# Functional Analysis (2006)

## Homework assignment 3

All students should solve the following problems:

**1.** (§4.8: Problem 4.) Show that if the sequence $(x_n)$ in a normed space $X$ is weakly convergent to $x_0 \in X$, then $\liminf_{n\to\infty} ||x_n|| \geqq ||x_0||$. (Hint: You may find Theorem 4.3-3 useful.)

**2.** Let $T_1, T_2, T_3, \cdots$ be the following bounded linear operators $\ell^1 \to \ell^\infty$:

$$T_1((\xi_1, \xi_2, \xi_3, \cdots)) = (\xi_1, \xi_1, \xi_1, \xi_1, \xi_1, \cdots);$$
$$T_2((\xi_1, \xi_2, \xi_3, \cdots)) = (\xi_1, \xi_2, \xi_2, \xi_2, \xi_2, \cdots);$$
$$T_3((\xi_1, \xi_2, \xi_3, \cdots)) = (\xi_1, \xi_2, \xi_3, \xi_3, \xi_3, \cdots);$$
*etc.*

Prove that the sequence $(T_n)$ is strongly operator convergent. Also prove that $(T_n)$ is not uniformly operator convergent.

**3.** Define $T : \ell^1 \to \ell^1$ by

$$T((\xi_1, \xi_2, \xi_3, ...)) = (\xi_2, \xi_3, ...).$$

Determine the four sets $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$.

**4.** Let $T : H \to H$ be a compact and self-adjoint operator on a Hilbert space $H$. We say that $T$ is *positive* if $\langle Tx, x\rangle \geqq 0$ holds for every $x \in H$. Prove that $T$ is positive if and only if each eigenvalue $\lambda$ of $T$ satisfies $\lambda \geqq 0$.

Students taking Functional Analysis as a 6 point course should also solve the following problems:

**5.** Let $S$ and $T$ be bounded self-adjoint operators on a Hilbert space $H$. Let $\mathcal{E} = (E_\lambda)$ be the spectral family associated with $T$ and let $\mathcal{F} = (F_\mu)$ be the spectral family associated with $S$. Prove that $TS = ST$ holds if and only if $E_\lambda F_\mu = F_\mu E_\lambda$ for all $\lambda, \mu \in \mathbb{R}$.

**6.** Let $T$ be a (possibly unbounded) symmetric operator on a Hilbert space $H$. Prove that the following three statements are equivalent:
(a) $T$ is self-adjoint.
(b) $T$ is closed and $\mathcal{N}(T^* - i) = \mathcal{N}(T^* + i) = \{0\}$.
(c) $\mathcal{R}(T - i) = \mathcal{R}(T + i) = H$.

[*Hints.* (Partial score is given for proof of any fact mentioned in the following hints.) To prove (b)⇒(c) one may first prove that $\mathcal{N}(T^* - i) = \{0\}$ implies $\mathcal{R}(T - i)^\perp = \{0\}$ so that $\mathcal{R}(T - i)$ is dense in $H$. To prove $\mathcal{R}(T - i) = H$ it then suffices to prove that $\mathcal{R}(T - i)$ is closed; to do this it may be useful to note $||(T + i)w||^2 = ||Tw||^2 + ||w||^2$, $\forall w \in \mathcal{D}(T)$ (proof?), and use the fact ((b)) that $T$ is closed.
To prove (c)⇒(a) one may first prove that $\mathcal{R}(T - i) = H$ implies $\mathcal{N}(T^* + i) = \{0\}$. One then uses this (plus other observations!) to prove $\mathcal{D}(T^*) \subset \mathcal{D}(T)$, which implies (a).]

**Solutions to problems 1-4 should be handed in by Monday, March 13. Solutions to problems 5-6 should be handed in by Tuesday, April 18.** (Either give the solutions to me directly or put them in my mailbox, third floor, House 3, Polacksbacken.)

**Functional Analysis**
**Solutions to homework assignment 3**

**1.** If $x_0 = 0$ then $||x_0|| = 0$ and the statement is obviously true. Now assume $x_0 \neq 0$. Then by Theorem 4.3-3 there is some $f \in X'$ such that $||f|| = 1$ and $f(x_0) = ||x_0||$. Since $(x_n)$ is weakly convergent to $x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0) = ||x_0||$. But $f(x_n) \leqq |f(x_n)| \leqq ||f|| \cdot ||x_n|| = ||x_n||$. Hence $\liminf_{n\to\infty} ||x_n|| \geqq \lim_{n\to\infty} f(x_n) = ||x_0||$, Q.E.D.

