Prop 18.1. For any function $f : \mathbb{Z}^+ \to \mathbb{C}$ and any real numbers $N, U, V \geq 2$ with $N \geq UV$, we have (with an absolute implied constant)

$$\Big|\sum_{n\leq N} f(n)\Lambda(n)\Big| \ll \sum_{n\leq U}\big|f(n)\big|\Lambda(n) + (\log UV)\sum_{t\leq UV}\Big|\sum_{r\leq N/t} f(rt)\Big|$$
$$+ (\log N)\sum_{d\leq V}\max_{1\leq w\leq N/d}\Big|\sum_{1\leq h\leq w} f(dh)\Big| + N^{\frac{1}{2}}(\log N)^3 \max_{U\leq M\leq N/V}\Delta(f, M, N, V),$$

where $\Delta(f, M, N, V)$ denotes any non-negative real number satisfying

$$\Big|\sum_{M<m\leq 2M} b_m \sum_{V<k\leq N/m} c_k f(mk)\Big| \leq \Delta(f, M, N, V)\Big(\sum_{M<m\leq 2M}|b_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\leq N/M}|c_k|^2\Big)^{\frac{1}{2}}$$

for all complex numbers b_m, c_k.

Outline of proof: Set $F(s) = \sum_{m \le U} \Lambda(m) m^{-s}$, $G(s) = \sum_{d \le V} \mu(d) d^{-s}$. Note

$$-\frac{\zeta'(s)}{\zeta(s)} = F(s) - \zeta(s)F(s)G(s) - \zeta'(s)G(s) + \left(-\frac{\zeta'(s)}{\zeta(s)} - F(s)\right) \cdot (1 - \zeta(s)G(s)).$$

$$\implies \Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n) \qquad \forall n \in \mathbb{Z}^+,$$

where

$$a_1(n) = \begin{cases} \Lambda(n) & \text{if } n \le U \\ 0 & \text{if } n > U; \end{cases}$$

$$a_2(n) = -\sum_{\substack{mdr=n \\ m \le U \\ d \le V}} \Lambda(m)\mu(d);$$

$$a_3(n) = \sum_{\substack{hd=n \\ d \le V}} \mu(d) \log h;$$

$$a_4(n) = -\sum_{\substack{mk=n \\ m>U \\ k>1}} \Lambda(m) \Big(\sum_{\substack{d|k \\ d \le V}} \mu(d) \Big).$$

Hence

$$\sum_{n\leq N} f(n)\Lambda(n) = S_1 + S_2 + S_3 + S_4,$$

where

$$S_i = \sum_{n\leq N} f(n)a_i(n).$$

Here

$$S_1 = \sum_{n\leq U} f(n)\Lambda(n);$$

thus

$$\left|S_1\right| \leq \sum_{n\leq U}\left|f(n)\right|\Lambda(n).$$

Next

$$S_2 = -\sum_{n\le N} \sum_{\substack{mdr=n \\ m\le U \\ d\le V}} f(n)\Lambda(m)\mu(d)$$

$$\cdots = -\sum_{t\le UV} \Bigg(\sum_{\substack{md=t \\ m\le U \\ d\le V}} \Lambda(m)\mu(d) \Bigg) \sum_{r\le N/t} f(rt).$$

Gives

$$\big|S_2\big| \le (\log UV) \sum_{t\le UV} \Bigg| \sum_{r\le N/t} f(rt) \Bigg|.$$

Next

$$S_3 = \sum_{n \le N} f(n) \sum_{\substack{hd=n \\ d \le V}} \mu(d) \log h$$

$$\cdots = \sum_{d \le V} \mu(d) \sum_{h \le N/d} f(hd) \log h.$$

Integration by parts

$$\Longrightarrow \left|S_3\right| \ll (\log N) \sum_{d \le V} \max_{1 \le w \le N/d} \left| \sum_{1 \le h \le w} f(dh) \right|.$$

Finally,

$$S_4 = -\sum_{n\le N} f(n) \sum_{\substack{mk=n\\ m>U\\ k>1}} \Lambda(m)\Big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\Big)$$

$$\cdots = -\sum_{U<m\le N/V} \Lambda(m) \sum_{V<k\le N/m} \Big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\Big) f(mk).$$

