Matematik MN2, analysdelen, HT 05.
Bonusövning 3 (partiella derivator, relaterat material)

• 1. Partiella derivator (4 x 2p = 8p)
Beräkna alla partiella derivator av ordning två av följande funktioner:

(a) $f(x, y) = x^2(x^4 + y^4)$ (b) $g(r, \theta) = r \cdot \cos(\theta + \pi/4)$

(c) $f(x, y, z) = \sqrt{1 + x} + \sqrt{1 + y} + \sqrt{1 + z}$ (d) $f(x, y) = \cos\Big(x + \sqrt{x + y}\Big)$

• 2. Kedjeregeln (3p + 3p = 6p)
(a) Antag att funktionen $f(x, y)$ har kontinuerliga partiella derivator av första ordningen. Beräkna $\frac{\partial}{\partial a} f\Big(f(a, b), af(b^3, a^2)\Big)$.

(b) Transformera den partiella differentialekvationen

$$\frac{x}{(x^2 + y^2)^2} \cdot \frac{\partial}{\partial x} g(x, y) + \frac{y}{(x^2 + y^2)^2} \cdot \frac{\partial}{\partial y} g(x, y) = 1$$

till polära koordinater.

• 3. Använding av definitionerna (4p)
Beräkna de partiella derivatorna av första ordningen till funktionen $f(x, y)$ i punkten $(0, 0)$, då

$$f(x, y) = \begin{cases} \dfrac{3x^3 - 2y^3}{x - y^2} & \text{om } (x, y) \neq (0, 0) \\ 0 & \text{om } (x, y) = (0, 0). \end{cases}$$

Är funktionen $f(x, y)$ kontinuerlig i punkten $(0, 0)$?

• 4. Gradient och förändringshastighet (3 x 2p=6p)
(a) Beräkna gradienten av $f(x, y) = (y + \sqrt{x + y})^2$ och ange riktningsderivatan av $f(x, y)$ i $(x, y) = (1, 0)$ i riktningen $(1, 1)$.

(b) Beräkna gradienten av $g(x, y) = (3x - y)e^{x^2+y^2}$ och ange riktningsderivatan av $g(x, y)$ i $(x, y) = (0, 0)$ i riktningen $(-2, 1)$.

(c) Beräkna gradienten av $h(x, y, z) = \sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}$ och ange riktningsderivatan av $h(x, y, z)$ i $(x, y, z) = (e, e, e)$ i riktningen $(1, 0, 0)$.

För godkänt: Minst 14p (av totalt 24p).

Inlämnas senast: 12 oktober, i mitt postfack före midnatt. (Mitt postfack är på tredje våningen i hus 3 på Polacksbacken.)

Matematik MN2, analysdelen, HT 05.

Lösningar till bonusövning nr 3

• 1.(a). Observera $f(x,y) = x^6 + x^2y^4$.

$$\frac{\partial}{\partial x} f(x,y) = 6x^5 + 2xy^4; \qquad \frac{\partial}{\partial y} f(x,y) = 4x^2y^3;$$

Alltså, **Svar:**

$$\frac{\partial^2}{\partial x^2} f(x,y) = 30x^4 + 2y^4; \qquad \frac{\partial^2}{\partial x \partial y} f(x,y) = 8xy^3; \qquad \frac{\partial^2}{\partial y^2} f(x,y) = 12x^2y^2.$$

1.(b).

$$\frac{\partial}{\partial r} g(r,\theta) = \cos(\theta + \pi/4); \qquad \frac{\partial}{\partial \theta} g(r,\theta) = -r \cdot \sin(\theta + \pi/4).$$

Alltså, **svar:**

$$\begin{aligned} \frac{\partial^2}{\partial r^2} g(r,\theta) &= 0, \\ \frac{\partial^2}{\partial \theta \partial r} g(r,\theta) &= -\sin(\theta + \pi/4), \\ \frac{\partial^2}{\partial \theta^2} g(r,\theta) &= -r\cos(\theta + \pi/4), \end{aligned}$$

1.(c).

$$\frac{\partial}{\partial x} f(x,y,z) = \frac{1}{2\sqrt{1+x}}; \qquad \frac{\partial}{\partial y} f(x,y,z) = \frac{1}{2\sqrt{1+y}}; \qquad \frac{\partial}{\partial z} f(x,y,z) = \frac{1}{2\sqrt{1+z}}.$$

