End of lecture 5 April 2006

These are the things I had planned to say today but didn't get time to. [I include some more proofs here than I had planned to do in my lecture.] Please don't hesitate to email me if you have any questions on this material!

The following two facts are the last things I wrote on the board; they contain the definition 10.3-4 (p.537) in the book, and give some extra information.

Fact 1. Given $T : \mathcal{D}(T) \to H$, there exists a *closed* linear extension of T if and only if $\overline{\mathcal{G}(T)}$ is the graph of an operator (i.e., if and only if $\forall x \in H : \sharp\{y \in H \mid (x,y) \in \overline{\mathcal{G}(T)}\} \leqq 1$).

Fact/def 2. If $T : \mathcal{D}(T) \to H$ has some closed linear extension, then there exists a unique *minimal*[1] closed linear extension of T; this operator is called $\overline{T} : \mathcal{D}(\overline{T}) \to H$, the *closuse of* T. Furthermore in this situation we have $\mathcal{G}(\overline{T}) = \overline{\mathcal{G}(T)}$.

Proof of fact 1. Assume that T has a closed linear extension T_1. Thus $T \subset T_1$ and T_1 is closed. It follows that $\mathcal{G}(T) \subset \mathcal{G}(T_1)$ and that $\mathcal{G}(T_1)$ is closed. Hence $\overline{\mathcal{G}(T)} \subset \mathcal{G}(T_1)$. But by definition we have $\mathcal{G}(T_1) = \{(x, T_1x) \mid x \in \mathcal{D}(T_1)\}$, and hence from $\overline{\mathcal{G}(T)} \subset \mathcal{G}(T_1)$ it follows that

$$\overline{\mathcal{G}(T)} = \{(x, T_1x) \mid x \in M\}$$

for some subset $M \subset \mathcal{D}(T_1)$. Note that $\mathcal{G}(T)$ is a linear subspace of $H \times H$; thus also $\overline{\mathcal{G}(T)}$ is a linear subspace of $H \times H$ (by exercise 6 on p. 70). Hence M must be a linear subspace of $\mathcal{D}(T_1)$, and $(T_1)_{|M}$ is a linear operator with graph $\mathcal{G}((T_1)_{|M}) = \{(x, T_1x) \mid x \in M\} = \overline{\mathcal{G}(T)}$. Hence $\overline{\mathcal{G}(T)}$ is the graph of an operator!

Conversely, suppose that $\overline{\mathcal{G}(T)}$ is the graph of an operator, i.e. $\overline{\mathcal{G}(T)} = \mathcal{G}(T_2)$ for some operator $T_2 : \mathcal{D}(T_2) \to H$. Then T_2 is closed, since $\mathcal{G}(T_2) = \overline{\mathcal{G}(T)}$ is closed by definition. [Note that T_2 is automatically a *linear* operator, since $\mathcal{G}(T_2)$ is a linear subset of $H \times H$.] Also $T \subset T_2$, since $\mathcal{G}(T) \subset \overline{\mathcal{G}(T)} = \mathcal{G}(T_2)$. Hence T_2 is a closed linear extension of T, i.e. T has a closed linear extension.

□

Proof of fact 2. Assume that T has some closed linear extension. Then by our fact 1 above, $\overline{\mathcal{G}(T)}$ is the graph of an operator T_2 (we

[1]The precise meaning of "$\overline{T}$ is a minimal closed linear extension of T" is the following: $\overline{T}$ is a closed linear extension of T, *and* for *every* closed linear extension T_1 of T we have $\overline{T} \subset T_1$.

use the same name as above), and as in the last paragraph of the above proof of fact 1, we see that T_2 is actually linear and closed, and $T \subset T_2$. We claim that T_2 is a *minimal* closed linear extension of T, i.e. that T_2 is the closure of T! Indeed, assume that T_1 is *any* closed linear extension of T. Then $\mathcal{G}(T) \subset \mathcal{G}(T_1)$, and since $\mathcal{G}(T_1)$ is closed it follows that $\overline{\mathcal{G}(T)} \subset \mathcal{G}(T_1)$, i.e. $\mathcal{G}(T_2) \subset \mathcal{G}(T_1)$. This implies $T_2 \subset T_1$, and the minimality of T_2 is proved!