**2.** Let $T : \ell^1 \to \ell^\infty$ be the bounded linear operator given by

$$T((\xi_1, \xi_2, \xi_3, \cdots)) = (\xi_1, \xi_2, \xi_3, \cdots).$$

($T$ is obviously linear, and it is bounded with norm $||T|| \leqq 1$, for if $(\xi_j) \in \ell^1$ then $\sum_{k=1}^\infty |\xi_k| = ||(\xi_j)||_{\ell^1} < \infty$ and hence $|\xi_k| \leqq ||(\xi_j)||_{\ell^1}$ for all $k$; hence $||(\xi_j)||_{\ell^\infty} \leqq ||(\xi_j)||_{\ell^1}$.) We claim that $(T_n)$ is strongly operator convergent to $T$.

Let $x = (\xi_j)$ be an arbitrary vector in $\ell^1$. Then

$$||T_n x - Tx||_{\ell^\infty} = \Big\|(0, 0, \cdots, 0, \xi_n - \xi_{n+1}, \xi_n - \xi_{n+2}, \xi_n - \xi_{n+3}, \cdots)\Big\|_{\ell^\infty} = \sup_{j \geqq 1} |\xi_n - \xi_{n+j}|.$$

But since $x = (\xi_j) \in \ell^1$ the sum $\sum_{k=1}^\infty |\xi_k|$ is convergent. In particular the individual terms tend to 0, i.e. $\lim_{k\to\infty} |\xi_k| = 0$. Hence, given any $\varepsilon > 0$ there is some $K \in \mathbb{Z}^+$ such that $|\xi_k| \leqq \varepsilon$ for all $k \geqq K$. Then if $n \geqq K$ we have for all $j \geqq 1$: $n + j \geqq K$, hence $|\xi_n| \leqq \varepsilon$ and $|\xi_{n+j}| \leqq \varepsilon$, and thus

$$|\xi_n - \xi_{n+j}| \leqq |\xi_n| + |\xi_{n+j}| \leqq \varepsilon + \varepsilon = 2\varepsilon.$$

This is true for all $j \geqq 1$. Hence

$$||T_n x - Tx||_{\ell^\infty} = \sup_{j \geqq 1} |\xi_n - \xi_{n+j}| \leqq 2\varepsilon.$$

This is true for all $n \geqq K$. Also, we have shown above that such a $K$ can be found for any given $\varepsilon > 0$. Hence:

$$\lim_{n\to\infty} ||T_n x - Tx||_{\ell^\infty} = 0. \qquad \text{(That is, } T_n x \to Tx \text{ in } \ell^\infty.\text{)}$$

This is true for every vector $x \in \ell^1$. Hence $(T_n)$ *is strongly operator convergent to* $T$.

It follows from this that if $(T_n)$ would be *uniformly* operator convergent, then the limit must be equal to $T$! Hence to prove that $(T_n)$ is *not* uniformly operator convergent, it suffices to prove that $(T_n)$ is not uniformly operator convergent to $T$. Now note that $T_n - T$ is the following operator:

$$(T_n - T)((\xi_1, \xi_2, \xi_3, \cdots)) = (0, 0, \cdots, 0, \xi_n - \xi_{n+1}, \xi_n - \xi_{n+2}, \xi_n - \xi_{n+3}, \cdots).$$

In particular if $x = (0, 0, \cdots, 0, 1, 0, 0, 0, \cdots)$ (with the "1" in the $n$th position) then

$$(T_n - T)(x) = (0, 0, \cdots, 0, 1, 1, 1, \cdots).$$

Here $||x||_{\ell^1} = 1$ and $||(0, 0, \cdots, 0, 1, 1, 1, \cdots)||_{\ell^\infty} = 1$. Hence $||T_n - T|| \geqq 1$. This is true for all $n$, and hence we do *not* have $\lim_{n\to\infty} ||T_n - T|| = 0$. Hence $(T_n)$ is not uniformly operator convergent to $T$, and hence, by our remarks above, $(T_n)$ *is not uniformly operator convergent.*

**3.** Solution: Note that $||T|| = 1$, and hence by Theorem 7.3-4 the spectrum $\sigma(T)$ is contained in the disk given by $|\lambda| \leqq 1$. In other words we now know that every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ belongs to the resolvent set, $\lambda \in \rho(T)$. Hence it only remains to analyze arbitary $\lambda \in \mathbb{C}$ with $|\lambda| \leqq 1$.