DYADIC DECOMPOSITION in the *m*-variable:

$$= \sum_{\substack{M\in\{2^0U,2^1U,2^2U,\ldots\}\\ M<N/V}} \Big\{ \sum_{M<m\le\min(N/V,2M)} \Lambda(m) \sum_{V<k\le N/m} \Big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\Big) f(mk) \Big\}$$

$$\Longrightarrow$$

$$|S_4| \le \sum_{\substack{M\in\{2^0U,2^1U,2^2U,\ldots\}\\ M<N/V}} \Big\{ \Delta(f, M, N, V) \Big(\sum_{M<m\le 2M} \Lambda(m)^2\Big)^{\frac12} \Big(\sum_{k\le N/M} \big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\big)^2\Big)^{\frac12} \Big\}$$

MISPRINT in eq. (579), p. 249; $M \to 2M$

Finally,

$$S_4 = -\sum_{n\le N} f(n) \sum_{\substack{mk=n\\ m>U\\ k>1}} \Lambda(m)\Big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\Big)$$

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$$\Longrightarrow$$

$$|S_4| \le \sum_{\substack{M\in\{2^0U,2^1U,2^2U,\ldots\}\\ M<N/V}} \left\{ \Delta(f,M,N,V) \underbrace{\Big(\sum_{M<m\le 2M} \Lambda(m)^2\Big)}_{\ll M\log M}^{\frac{1}{2}} \Big(\sum_{k\le N/M} \underbrace{\big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\big)^2}_{\le d(k)^2}\Big)^{\frac{1}{2}} \right\}$$

Finally,

$$S_4 = -\sum_{n\le N} f(n) \sum_{\substack{mk=n\\ m>U\\ k>1}} \Lambda(m)\Big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\Big)$$

$$\cdots = -\sum_{U<m\le N/V} \Lambda(m) \sum_{V<k\le N/m} \Big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\Big) f(mk).$$

DYADIC DECOMPOSITION in the *m*-variable:

$$= \sum_{\substack{M\in\{2^0U,2^1U,2^2U,\dots\}\\ M<N/V}} \Big\{ \sum_{M<m\le \min(N/V,2M)} \Lambda(m) \sum_{V<k\le N/m} \Big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\Big) f(mk)\Big\}$$

$$\Longrightarrow$$

$$|S_4| \le \sum_{\substack{M\in\{2^0U,2^1U,2^2U,\dots\}\\ M<N/V}} \Big\{\Delta(f,M,N,V)\Big(\underbrace{\sum_{M<m\le 2M} \Lambda(m)^2}_{\ll M\log M}\Big)^{\frac{1}{2}} \underbrace{\Big(\sum_{k\le N/M} \underbrace{\big(\sum_{\substack{d|k\\ d\le V}} \mu(d)\big)^2}_{\le d(k)^2}\Big)}_{\ll (N/M)\log^3(N/M)}^{\frac{1}{2}}\Big\}$$

$$\ll N^{\frac{1}{2}}(\log N)^3 \max_{U\le M\le N/V} \Delta(f,M,N,V).$$

Example: $S(\alpha) = \sum_{n \le N} \Lambda(n) e(n\alpha)$

Prop 18.3. If $\alpha \in \mathbb{R}$ and

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2} \qquad (q \in \mathbb{Z}^+,\ a \in \mathbb{Z},\ (a,q) = 1), \qquad (*)$$

then

$$\left|S(\alpha)\right| \ll (Nq^{-\frac{1}{2}} + N^{\frac{4}{5}} + N^{\frac{1}{2}} q^{\frac{1}{2}})(\log N)^4,$$

where the implied constant is absolute.

Remark 18.5:
Prop 18.3 Gives a *power saving* versus the trivial bound $\left|S(\alpha)\right| \ll N$ if we can find a rational approximation $\frac{a}{q}$ to α satisfying $(*)$ and $N^{\varepsilon} \le q \le N^{1-\varepsilon}$.
(Here $\varepsilon > 0$ is some fixed, small constant.)