To prove the *uniqueness* of the closure, let us assume that T_3 is *also* a minimal closed linear extension of T. Then since $T_2 \subset T_3$ (by the minimality of T_2) and $T_3 \subset T_2$ (by the minimality of T_3), and this clearly implies $T_3 = T_2$.

Finally, note that the last claim in our fact/def 2, $\mathcal{G}(\overline{T}) = \overline{\mathcal{G}(T)}$, is already contained in our construction, for we constructed $\overline{T}$ as the operator $\overline{T} = T_2$ with graph $\mathcal{G}(T_2) = \overline{\mathcal{G}(T)}$.

□

Theorem 10.3-5. Assume that $T : \mathcal{D}(T) \to H$ is symmetric (and thus densely defined). Then the closure $\overline{T}$ exists.

The proof of this theorem in the book is very detailed, and well worth studying! We here give an alternative, much shorter proof, using our Fact 1 and Fact 2 from above!

[Actually, our argument is the same thing as on p.538(a) in the book, but using a language involving the graph $\mathcal{G}(T)$ much more explicitly.]

Proof of Theorem 10.3-5. By our Fact 2 it suffices to prove that T has *some* closed linear extension, and by Fact 1 this will follow if we can show that

$$(*) \qquad \forall x \in H : \sharp\{y \in H \mid (x,y) \in \overline{\mathcal{G}(T)}\} \leqq 1.$$

To prove this, let us assume that we have $(x,y) \in \overline{\mathcal{G}(T)}$ *and* $(x,\tilde{y}) \in \overline{\mathcal{G}(T)}$ for some $x, y, \tilde{y} \in H$. We then wish to prove $y = \tilde{y}$.

Since $(x,y) \in \overline{\mathcal{G}(T)}$ there is a sequence $(x_1,y_1),(x_2,y_2),(x_3,y_3),\ldots$ of vectors in $\mathcal{G}(T)$ with $(x_n,y_n) \to (x,y)$ in $H \times H$. Note that $(x_n,y_n) \in \mathcal{G}(T)$ implies $x_n \in \mathcal{D}(T)$, $y_n = Tx_n$, and $(x_n,y_n) \to (x,y)$ implies (using the definition of the norm in $H \times H$) that $x_n \to x$ (in H) and $Tx_n = y_n \to y$ (in H) as $n \to \infty$. Similarly, since $(x,\tilde{y}) \in \overline{\mathcal{G}(T)}$ there is a sequence $(\tilde{x}_1,\tilde{y}_1),(\tilde{x}_2,\tilde{y}_2),(\tilde{x}_3,\tilde{y}_3),\ldots$ of vectors in $\mathcal{G}(T)$ with $(\tilde{x}_n,\tilde{y}_n) \to (x,\tilde{y})$ in $H \times H$, and this implies $\tilde{x}_n \in \mathcal{D}(T)$, $\tilde{y}_n = T\tilde{x}_n$, $\tilde{x}_n \to x$ (in H) and $T\tilde{x}_n = \tilde{y}_n \to \tilde{y}$ (in H) as $n \to \infty$.

Now for every $v \in \mathcal{D}(T)$ we have

$$\langle v, y - \tilde{y} \rangle = \langle v, \lim_{n\to\infty}(y_n - \tilde{y}_n)\rangle = \lim_{n\to\infty}\langle v, y_n - \tilde{y}_n\rangle = \lim_{n\to\infty}\langle v, T(x_n - \tilde{x}_n)\rangle =$$

$\Big[$Use that T *is symmetric* and $v \in \mathcal{D}(T)\Big]$

$$= \lim_{n\to\infty}\langle Tv, x_n - \tilde{x}_n\rangle = \langle Tv, \lim_{n\to\infty}(x_n - \tilde{x}_n)\rangle = \langle Tv, x - x\rangle = 0.$$

Hence $y - \tilde{y} \in \mathcal{D}(T)^{\perp} = \overline{\mathcal{D}(T)}^{\perp} = H^{\perp} = \{0\}$, i.e. $y - \tilde{y} = 0$, Q.E.D.