It is easy to determine the point spectrum $\sigma_p(T)$: Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $T$; then $Tx = \lambda x$ for some vector $x = (\xi_j) \in \ell^1$, thus

$$(\xi_2, \xi_3, \xi_4, \cdots) = \lambda(\xi_1, \xi_2, \xi_3, \cdots).$$

This implies $\xi_2 = \lambda\xi_1$, $\xi_3 = \lambda\xi_2 = \lambda^2\xi_1$, etc, thus $\xi_n = \lambda^{n-1}\xi_1$ for all $n = 2, 3, \cdots$, i.e.

$$x = \xi_1(1, \lambda, \lambda^2, \lambda^3, \cdots).$$

But if $|\lambda| \geqq 1$ then $(1, \lambda, \lambda^2, \lambda^3, \cdots) \notin \ell^1$, hence the above equation forces $\xi_1 = 0$ and $x = 0$. Hence no $\lambda$ with $|\lambda| \geqq 1$ belongs to $\sigma_p(T)$. On the other hand, if $|\lambda| < 1$ then $(1, \lambda, \lambda^2, \lambda^3, \cdots) \notin \ell^1$, since $\sum_{j=1}^{\infty} |\lambda^j| < \infty$, and this vector $(1, \lambda, \lambda^2, \lambda^3, \cdots)$ is an eigenvector of $T$ with eigenvalue $\lambda$. Hence

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \,:\, |\lambda| < 1\}.$$

It now remains to analyze the case $|\lambda| = 1$. Let us fix an arbitrary $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. We then know that $T_\lambda^{-1}$ exists, since $\lambda \notin \sigma_p(T)$. Note that

$$T_\lambda((\xi_n)) = (\xi_2 - \lambda\xi_1, \xi_3 - \lambda\xi_2, \xi_4 - \lambda\xi_3, ...).$$

Hence if $(\eta_n) = T_\lambda((\xi_n))$ for some $(\xi_n) \in \ell^1$, $(\eta_n) \in \ell^1$ then

$$(*) \qquad \begin{cases} \xi_2 = \eta_1 + \lambda\xi_1 \\ \xi_3 = \eta_2 + \lambda\eta_1 + \lambda^2\xi_1 \\ \xi_4 = \eta_3 + \lambda\eta_2 + \lambda^2\eta_1 + \lambda^3\xi_1 \\ \ldots \\ \xi_n = \sum_{j=1}^{n-1} \lambda^{n-1-j}\eta_j + \lambda^{n-1}\xi_1 \\ \ldots \end{cases}$$

Here it follows from $(\xi_n) \in \ell^1$ that $\lim_{n\to\infty} \xi_n = 0$, hence also $\lim_{n\to\infty} \lambda^{1-n}\xi_n = 0$ (since $|\lambda| = 1$). Using the above equations this gives:

$$\lim_{n\to\infty} \Big( \sum_{j=1}^{n-1} \lambda^{-j}\eta_j + \xi_1 \Big) = 0,$$

i.e.

$$\xi_1 = -\lim_{n\to\infty} \sum_{j=1}^{n-1} \lambda^{-j}\eta_j = -\sum_{j=1}^{\infty} \lambda^{-j}\eta_j.$$

(Note that this sum is absolutely convergent, since $\sum_{j=1}^{\infty} |\lambda^{-j}\eta_j| = \sum_{j=1}^{\infty} |\eta_j| < \infty$.) Using this together with (*) we now see, for each $n \geqq 2$:

$$\xi_n = \sum_{j=1}^{n-1} \lambda^{n-1-j}\eta_j + \lambda^{n-1} \left( -\sum_{j=1}^{\infty} \lambda^{-j}\eta_j \right) = \sum_{j=1}^{n-1} \lambda^{n-1-j}\eta_j - \sum_{j=1}^{\infty} \lambda^{n-1-j}\eta_j$$
$$= -\sum_{j=n}^{\infty} \lambda^{n-1-j}\eta_j.$$