Alltså, **svar:**

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2} f(x,y,z) = -\frac{1}{4}(1+x)^{-\frac{3}{2}};\\
&\frac{\partial^2}{\partial x \partial y} f(x,y,z) = 0;\\
&\frac{\partial^2}{\partial x \partial z} f(x,y,z) = 0;\\
&\frac{\partial^2}{\partial y^2} f(x,y,z) = -\frac{1}{4}(1+y)^{-\frac{3}{2}};\\
&\frac{\partial^2}{\partial y \partial z} f(x,y,z) = 0;\\
&\frac{\partial^2}{\partial z^2} f(x,y,z) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}.
\end{aligned}$$

1.(d).

$$\begin{aligned}
&\frac{\partial}{\partial x} f(x,y) = -\sin(x+\sqrt{x+y}) \cdot \left(1 + \frac{1}{2\sqrt{x+y}}\right);\\
&\frac{\partial}{\partial y} f(x,y) = -\sin(x+\sqrt{x+y}) \cdot \frac{1}{2\sqrt{x+y}}.
\end{aligned}$$

Alltså, **svar:**

$$\frac{\partial^2}{\partial x^2} f(x,y) = -\cos(x+\sqrt{x+y}) \cdot \left(1 + \frac{1}{2\sqrt{x+y}}\right)^2 + \sin(x+\sqrt{x+y}) \cdot \frac{1}{4} \cdot (x+y)^{-\frac{3}{2}};$$

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y} f(x,y) = &-\cos(x+\sqrt{x+y}) \cdot \left(1 + \frac{1}{2\sqrt{x+y}}\right) \cdot \frac{1}{2\sqrt{x+y}}\\
&+ \sin(x+\sqrt{x+y}) \cdot \frac{1}{4} \cdot (x+y)^{-\frac{3}{2}};
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} f(x,y) &= -\cos(x+\sqrt{x+y}) \cdot \left(\frac{1}{2\sqrt{x+y}}\right)^2 + \sin(x+\sqrt{x+y}) \cdot \frac{1}{4} \cdot (x+y)^{-\frac{3}{2}}\\
&= -\cos(x+\sqrt{x+y}) \cdot \frac{1}{4(x+y)} + \sin(x+\sqrt{x+y}) \cdot \frac{1}{4} \cdot (x+y)^{-\frac{3}{2}}.
\end{aligned}$$

• 2.(a)

$$\frac{\partial}{\partial a} f\Big(f(a,b), af(b^3,a^2)\Big) = f_1\big(f(a,b), af(b^3,a^2)\big)\cdot\frac{\partial}{\partial a}\Big(f(a,b)\Big)$$
$$+ f_2\big(f(a,b), af(b^3,a^2)\big)\cdot\frac{\partial}{\partial a}\Big(af(b^3,a^2)\Big)$$
$$= f_1\big(f(a,b), af(b^3,a^2)\big)\cdot f_1(a,b) + f_2\big(f(a,b), af(b^3,a^2)\big)\cdot\Big(f(b^3,a^2) + a\cdot 2af_2(b^3,a^2)\Big).$$

2.(b). Byte till polära koordinater: $(x,y) \longleftrightarrow (r,\varphi)$; $x = r\cos\varphi$, $y = r\sin\varphi$.

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} = (\cos\varphi)\cdot\frac{\partial}{\partial x} + (\sin\varphi)\cdot\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \varphi} = -r(\sin\varphi)\cdot\frac{\partial}{\partial x} + r(\cos\varphi)\cdot\frac{\partial}{\partial y}$$

$r = \sqrt{x^2+y^2}$, $\varphi = \arctan(y/x)$. Alltså har vi

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{1}{1+(y/x)^2}\cdot\frac{-y}{x^2}\cdot\frac{\partial}{\partial\varphi}$$
$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} - \frac{y}{x^2+y^2}\cdot\frac{\partial}{\partial\varphi};$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}\cdot\frac{\partial}{\partial\varphi}$$
$$= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{x}{x^2+y^2}\cdot\frac{\partial}{\partial\varphi}.$$

Alltså:

$$\frac{x}{(x^2+y^2)^2}\cdot\frac{\partial}{\partial x}g(x,y) + \frac{y}{(x^2+y^2)^2}\cdot\frac{\partial}{\partial y}g(x,y)$$
$$= \frac{x}{(x^2+y^2)^2}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} - \frac{y}{x^2+y^2}\cdot\frac{\partial}{\partial\varphi}\right)g$$
$$+ \frac{y}{(x^2+y^2)^2}\cdot\left(\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{x}{x^2+y^2}\cdot\frac{\partial}{\partial\varphi}\right)g$$
$$= \frac{x^2+y^2}{(x^2+y^2)^{5/2}}\frac{\partial}{\partial r}g = \frac{1}{r^3}\frac{\partial}{\partial r}g$$

Svar: $\frac{1}{r^3}\frac{\partial}{\partial r}g = 1$.

