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Matematiska Institutionen
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Prov i matematik
Funktionalanalys
Kurs: NVP 6p-delen
2006-04-21

Skrivtid: 9.00–11.30
Tillåtna hjälpmedel: Manuella skrivdon, Kreyszigs bok *Introductory Functional Analysis with Applications* och Strömbergssons häften *Spectral theorem for compact, self-adjoint operators* and *Mathematical statements and proofs*.

1. Let $T : \mathcal{D}(T) \to H$ be a (possibly unbounded) densely defined linear operator in a complex Hilbert space H. Recall the definition of the Hilbert-adjoint operator $T^* : \mathcal{D}(T^*) \to H$ (Definition 10.1-2). Give a careful proof that $\mathcal{D}(T^*)$ is a vector subspace of H and that T^* is a linear operator. (6p)

2. Let $T : \mathcal{D}(T) \to Y$ be a closed linear operator, where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Let $C \subset \mathcal{D}(T)$ be a compact set. Prove that the image $T(C) = \{T(x) \mid x \in C\}$ is a closed subset of Y. (7p)

3. Let $T : \ell^2 \to \ell^2$ be the self-adjoint bounded linear operator

$$T\Big((\xi_1, \xi_2, \xi_3, \ldots)\Big) = (\xi_1, \tfrac{1}{2}\xi_2, \xi_3, \tfrac{1}{4}\xi_4, \xi_5, \tfrac{1}{6}\xi_6, \xi_7, \ldots).$$

What is the spectral family (E_λ) associated with T?
(You get several points for giving the correct formula for E_λ for each $\lambda \in \mathbb{R}$, even if found by an intuitive (non-rigorous) argument. However, for full score, please give a careful verification of the fact $T = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$ for your family (E_λ).) (7p)

GOOD LUCK!

Solutions

1. By Definition 10.1-2 we have

$$D(T^*) = \Big\{ y \in H \mid \exists y^* \in H : \forall x \in D(T) : \langle Tx, y \rangle = \langle x, y^* \rangle \Big\},$$

and for each $y \in D(T^*)$ we define $T^*y := y^*$ where $y^* \in H$ is the vector as above, i.e. the vector which has the property $\forall x \in D(T) : \langle Tx, y \rangle = \langle x, y^* \rangle$ (this vector y^* is *unique* since $D(T)$ is dense in H).

Now let y_1, y_2 be arbitrary vectors in $D(T^*)$ and let α, β be arbitrary complex numbers. Then by the definition of $D(T^*)$ there exist vectors y_1^*, y_2^* such that

$$\forall x \in D(T) : \ \langle Tx, y_1 \rangle = \langle x, y_1^* \rangle \text{ and } \langle Tx, y_2 \rangle = \langle x, y_2^* \rangle.$$

Now note that for all $x \in D(T)$ we have

$$\langle Tx, \alpha y_1 + \beta y_2 \rangle = \overline{\alpha} \langle Tx, y_1 \rangle + \overline{\beta} \langle Tx, y_2 \rangle = \overline{\alpha} \langle x, y_1^* \rangle + \overline{\beta} \langle x, y_2^* \rangle = \langle x, \alpha y_1^* + \beta y_2^* \rangle.$$

This proves that $\alpha y_1 + \beta y_2 \in D(T^*)$, and also that $T^*(\alpha y_1 + \beta y_2) = \alpha y_1^* + \beta y_2^* = \alpha T^*(y_1) + \beta T^*(y_2)$. Since these two properties hold for all $y_1, y_2 \in D(T^*)$ and all $\alpha, \beta \in \mathbb{C}$ it follows that $D(T^*)$ is a vector subspace of H and that T^* is a linear operator.

2. Let $y_1, y_2, y_3, \ldots$ be a sequence of points in $T(C)$ such that $y = \lim_{j \to \infty} y_j$ exists in Y. We must prove that $y \in T(C)$.