(Clearly this is also true for $n = 1$.) Conversely, let us note that if $(\xi_n) \in \ell^1$, $(\eta_n) \in \ell^1$ and $\xi_n = -\sum_{j=n}^{\infty} \lambda^{n-1-j}\eta_j$ holds for all $n \geqq 1$, then $(\eta_n) = T_\lambda((\xi_n))$. [Proof: Under the stated assumptions, the $n$th entry in $T_\lambda((\xi_n))$ is: $\xi_{n+1} - \lambda\xi_n = -\sum_{j=n+1}^{\infty} \lambda^{n-j}\eta_j + \lambda \sum_{j=n}^{\infty} \lambda^{n-1-j}\eta_j = -\sum_{j=n+1}^{\infty} \lambda^{n-j}\eta_j + \sum_{j=n}^{\infty} \lambda^{n-j}\eta_j = \eta_n$.]

Let $M$ be the set of those $(\eta_n) \in \ell^1$ which have only finitely many nonzero entries. We then have $M \subset \mathcal{D}(T_\lambda^{-1})$. Proof: If $(\eta_n) \in M$ then there is some $N \in \mathbb{Z}^+$ such that $\eta_n = 0$ for all $n \geqq N$, and then if we define $\xi_n$ by $\xi_n = -\sum_{j=n}^{\infty} \lambda^{n-1-j}\eta_j$, we get $\xi_n = 0$ for all $n \geqq N$. Hence $(\xi_n) \in \ell^1$, and by assumption $(\eta_n) \in M \subset \ell^1$; hence $T_\lambda((\xi_n)) = (\eta_n)$ and $(\eta_n) \in \mathcal{D}(T_\lambda^{-1})$. This is true for all $(\eta_n) \in M$, hence we have proved the claim, $M \subset \mathcal{D}(T_\lambda^{-1})$.

But $M$ is dense in $\ell^1$! (Proof: Given $x = (\xi_j) \in \ell^1$ we may define the sequence $v_1, v_2, v_3, \cdots \in M$ by $v_n = (\xi_1, \xi_2, \cdots, \xi_n, 0, 0, 0 \cdots)$. Then $||v_n - x|| = \sum_{j=n+1}^{\infty} |\xi_j| \to 0$ as $n \to \infty$. Hence $x$ is a limit point of $M$. This is true for all $x \in \ell^1$. Hence $\overline{M} = \ell^1$, as claimed.) From the facts $M \subset \mathcal{D}(T_\lambda^{-1})$ and $M$ dense in $\ell^1$ it follows that $\mathcal{D}(T_\lambda^{-1})$ is dense in $\ell^1$.

We can now complete the solution using only general principles: We have seen above that the spectrum $\sigma(T)$ contains the set $\sigma_p(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, i.e. the open unit disk. But we know from Theorem 7.3-4 that $\sigma(T)$ is *compact*; hence $\sigma(T)$ must also contain every boundary point of the unit disk, i.e. every $\lambda$ with $|\lambda| = 1$ belongs to $\sigma(T)$. We have also seen that for these $\lambda$ we have $\lambda \notin \sigma_p(T)$ and $\lambda \notin \sigma_r(T)$ (since $T_\lambda^{-1}$ exists and $\mathcal{D}(T_\lambda^{-1})$ is dense in $\ell^1$). Hence the only remaining possibility is $\lambda \in \sigma_c(T)$.

**Answer:** $\rho(T) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$, $\sigma_p(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, $\sigma_c(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\sigma_r(T) = \emptyset$.

**Alternative proof that every $\lambda$ with $|\lambda| = 1$ belongs to $\sigma_c(T)$.** Fix an arbitrary $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. We have proved that $T_\lambda^{-1}$ exists and that $\mathcal{D}(T_\lambda^{-1})$ is dense in $\ell^1$. Hence it remains to prove that $T_\lambda^{-1}$ is unbounded. To do this we let $v_n = (n\lambda^0, (n-1)\lambda^1, (n-2)\lambda^2, \cdots, 1 \cdot \lambda^{n-1}, 0, 0, 0, \cdots) \in \ell^1$, for each $n \geqq 1$. We then compute $w_n = T_\lambda(v_n) = (-\lambda^1, -\lambda^2, -\lambda^3, \cdots, -\lambda^n, 0, 0, 0, \cdots)$. Here $||v_n|| = \sum_{j=1}^n |n+1-j||\lambda^{j-1}| = \sum_{j=1}^n (n+1-j) = \sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}$, and $||w_n|| = \sum_{j=1}^n |-\lambda^j| = \sum_{j=1}^n 1 = n$. But $v_n = T_\lambda^{-1}(w_n)$; hence if $T_\lambda^{-1}$ were bounded then we would have $||v_n|| \leqq ||T_\lambda^{-1}|| \cdot ||w_n||$, i.e. $\frac{n(n-1)}{2} \leqq ||T_\lambda^{-1}|| \cdot n$, i.e. $||T_\lambda^{-1}|| \geqq \frac{n-1}{2}$. This would be true for every $n \in \mathbb{Z}^+$. This is a contradiction, since no real number can be larger than $\frac{n-1}{2}$ for all $n \in \mathbb{Z}^+$. Hence $T_\lambda^{-1}$ is unbounded, Q.E.D.

