtain $||r_1 x|| = r_1 < r$, i.e. $r_1 x \in B(0; r)$. We also obtain $||r_1 x - r_1 y|| \geqq r_1 \theta > 1$ for all $y \in Y$. In particular, taking $y = r_1^{-1} v_j \in Y$ we see that $||r_1 x - v_j|| > 1$ for each $j = 1, 2, \cdots, n$. This means that $r_1 x \notin v_j + B(0; 1)$, for each $j = 1, 2, \cdots, n$. Hence the sets $v_1 + B(0; 1)$, $v_2 + B(0; 1), \cdots, v_n + B(0; 1)$ do *not* cover $B(0; r)$, a contradiction to our assumption above.

Hence the assumption $Y \neq X$ must be false; thus $Y = X$! In other words, $X = \text{Span}\{v_1, \cdots, v_n\}$, and this proves that X is finite dimensional, $\dim X \leqq n$.

6. Let $y \in C[0, 1]$ and assume that there are numbers $\alpha_1, \alpha_2, \cdots \in \mathbb{R}$ such that

$$||y - (\alpha_1 x_1 + \cdots + \alpha_n x_n)|| \to 0 \qquad \text{as } n \to \infty.$$

Fix some $j \in \{1, 2, \cdots\}$. Note that $x_j(t_j) = 1$ and $x_k(t_j) = 0$ for all $k > j$, hence for all $n \geqq j$ we have $(\alpha_1 x_1 + \cdots + \alpha_n x_n)(t_j) = \alpha_j + \sum_{k=1}^{j-1} \alpha_k x_k(t_j)$. Now by the definition of the norm $|| \cdot ||$ in $C[0, 1]$, $||y - (\alpha_1 x_1 + \cdots + \alpha_n x_n)|| \to 0$ implies that

$$\lim_{n \to \infty} |y(t_j) - (\alpha_1 x_1 + \cdots + \alpha_n x_n)(t_j)| = 0,$$

i.e. $\lim_{n \to \infty} \left| y(t_j) - \left(\alpha_j + \sum_{k=1}^{j-1} \alpha_k x_k(t_j) \right) \right| = 0$. Since the expression here does not depend on n, this implies $\alpha_j = y(t_j) - \sum_{k=1}^{j-1} \alpha_k x_k(t_j)$. This is true for each $j = 1, 2, \cdots$, i.e.:

$$(*) \qquad \begin{cases} \alpha_1 = y(t_1); \\ \alpha_2 = y(t_2) - \alpha_1 x_1(t_2); \\ \alpha_3 = y(t_3) - \alpha_1 x_1(t_3) - \alpha_2 x_2(t_3); \\ \cdots \\ \alpha_j = y(t_j) - \alpha_1 x_1(t_j) - \alpha_2 x_2(t_j) - \cdots - \alpha_{j-1} x_{j-1}(t_j); \\ \cdots \end{cases}$$

This proves that for any $y \in C[0, 1]$ there is *at most one* choice of scalars $\alpha_1, \alpha_2, \cdots \in \mathbb{R}$ such that $\lim_{n \to \infty} ||y - (\alpha_1 x_1 + \cdots + \alpha_n x_n)|| = 0$.

We now prove that the choice of scalars in (*) above really works, i.e. that if we choose $\alpha_1, \alpha_2, \cdots \in \mathbb{R}$ as in (*) then we indeed have $\lim_{n \to \infty} ||y - (\alpha_1 x_1 + \cdots + \alpha_n x_n)|| = 0$.

Fix any $n \geqq 3$, and let $S_n = \alpha_1 x_1 + \cdots + \alpha_n x_n \in C[0,1]$. Then $S_n(t_1) = \alpha_1 \cdot 1 + 0 + \cdots + 0 = \alpha_1 = y(t_1)$ and $S_n(t_2) = \alpha_1 x_1(t_2) + \alpha_2 + 0 + \cdots + 0 = y(t_2)$, and for all $3 \leqq k \leqq n$:

$$S_n(t_k) = \sum_{j=1}^{k-1} \alpha_j x_j(t_k) + \alpha_k \cdot 1 + 0 + \cdots + 0$$
$$= \sum_{j=1}^{k-1} \alpha_j x_j(t_k) + \Big(y(t_k) - \alpha_1 x_1(t_k) - \alpha_2 x_2(t_k) - \cdots - \alpha_{k-1} x_{k-1}(t_k) \Big) = y(t_k).$$

In conclusion, we have $S_n(t_k) = y(t_k)$ for all $k \in \{1, 2, \cdots, n\}$. Furthermore, since each function $x_1, x_2, \cdots, x_n$ is linear at all points $t \notin \{t_1, \cdots, t_n\}$, so is the function $S_n(t)$. Hence: $S_n(t)$ is in fact *the* piecewise linear function which satisfies $S_n(t_k) = y(t_k)$ for all $k \in \{1, 2, \cdots, n\}$ and which is linear at all points $t \notin \{t_1, \cdots, t_n\}$.