For each $j \geqq 1$ there exists some $x_j \in C$ with $y_j = T(x_j)$, since $y_j \in T(C)$. We now assume that such a vector x_j has been chosen for each $j \geqq 1$. Since C is compact and $x_1, x_2, x_3, \ldots \in C$ there exists a subsequence $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ (where $1 \leqq j_1 < j_2 < j_3 < \ldots$) which converges to an element $x \in C$, i.e. $\lim_{n \to \infty} x_{j_n} = x \in C$. Now $\lim_{n \to \infty} T(x_{j_n}) = \lim_{n \to \infty} y_{j_n} = y$ (since $\lim_{j \to \infty} y_j = y$). Hence by Theorem 4.13-3, since T is a closed linear operator, we have $Tx = y$. But $x \in C$, hence $y \in T(C)$.

Hence we have proved that for every sequence $y_1, y_2, y_3, \ldots$ in $T(C)$ which converges to some $y \in Y$, we actually have $y \in T(C)$. Hence $T(C)$ is closed in Y, by Theorem 1.4-6.

3. Recall the intuitive formula $E_\lambda =$[projection on all part of ℓ^2 which have "eigenvalues" $\leqq \lambda$]. Note that the given operator T has the property that all of ℓ^2 is (Hilbert-)spanned by eigenvectors; for note that all the vectors[1] $e_1, e_2, e_3, e_4, \ldots$ are eigenvectors of T, and these vectors span ℓ^2 in the Hilbert sense, i.e. $\ell^2 = \overline{\text{Span}\{e_1, e_2, e_3, \ldots\}}$. Hence it

[1] We use the standard notation $e_j = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$, where the "1" is in position j.

seems reasonable to expect that the above intuitive formula is in fact rigorously true in our case.

This leads to the following guess: For $\lambda \leqq 0$: $E_\lambda = 0$. For $0 < \lambda < 1$: Let k be the smallest positive integer such that $\lambda \geqq \frac{1}{2k}$:

$$E_\lambda\Big((\xi_1, \xi_2, \xi_3, \ldots)\Big) = (0, 0, \ldots, 0, \xi_{2k}, 0, \xi_{2k+2}, 0, \xi_{2k+4}, 0, \ldots)$$

(where the first non-zero entry is in position $2k$). Finally for $\lambda \geqq 1$ we should have $E_\lambda := I$.

We now prove that (E_λ) as specified above satisfies all the desired properties.

One easily checks that (E_λ) is a spectral family on $[0,1]$. Indeed, properties (7) and (8*) on p. 495 are directly clear from our definition of (E_λ). It thus remains to check (9) on p. 495. Note that by our definition of (E_λ) we have that for each $\lambda \neq 0 \in \mathbb{R}$ there exists some $\varepsilon > 0$ such that $E_\mu = E_\lambda$ for all $\mu \in [\lambda, \lambda + \varepsilon]$, and this property immediately implies $\lim_{\mu \to \lambda + 0} E_\mu x = E_\lambda x$ for all $x \in H$. Hence it only remains to verify that (9) on p. 495 holds when $\lambda = 0$, i.e. that $\lim_{\mu \to 0+0} E_\mu x = E_\lambda x$ holds for all $x \in H$. By our definition of (E_λ) this is the same as proving:

$$\lim_{k \to \infty} (0, 0, \ldots, 0, \xi_{2k}, 0, \xi_{2k+2}, 0, \ldots) = 0, \qquad \forall x = (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^2.$$

This is clear from the fact that for all $x = (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^2$ we have

$$\Big\|(0, 0, \ldots, 0, \xi_{2k}, 0, \xi_{2k+2}, 0, \ldots)\Big\|^2 = \sum_{j=k}^{\infty} |\xi_{2j}|^2 \to 0, \qquad \text{as } k \to \infty.$$

Hence we have proved completely that (E_λ) is a spectral family on $[0,1]$.