**4.** Since $T$ is self-adjoint we know that each eigenvalue $\lambda$ is real. Suppose there is some negative eigenvalue; $Tv = \lambda v$ with $\lambda < 0$, $v \neq 0$. Then $\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda ||v||^2 < 0$, i.e. $T$ is not positive.

Conversely, assume that all eigenvalues of $T$ are $\geqq 0$. Let $\{\lambda_n\}_{n=1}^N$ be the nonzero eigenvalues of $T$, and $\{e_n\}_{n=1}^N$ the corresponding orthonormal sequence of eigenvectors, as in Theorem 1 in the text about compact self-adjoint operators. (Here $N \in \mathbb{Z}^+$ or $N = \infty$.) Then our assumption says that $\lambda_n > 0$ for all $n$. We will prove that $T$ is positive. Let $x \in H$ be an arbitrary vector. We may then write $x = \left( \sum_{n=1}^N \alpha_n e_n \right) + z$ for some $\alpha_n \in \mathbb{C}$ with $\sum_{n=1}^N |\alpha_n|^2 < \infty$ and

$z \in \mathcal{N}(T)$. Now

$$\langle Tx, x\rangle = \langle \sum_{n=1}^{N} \lambda_n \alpha_n e_n, \sum_{n=1}^{N} \alpha_n e_n + z\rangle = \sum_{n=1}^{N} \lambda_n |\alpha_n|^2 \geqq 0,$$

since all $\lambda_n \geqq 0$. Hence we have proved $\langle Tx, x\rangle \geqq 0$ for all $x \in H$. Hence $T$ is positive.

**Alternative proof using Theorems from §9.2 in the book.** (Mainly of interest for students taking the 6p course.)

We will use the following fact, which we prove below: If $T$ is an arbitrary compact self-adjoint operator $T$ with spectral decomposition as in Theorem 1 in the text about compact self-adjoint operators, then we have

$$(*) \qquad \sigma(T) = \overline{E} \quad \text{where} \quad E = \begin{cases} \{\lambda_n\}_{n=1}^{N} & \text{if } \mathcal{N}(T) = \{0\} \\ \{0\} \cup \{\lambda_n\}_{n=1}^{N} & \text{if } \mathcal{N}(T) \neq \{0\}. \end{cases}$$

(The splitting in two cases is rather natural, for note that $\mathcal{N}(T) \neq \{0\}$ holds if and only if 0 is an eigenvalue of $T$.)

Using (*), we may now solve the given problem using Theorem 9.2-1 and Theorem 9.2-3 in the book. First, if $T$ is positive then $m \geqq 0$ in Theorem 9.2-1, and hence that theorem implies that $\sigma(T) \subset [0, \infty)$, and in particular $\sigma_p(T) \subset [0, \infty)$, i.e. each eigenvalue of $T$ is $\geqq 0$. Conversely, if $T$ is not positive then $m < 0$ in Theorem 9.2-1, and Theorem 9.2-3 says that $m \in \sigma(T)$. Then by (*) there exists some $\lambda_n$ which is $< 0$, i.e. $T$ has a negative eigenvalue.

Finally, we give a proof of (*) (see Theorem 8.4-4 in the book for an alternative proof in a more general case): Note that each $\lambda \in E$ is an eigenvalue of $T$, i.e. $E \subset \sigma_p(T)$. Hence $E \subset \sigma(T)$, and since $\sigma(T)$ is a closed set (Theorem 7.3-4) it follows that $\overline{E} \subset \sigma(T)$.