From this, we can now prove that $\lim_{n\to\infty} ||y - S_n|| = 0$: Since $y(t)$ is continuous and $[0,1]$ is compact, $y(t)$ is actually uniformly continuous over $[0,1]$. Hence, given $\varepsilon > 0$ there is some integer $M \in \mathbb{Z}^+$ such that for all $t, t' \in [0,1]$ with $|t - t'| < M^{-1}$ we have $|y(t) - y(t')| < \varepsilon$. Since the set $\{t_1, t_2, \cdots\}$ is dense in $[0,1]$ there is some number $N \in \mathbb{Z}^+$ such that each of the intervals

$$[0, \tfrac{1}{3M}],\ [\tfrac{1}{3M}, \tfrac{2}{3M}], [\tfrac{2}{3M}, \tfrac{3}{3M}], \cdots, [\tfrac{3M-1}{3M}, 1]$$

contains some point in $\{t_1, t_2, ..., t_N\}$.

Let n be any number $n \geqq N$. Then for any $t \in [0,1]$, if we let t_j be the point in $\{t_1, t_2, ..., t_n\}$ which lies closest *below* t, and let t_k be the point in $\{t_1, t_2, ..., t_n\}$ which lies closest *above* t, we have $t_j \leqq t \leqq t_k$ and $|t_k - t_j| < M^{-1}$. [Proof: t belongs to some interval $[\frac{a}{3M}, \frac{a+1}{3M}]$, $a \in \{0, 1, \cdots, 3M-1\}$ and we know that both $[\frac{a+1}{3M}, \frac{a+2}{3M}]$ and $[\frac{a-1}{3M}, \frac{a}{3M}]$ contain some points from $\{t_1, \cdots, t_n\}$ (exceptional cases: If $a = 0$, use $t_1 = 0$. If $a = 3M - 1$, use $t_2 = 1$.); hence we certainly have $\frac{a-1}{3M} \leqq t_j$ and $t_k \leqq \frac{a+1}{3M}$; thus $0 \leqq t_k - t_j \leqq \frac{2}{3M} < M^{-1}$.]

It follows that $|y(t) - y(t_j)| < \varepsilon$ and $|y(t) - y(t_k)| < \varepsilon$. Now $S_n(t_j) = y(t_j)$ and $S_n(t_k) = y(t_k)$, and the function S_n is linear in the interval $[t_j, t_k]$, since by construction there are no other points from $\{t_1, t_2, ..., t_n\}$ in $[t_j, t_k]$. Thus:

$$S_n(t) = \frac{t_k - t}{t_k - t_j} S_n(t_j) + \frac{t - t_j}{t_k - t_j} S_n(t_k).$$

Hence:

$$\begin{aligned}|S_n(t) - y(t)| &= \left|\frac{t_k - t}{t_k - t_j}(S_n(t_j) - y(t)) + \frac{t - t_j}{t_k - t_j}(S_n(t_k) - y(t))\right| \\ &= \left|\frac{t_k - t}{t_k - t_j}(y(t_j) - y(t)) + \frac{t - t_j}{t_k - t_j}(y(t_k) - y(t))\right| \\ &\leqq \frac{t_k - t}{t_k - t_j} \cdot \varepsilon + \frac{t - t_j}{t_k - t_j} \cdot \varepsilon = \varepsilon.\end{aligned}$$

The above argument works for any $t \in [0, 1]$. Hence $|S_n(t) - y(t)| \leqq \varepsilon$ for any $t \in [0, 1]$. Hence $||S_n - y|| \leqq \varepsilon$. This is true for any $n \geqq N$.

We have proved that for any $\varepsilon > 0$ there is some $N \in \mathbb{Z}^+$ such that $||S_n - y|| \leqq \varepsilon$ for all $n \geqq N$. Hence $\lim_{n\to\infty} ||y - S_n|| = 0$. In other words: $\lim_{n\to\infty} ||y - (\alpha_1 x_1 + \cdots + \alpha_n x_n)|| = 0$.

Hence we have proved that for every $y \in C[0, 1]$ there is a unique choice of scalars $\alpha_1, \alpha_2, \cdots \in \mathbb{R}$ such that

$$\lim_{n\to\infty} ||y - (\alpha_1 x_1 + \cdots + \alpha_n x_n)|| = 0.$$

This proves that $x_1, x_2, x_3, \cdots$ is a Schauder basis for $C[0, 1]$.