We now prove $T = \int_{0-0}^{1} t\, dE_t$. Given $n \in \mathbb{Z}^+$, let $P_n : 0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$ be any partition of $[0,1]$ which has $t_1 = \frac{1}{2n}$ and for which all the points $\frac{1}{2n}, \frac{1}{2n-2}, \frac{1}{2n-4}, \ldots, \frac{1}{4}, \frac{1}{2}$ occur among the t_j's, and with more t_j-points inserted in a way such that $\eta(P_n) \leqq \frac{1}{2n}$. Then the Riemann-Stieltjes sum for the integral $\int_{0-0}^{1} t\, dE_t$ corresponding to P_n is:

$$s(P_n) = \sum_{j=1}^{m} t_j \cdot \big(E_{t_j} - E_{t_{j-1}}\big)$$

(Since we have lower limit of integration "$0 - 0$" we should in fact add a term $t_0 \cdot E_{t_0}$ to this sum, but note that this term is anyway 0.) The contribution from $j = 1$ in the above sum is $t_1 \cdot (E_{t_1} - E_{t_0}) = \frac{1}{2n}(E_{\frac{1}{2n}} - 0)$.

Note that this operator acts as follows on ℓ^2:

$$\tfrac{1}{2n}(E_{\frac{1}{2n}} - 0)\Big((\xi_1, \xi_2, \xi_3, \ldots)\Big) = (0, 0, \ldots, 0, \tfrac{1}{2n}\xi_{2n}, 0, \tfrac{1}{2n}\xi_{2n+2}, 0, \tfrac{1}{2n}\xi_{2n+4}, 0, \ldots).$$

Furthermore, if j in the above sum is such that $t_j = \frac{1}{2k}$ for some $k = 1, 2, \ldots, n-1$ then by construction we have $\frac{1}{2k+2} \leqq t_{j-1} < \frac{1}{2k}$, and thus $E_{t_j} - E_{t_{j-1}}$ is projection onto the $2k$:th coordinate, and the contribution from j to our sum is $\frac{1}{2k}(E_{t_j} - E_{t_{j-1}})$, which acts as follows:

$$\tfrac{1}{2k}(E_{t_j} - E_{t_{j-1}})\Big((\xi_1, \xi_2, \xi_3, \ldots)\Big) = (0, 0, \ldots, 0, \tfrac{1}{2k}\xi_{2k}, 0, 0, 0, \ldots).$$

(where the non-zero entry is in position $2k$). Next, for $j = m$ we have $t_m = 1$ and $t_{m-1} \geqq \frac{1}{2}$, hence the contribution from this j is $1 \cdot (E_1 - E_{\frac{1}{2}})$. Note that this operator acts as follows:

$$(E_1 - E_{\frac{1}{2}})\Big((\xi_1, \xi_2, \xi_3, \ldots)\Big) = (\xi_1, 0, \xi_3, 0, \xi_5, 0, \ldots).$$

Finally, for every *other* j in the sum we see from our construction that $E_{t_j} = E_{t_{j-1}}$, hence the contribution for this j is 0. Hence:

$$s(P_n)\Big((\xi_1, \xi_2, \xi_3, \ldots)\Big)$$
$$= (\xi_1, \tfrac{1}{2}\xi_2, \xi_3, \tfrac{1}{4}\xi_4, \xi_5, \ldots, \tfrac{1}{2n}\xi_{2n}, \xi_{2n+1}, \tfrac{1}{2n}\xi_{2n+2}, \xi_{2n+3}, \tfrac{1}{2n}\xi_{2n+4}, \xi_{2n+5}, \ldots).$$

Hence we easily see $||s(P_n) - T|| \leqq \frac{1}{2n} - \frac{1}{2n+2} \to 0$ as $n \to \infty$. Hence $T = \int_{0-0}^{1} t\, dE_t$.

We have now proved that (E_λ) is a spectral family on $[0, 1]$ and that $T = \int_{0-0}^{1} t\, dE_t$. Hence by the uniqueness part of Theorem 9.9-1 (which I told about in class) we have that (E_λ) is *the* spectral family associated with T.

Answer: For $\lambda \leqq 0$: $E_\lambda = 0$. For $0 < \lambda < 1$: Let k be the smallest positive integer such that $\lambda \geqq \frac{1}{2k}$:

$$E_\lambda\Big((\xi_1, \xi_2, \xi_3, \ldots)\Big) = (0, 0, \ldots, 0, \xi_{2k}, 0, \xi_{2k+2}, 0, \xi_{2k+4}, 0, \ldots)$$

(where the first non-zero entry is in position $2k$). For $\lambda \geqq 1$: $E_\lambda := I$.