Conversely, let $\mu$ be any complex number which does not belong to $\overline{E}$. Then there is some $r > 0$ such that $|\mu - \lambda| > r$ for all $\lambda \in E$. Recall from the text about compact self-adjoint operators that every vector $v \in H$ can be uniquely written $v = \left(\sum_{n=1}^{N} \alpha_n e_n\right) + z$, where $\alpha_n \in \mathbb{C}$, $\sum_{n=1}^{N} |\alpha_n|^2 < \infty$ and $z \in \mathcal{N}(T)$. We now define a linear operator $A : H \to H$ by defining [we here assume $\mathcal{N}(T) \neq \{0\}$, and thus in particular $|\mu| > r$]:

$$A(v) = A\left(\left(\sum_{n=1}^{N} \alpha_n e_n\right) + z\right) := \left(\sum_{n=1}^{N} (\lambda_n - \mu)^{-1} \alpha_n e_n\right) - \mu^{-1} z.$$

This is in fact a bounded linear operator on $H$, since

$$||A(v)||^2 = \left( \sum_{n=1}^{N} |\lambda_n - \mu|^{-2} |\alpha_n|^2 \right) + |\mu|^{-2} ||z||^2$$
$$\leqq r^{-2} \left( \left( \sum_{n=1}^{N} |\alpha_n|^2 \right) + ||z||^2 \right) = r^{-2} ||v||^2.$$

(Thus $||A|| \leqq r^{-2}$.) We compute that for all $v \in H$ represented as above we have

$$AT_\mu(v) = A \left( \left( \sum_{n=1}^{N} (\lambda_n - \mu) \alpha_n e_n \right) - \mu z \right) = \left( \sum_{n=1}^{N} \alpha_n e_n \right) + z = v$$

and

$$T_\mu A(v) = T_\mu \left( \left( \sum_{n=1}^{N} (\lambda_n - \mu)^{-1} \alpha_n e_n \right) - \mu^{-1} z \right) = \left( \sum_{n=1}^{N} \alpha_n e_n \right) + z = v.$$

Hence $AT_\mu = T_\mu A = I$, the identity operator, and thus $A = T_\mu^{-1}$. Hence $T_\mu^{-1}$ exists and is defined on all $H$, and is bounded. This proves $\mu \in \rho(T)$, i.e. $\mu \notin \sigma(T)$. [In the case $\mathcal{N}(T) = \{0\}$ the proof is simply modified by removing the $z$-term in all sums above.] This is true for all $\mu \notin \overline{E}$. Hence $\sigma(T) \subset \overline{E}$, and in total we have proved $\sigma(T) = \overline{E}$, i.e. (*)!

**5.** First assume $TS = ST$. Then by Lemma 9.8-2 (carrying over Lemma 9.8-1(b)) we have $E_\lambda S = S E_\lambda$ for all $\lambda \in \mathbb{R}$. Fixing any $\lambda \in \mathbb{R}$ and again applying Lemma 9.8-2 (carrying over Lemma 9.8-1(b)), but this time applied to the operator $S$ in place of $T$, we obtain $E_\lambda F_\mu = F_\mu E_\lambda$ for all $\mu \in \mathbb{R}$.

Conversely, assume that $E_\lambda F_\mu = F_\mu E_\lambda$ holds for all $\lambda, \mu \in \mathbb{R}$. We know that $T = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$ and $S = \int_{-\infty}^{\infty} \mu \, dF_\mu$, and in fact there is a number $A > 0$ such that $T = \int_{-A}^{A} \lambda \, dE_\lambda$ and $S = \int_{-A}^{A} \mu \, dF_\mu$ (for example we may take $A = 1 + \max(||T||, ||S||)$). This means that $T$ is obtained as the uniform limit of a sequence of operator sums of the form (for a given partition $-A = t_0 < t_1 < \cdots < t_n = A$)

$$T' = \sum_{j=1}^{n} t_j (E_{t_j} - E_{t_{j-1}}),$$

and similarly for $S$. Hence, given $\varepsilon > 0$ there exists a partition $-A = t_0 < t_1 < \cdots < t_n = A$ such that if

$$T' = \sum_{j=1}^{n} t_j (E_{t_j} - E_{t_{j-1}})$$

$$\text{and } \ S' = \sum_{j=1}^{n} t_j (F_{t_j} - F_{t_{j-1}}),$$

then

$$||T' - T|| < \varepsilon \quad \text{and} \quad ||S' - S|| < \varepsilon.$$

The point now is that it follows from our assumption $E_\lambda F_\mu = F_\mu E_\lambda$, $\forall \lambda, \mu \in \mathbb{R}$, that $T'S' = S'T'$! Proof:

$$\begin{aligned} T'S' &= \left( \sum_{j=1}^{n} t_j (E_{t_j} - E_{t_{j-1}}) \right) \left( \sum_{k=1}^{n} t_k (F_{t_k} - F_{t_{k-1}}) \right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} t_j (E_{t_j} - E_{t_{j-1}}) t_k (F_{t_k} - F_{t_{k-1}}) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} t_k (F_{t_k} - F_{t_{k-1}}) t_j (E_{t_j} - E_{t_{j-1}}) \\ &= \left( \sum_{k=1}^{n} t_k (F_{t_k} - F_{t_{k-1}}) \right) \left( \sum_{j=1}^{n} t_j (E_{t_j} - E_{t_{j-1}}) \right) = S'T'. \end{aligned}$$

We also have

$$\begin{aligned} &||TS - T'S'|| \leqq ||T(S - S')|| + ||(T - T')S'|| \\ &\leqq ||T|| \cdot ||S - S'|| + ||T - T'|| \cdot ||S'|| \\ &< ||T||\varepsilon + ||S'||\varepsilon \leqq ||T||\varepsilon + (||S|| + ||S' - S||)\varepsilon \\ &< ||T||\varepsilon + ||S||\varepsilon + \varepsilon^2, \end{aligned}$$

and similarly

$$||S'T' - ST|| < ||T||\varepsilon + ||S||\varepsilon + \varepsilon^2.$$

Hence

$$\begin{aligned} ||TS - ST|| &= ||TS - T'S' + S'T' - ST|| \\ &\leqq ||TS - T'S'|| + ||S'T' - ST|| \\ &< 2\big(||T||\varepsilon + ||S||\varepsilon + \varepsilon^2\big). \end{aligned}$$

But here $\varepsilon > 0$ is arbitrary, and by taking $\varepsilon$ small we can make the right hand side arbitrarily small. Hence $||TS - ST|| = 0$, i.e. $TS = ST$, Q.E.D.

**6.**
**(a) $\Longrightarrow$ (b).** Assume (a), i.e. that $T$ is self-adjoint. Then $T^* = T$ and $T$ is closed (Theorem 10.3-3). Assume $v \in \mathcal{N}(T^* - i)$. Then $v \in \mathcal{D}(T^*) = \mathcal{D}(T)$ and $(T^* - i)v = 0$, hence $(T - i)v = 0$, hence $Tv = iv$, and this implies:

$$i\langle v, v\rangle = \langle iv, v\rangle = \langle Tv, v\rangle = \langle v, T^*v\rangle = \langle v, Tv\rangle = \langle v, iv\rangle = -i\langle v, v\rangle.$$

Hence $\langle v, v\rangle = 0$, i.e. $v = 0$. This prove that $\mathcal{N}(T^* - i) = \{0\}$. Similarly one proves $\mathcal{N}(T^* + i) = \{0\}$. Hence (b) holds.

**(b) $\Longrightarrow$ (c).** Assume (b), i.e. that $T$ is closed and $\mathcal{N}(T^* - i) = \mathcal{N}(T^* + i) = \{0\}$. We first prove $\mathcal{R}(T + i)^{\perp} = \{0\}$: Let $v$ be an arbitrary vector in $\mathcal{R}(T+i)^{\perp}$. Then $\langle w, v\rangle = 0$ for all $w \in \mathcal{R}(T+i)$, i.e. $\langle (T + i)x, v\rangle = 0$ for all $x \in \mathcal{D}(T)$. Hence $\langle Tx, v\rangle = -\langle ix, v\rangle = \langle x, iv\rangle$ for all $x \in \mathcal{D}(T)$. This implies that $v \in \mathcal{D}(T^*)$ and $T^*v = iv$, i.e. $(T^* - i)v = 0$, i.e. $v \in \mathcal{N}(T^* - i)$. By our assumption (b), this implies $v = 0$. Hence we have proved $\mathcal{R}(T + i)^{\perp} = \{0\}$.