Fact 3. If $T : \mathcal{D}(T) \to H$ is a densely defined operator which has a closed linear extension (so that $\overline{T}$ exists), then $(\overline{T})^* = T^*$. [2]

Remark: This Fact 3 is a stronger statement than Theorem 10.3-6 in the book, which says that if T is a symmetric operator, then $(\overline{T})^* = T^*$. (This follows from Fact 3 for if T is symmetric then $\overline{T}$ exists by Theorem 10.3-5 above.)

Proof of Fact 3. Since $T \subset \overline{T}$ we have $(\overline{T})^* \subset T^*$, by Theorem 10.2-1.

Conversely, take any $x \in \mathcal{D}(T^*)$; we wish to prove that $x \in \mathcal{D}((\overline{T})^*)$ and $(\overline{T})^*x = T^*x$. Note that $x \in \mathcal{D}(T^*)$ implies, by the definition of $\mathcal{D}(T^*)$, that

$$\forall v \in \mathcal{D}(T) : \langle Tv, x\rangle = \langle v, T^*x\rangle.$$

Now take an arbitrary $w \in \mathcal{D}(\overline{T})$. Then $(w, \overline{T}w) \in \mathcal{G}(\overline{T}) = \overline{\mathcal{G}(T)}$ and thus there is a sequence $(w_1, u_1), (w_2, u_2), (w_3, u_3), \ldots$ in $\mathcal{G}(T)$ with $(w_n, u_n) \to (w, \overline{T}w)$ in $H \times H$. Hence $w_n \in \mathcal{D}(T)$, $u_n = Tw_n$, $w_n \to w$ in H and $Tw_n = u_n \to \overline{T}w$ in H. Hence

$$\langle \overline{T}w, x\rangle = \langle \lim_{n\to\infty} Tw_n, x\rangle = \lim_{n\to\infty}\langle Tw_n, x\rangle = \qquad \Big[\text{use } x \in \mathcal{D}(T^*)\Big]$$

$$= \lim_{n\to\infty}\langle w_n, T^*x\rangle = \langle \lim_{n\to\infty} w_n, T^*x\rangle = \langle w, T^*x\rangle.$$

We have thus proved that

$$\langle \overline{T}w, x\rangle = \langle w, T^*x\rangle$$

holds for every $w \in \mathcal{D}(\overline{T})$, and this means that $x \in \mathcal{D}((\overline{T})^*)$ and $(\overline{T})^*x = T^*x$. Since this holds for every $x \in \mathcal{D}(T^*)$ we have proved that $T^* \subset (\overline{T})^*$. Since we have also noted $(\overline{T})^* \subset T^*$, it follows that $(\overline{T})^* = T^*$, Q.E.D.

□

[2] Note that $(\overline{T})^*$ certainly exists, for $\overline{T}$ is densely defined since T is densely defined.

As I said in class, a very important and often difficult problem is to prove that a given symmetric operator T is in fact self-adjoint. The reason is that it is only for self-adjoint operators that we have access to really good theorems about spectral decomposition (cf. Chapter 9 and also Theorem 10.6-3).

More generally, given a symmetric operator T, one often wants to prove that T *has some self-adjoint extension.* One of the most important lessons which we learn from our results above (i.e., the results of §10.3 in the book) is that when we study this question, we can always start by replacing T with the closed operator $\overline{T}$, for we have the following:

Fact 4. *If T is a symmetric operator then $\overline{T}$ is also symmetric, and T and $\overline{T}$ have exactly the same self-adjoint extensions!*

Proof. Note that $\overline{T}$ exists by Theorem 10.3-5, and $(\overline{T})^* = T^*$ by Fact 3 (or Theorem 10.3-6). Since T is symmetric we have $T \subset T^*$, i.e. $T \subset (\overline{T})^*$. But $(\overline{T})^*$ is closed by Theorem 10.3-3, hence $\overline{T} \subset (\overline{T})^*$. This means that $\overline{T}$ is symmetric!

Next, to see that T and $\overline{T}$ have exactly the same self-adjoint extensions, suppose that T_1 is a self-adjoint extension of T. Then T_1 is closed (by Theorem 10.3-3), hence $\overline{T} \subset T_1$, i.e. T_1 is also an extension of $\overline{T}$.

□