Example: $S(\alpha) = \sum_{n \le N} \Lambda(n) e(n\alpha)$

To better appreciate Prop 18.3, we recall

Lemma 18.4 = Dirichlet's Theorem on Diophantine approximation.

For every $\alpha \in \mathbb{R}$ and every real $Q \ge 1$, there is a rational number $\frac{a}{q}$ such that

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{qQ}; \qquad 1 \le q \le Q; \qquad (a, q) = 1.$$

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Also, Remark 18.6 * (External reading).
An irrational number $\alpha \in \mathbb{R}$ is said to be of *diophantine type* κ $(\kappa \ge 2)$ if there is $C > 0$ such that

$$\left|\alpha - \frac{a}{q}\right| > \frac{C}{q^\kappa}, \qquad \text{for all } a \in \mathbb{Z},\ q \in \mathbb{Z}^+.$$

For such α, one can prove $\left|S(\alpha)\right| \ll N^\tau$ as $N \to \infty$, where τ is any fixed number with

$$\tau > \max\left(\frac{4}{5}, \frac{2\kappa - 1}{2\kappa}\right).$$

Outline of proof of Prop 18.3:

Assume $N \geq 10$.

By Prop 18.2, for any $U, V \geq 2$ with $UV \leq N$:

$$\left|S(\alpha)\right| = \left|\sum_{n \leq N} \Lambda(n) e(n\alpha)\right| \ll U + (\log N) \sum_{t \leq UV} \max_{w} \left| \sum_{w \leq r \leq N/t} e(rt\alpha)\right|$$

$$+ N^{\frac{1}{2}} (\log N)^3 \max_{U \leq M \leq N/V} \max_{V \leq j \leq N/M} \Bigg(\sum_{V < k \leq N/M} \Bigg| \sum_{\substack{M < m \leq 2M \\ m \leq N/k \\ m \leq N/j}} e(m(j-k)\alpha) \Bigg| \Bigg)^{\frac{1}{2}}.$$

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$$+ N^{\frac{1}{2}}(\log N)^3 \max_{U\leq M\leq N/V} \max_{V\leq j\leq N/M}\Bigg(\sum_{V<k\leq N/M}\Bigg|\sum_{\substack{M<m\leq 2M\\ m\leq N/k\\ m\leq N/j}} e(m(j-k)\alpha)\Bigg|\Bigg)^{\frac{1}{2}}.$$

Note general bound (for any $\beta \in \mathbb{R}$ and any integers $N_1 < N_2$):

$$\left|\sum_{n=N_1}^{N_2} e(n\beta)\right| = \left|\frac{e((N_2+1)\beta) - e(N_1\beta)}{e(\beta) - 1}\right| \ll \min\left(N_2 - N_1, \frac{1}{\|\beta\|}\right),$$

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Hence

$$(\log N) \sum_{t\leq UV} \max_{w} \Big| \sum_{w\leq r\leq N/t} e(rt\alpha)\Big| \ll (\log N) \underbrace{\sum_{t\leq UV} \min\Big(\frac{N}{t}, \frac{1}{\|t\alpha\|}\Big)}_{\textbf{NEED TO BOUND!}}$$

Lemma 18.5. If $\alpha \in \mathbb{R}$ and

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2} \qquad (q \in \mathbb{Z}^+,\ a \in \mathbb{Z},\ (a, q) = 1),$$

then for any $N, T \geq 1$ we have

$$\sum_{t \leq T} \min\left(\frac{N}{t}, \frac{1}{\|t\alpha\|}\right) \ll \left(\frac{N}{q} + T + q\right) \log(2qT).$$

Proof of Lemma 18.5:

Substitute $t = hq + r$:

$$\sum_{t \le T} \min\left(\frac{N}{t}, \frac{1}{\|t\alpha\|}\right) \le \sum_{0 \le h \le T/q} \sum_{r=1}^{q} \min\left(\frac{N}{hq+r}, \frac{1}{\|(hq+r)\alpha\|}\right).$$

Proof of Lemma 18.5:

Set $\beta = \alpha - \frac{a}{q}$; thus $|\beta| \le q^{-2}$!