We next prove that $\mathcal{R}(T + i)$ is closed in $H$: Let $v_1, v_2, v_3, \cdots$ be an arbitrary sequence of points in $\mathcal{R}(T+i)$ such that $v = \lim_{n\to\infty} v_n$ exists in $H$. We have to prove $v \in \mathcal{R}(T + i)$. By definition there are vectors $w_1, w_2, w_3, \cdots \in \mathcal{D}(T)$ such that $v_n = (T + i)w_n$. Since $T$ is symmetric we have for every $w \in \mathcal{D}(T)$:

$$||(T + i)w||^2 = \langle (T + i)w, (T + i)w\rangle = ||Tw||^2 + i\langle w, Tw\rangle - i\langle Tw, w\rangle + ||iw||^2$$
$$= ||Tw||^2 + i\Big(\langle T^*w, w\rangle - \langle Tw, w\rangle\Big) + ||w||^2 = ||Tw||^2 + ||w||^2.$$

In particular

$$||v_n - v_m||^2 = ||(T + i)(w_n - w_m)||^2 = ||T(w_n - w_m)||^2 + ||w_n - w_m||^2.$$

Now $(v_n)$ is a Cauchy sequence, i.e. $||v_n - v_m|| \to 0$ as $n, m \to \infty$, and the above equality shows $||w_n - w_m||^2 \leqq ||v_n - v_m||^2$; hence also $||w_n - w_m|| \to 0$ as $n, m \to \infty$, i.e. $(w_n)$ is a Cauchy sequence. Since $H$ is complete it follows that $w = \lim_{n\to\infty} w_n \in H$ exists. Similarly, the above equality also shows that $u = \lim_{n\to\infty} Tw_n \in H$ exists. *Since $T$ is closed* (by assumption (b)) it follows that $w \in \mathcal{D}(T)$ and $Tw = u$, and thus

$$(T+i)w = u + iw = \left(\lim_{n\to\infty} Tw_n\right) + \left(\lim_{n\to\infty} iw_n\right) = \lim_{n\to\infty}(Tw_n + iw_n)$$
$$= \lim_{n\to\infty}(T+i)w_n = \lim_{n\to\infty} v_n = v$$

Hence $v \in \mathcal{R}(T+i)$. This proves that $\mathcal{R}(T+i)$ is closed in $H$.

Since $\mathcal{R}(T+i)$ is both dense and closed in $H$ it follows that $\mathcal{R}(T+i) = H$. Similarly one proves $\mathcal{R}(T-i) = H$. Hence (c) holds.

**(c) $\Longrightarrow$ (a).** Assume (c), i.e. that $\mathcal{R}(T-i) = \mathcal{R}(T+i) = H$. Let us first prove $\mathcal{N}(T^*+i) = \{0\}$: Assume $v \in \mathcal{N}(T^*+i)$, i.e. $v \in \mathcal{D}(T^*)$ and $(T^*+i)v = 0$. Then for all $w \in \mathcal{D}(T)$ we have

$$0 = \langle (T^*+i)v, w\rangle = \langle T^*v, w\rangle + i\langle v, w\rangle = \langle v, Tw\rangle - \langle v, iw\rangle = \langle v, (T-i)w\rangle.$$

But *every* vector in $H$ can be expressed as $(T-i)w$, since $\mathcal{R}(T-i) = H$ (assumption (c)). Hence $v$ is orthogonal to all $H$, and hence $v = 0$. This completes the proof that $\mathcal{N}(T^*+i) = \{0\}$.

Now let $v$ be an arbitrary vector in $\mathcal{D}(T^*)$. Since $\mathcal{R}(T+i) = H$ there exists a vector $w \in \mathcal{D}(T)$ such that $(T+i)w = (T^*+i)v$. Now since $T$ is symmetric we have $w \in \mathcal{D}(T^*)$ and $T^*w = Tw$. Hence also $v - w \in \mathcal{D}(T^*)$, and

$$(T^*+i)(v-w) = (T^*+i)v - (T^*+i)w = 0.$$

Hence $v-w \in \mathcal{N}(T^*+i)$. But we have seen above that $\mathcal{N}(T^*+i) = \{0\}$. Hence $v - w = 0$, i.e. $v = w \in \mathcal{D}(T)$. But $v$ was an arbitrary vector in $\mathcal{D}(T^*)$; hence we have proved $\mathcal{D}(T^*) \subset \mathcal{D}(T)$. On the other hand we have $T \subset T^*$ since $T$ is symmetric. Hence we actually have $T = T^*$, i.e. $T$ is self-adjoint. Hence (a) holds.