• 3. Enligt definitionen av partiell derivata är

$$\frac{\partial}{\partial x}f(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{3h^3/h-0}{h}=\lim_{h\to 0}\frac{3h^2}{h}=0.$$

$$\frac{\partial}{\partial y}f(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{2h^3/h^2-0}{h}=\lim_{h\to 0}\frac{2h}{h}=2.$$

Observera att funktionen f har definitionsmängd $\{(x,y)\mid x\neq y^2\}$. Tag kurvan $x=y^2+y^{10}$, låt $y\to 0$:

$$f(y^2+y^{10},y)=\frac{3(y^2+y^{10})^3-2y^3}{y^{10}}=\frac{3y^6+9y^{14}+9y^{22}+3y^{30}-2y^3}{y^{10}}\to\infty$$

Svar: Ej kontinuerlig.

• 4.(a).

$$\frac{\partial}{\partial x}f(x,y)=2(y+\sqrt{x+y})\frac{1}{2\sqrt{x+y}}=\frac{y+\sqrt{x+y}}{\sqrt{x+y}}$$

$$\frac{\partial}{\partial y}f(x,y)=2(y+\sqrt{x+y})\left(1+\frac{1}{2\sqrt{x+y}}\right).$$

Alltså är gradienten:

$$\nabla f(x,y)=\left(\frac{y+\sqrt{x+y}}{\sqrt{x+y}},2(y+\sqrt{x+y})\left(1+\frac{1}{2\sqrt{x+y}}\right)\right).$$

Då $(x,y)=(1,0)$ är detta:

$$\nabla f(1,0)=(1,3).$$

Enhetsvektor med samma riktning som $(1,1)$:

$$\mathbf{u}=\frac{(1,1)}{|(1,1)|}=\frac{1}{\sqrt{2}}(1,1).$$

Alltså blir den sökta riktningsderivatan:

$$D_{\mathbf{u}}f(1,0)=\frac{1}{\sqrt{2}}(1,1)\cdot(1,3)=2\sqrt{2}.$$

4.(b)

$$\frac{\partial}{\partial x}g(x,y)=3e^{x^2+y^2}+(3x-y)e^{x^2+y^2}\cdot 2x,$$

$$\frac{\partial}{\partial y}g(x,y)=(-1)e^{x^2+y^2}+(3x-y)e^{x^2+y^2}\cdot 2y.$$

Alltså:

$$\nabla g(x,y)=\left(e^{x^2+y^2}(3+6x^2-2xy),\ e^{x^2+y^2}(-1-2y^2+6xy)\right).$$

Då $(x, y) = (0, 0)$ är detta:

$$\nabla g(0,0) = (3, -1).$$

Enhetsvektor med samma riktning som $(-2, 1)$:

$$\mathbf{u} = \frac{(-2,1)}{|(-2,1)|} = \frac{1}{\sqrt{5}}(-2,1).$$

Alltså blir den sökta riktningsderivatan:

$$D_{\mathbf{u}} f(0,0) = \frac{1}{\sqrt{5}}(-2,1)\cdot(3,-1) = -\frac{7}{\sqrt{5}}.$$

4.(c)

$$\frac{\partial}{\partial x} h(x,y,z) = \frac{1}{2\sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}} \cdot 2(\ln x) \cdot \frac{1}{x} = \frac{\ln x}{x\sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}}$$

och analogt:

$$\frac{\partial}{\partial y} h(x,y,z) = \frac{\ln y}{y\sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}}$$

$$\frac{\partial}{\partial z} h(x,y,z) = \frac{\ln z}{z\sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}}.$$

Alltså:

$$\nabla h(x,y,z) = \left(\frac{\ln x}{x\sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}}, \frac{\ln y}{y\sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}}, \frac{\ln z}{z\sqrt{(\ln x)^2 + (\ln y)^2 + (\ln z)^2}} \right).$$

Alltså är gradienten i (e, e, e):

$$\nabla h(e,e,e) = (\frac{1}{e\sqrt{3}}, \frac{1}{e\sqrt{3}}, \frac{1}{e\sqrt{3}}).$$

Observera att $\mathbf{u} = (1, 0, 0)$ är en enhetsvektor. Alltså blir den sökta riktningsderivatan:

$$D_{\mathbf{u}} h(e,e,e) = (1,0,0)\cdot(\frac{1}{e\sqrt{3}}, \frac{1}{e\sqrt{3}}, \frac{1}{e\sqrt{3}}) = \frac{1}{e\sqrt{3}}.$$