Substitute $t = hq + r$:

$$\sum_{t \le T} \min\left(\frac{N}{t}, \frac{1}{\|t\alpha\|}\right) \le \sum_{0 \le h \le T/q} \sum_{r=1}^{q} \min\left(\frac{N}{hq+r}, \underbrace{\frac{1}{\|(hq+r)\alpha\|}}\right).$$

$$= \|ra/q + hq\beta + r\beta\|$$

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$$= \|ra/q + hq\beta + r\beta\|$$

Contribution for terms with $h = 0$, $1 \le r \le \frac{1}{2}q$:

$\Rightarrow |r\beta| \le \frac{1}{2q}$

$$\ll \sum_{1 \le r \le q/2} \frac{1}{\left\|\frac{ra}{q}\right\| - \frac{1}{2q}} \le \sum_{\substack{m \in (\mathbb{Z}/q\mathbb{Z}) \\ m \not\equiv 0 \pmod q}} \frac{1}{\left\|\frac{m}{q}\right\| - \frac{1}{2q}} \le 2 \sum_{1 \le m \le q/2} \frac{1}{\frac{m}{q} - \frac{1}{2q}} \ll q \sum_{n=1}^{q-1} \frac{1}{n} \ll q \log(2q).$$

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Substitute $t = hq + r$:

$$\sum_{t \le T} \min\left(\frac{N}{t}, \frac{1}{\|t\alpha\|}\right) \le \sum_{0 \le h \le T/q} \sum_{r=1}^{q} \min\left(\frac{N}{hq + r}, \underbrace{\frac{1}{\|(hq + r)\alpha\|}}\right).$$

$$= \|ra/q + hq\beta + r\beta\|$$

Remaining part:

$$\ll \sum_{0 \le h \le T/q} \sum_{r=1}^{q} \min\left(\frac{N}{(h+1)q}, \frac{1}{\|ra/q + hq\beta + r\beta\|}\right).$$

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Set $\beta = \alpha - \frac{a}{q}$; thus $|\beta| \le q^{-2}$!

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$$\ll \sum_{0 \le h \le T/q} \sum_{r=1}^{q} \min\Big(\frac{N}{(h+1)q}, \frac{1}{\|ra/q + hq\beta + r\beta\|} \Big).$$

For fixed h: $ra/q + hq\beta + r\beta$ "well spread" mod 1 as $r = 1, 2, \ldots, q$!

Hence

$$\ll \sum_{0 \le h \le T/q} \Big(\frac{N}{(h+1)q} + \sum_{j=1}^{\lceil q/2 \rceil} \frac{q}{j} \Big) \ll \frac{N}{q} \log(2T) + \Big(\frac{T}{q} + 1 \Big) q \log(2q).$$

□

Theorem 19.2.
For any fixed $A > 0$ we have

$$r(N) = \frac{1}{2}\mathfrak{S}(N)N^2 + O(N^2(\log N)^{-A}), \qquad (1)$$

for all integers $N \geq 2$, where the implied constant only depends on A, and where

$$\mathfrak{S}(N) = \Big(\prod_{p|N}\Big(1 - \frac{1}{(p-1)^2}\Big)\Big)\Big(\prod_{p\nmid N}\Big(1 + \frac{1}{(p-1)^3}\Big)\Big). \qquad (2)$$

"Easy consequence":

Theorem 19.3.
There exist some positive constants X and c such that every odd integer $N > X$ can be expressed as a sum of three odd primes in $> cN^2(\log N)^{-3}$ ways.

Proof: Let P=the set of prime numbers. Then:

$$\sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) = r(n) - \underbrace{\sum_{\substack{k_1,k_2,k_3 \\ k_1+k_2+k_3=n \\ \{k_1,k_2,k_3\}\not\subset P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)}_{\text{(*) NEED TO BOUND!}} .$$

$$\sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) = r(n) - \underbrace{\sum_{\substack{k_1,k_2,k_3 \\ k_1+k_2+k_3=n \\ \{k_1,k_2,k_3\}\not\subset P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)}_{\text{(*) NEED TO BOUND!}} .$$

Here

$$\text{(*)} \leq 3 \sum_{\substack{k_1,k_2,k_3 \\ k_1+k_2+k_3=n \\ k_1\notin P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3) = \qquad \left\{\text{Subst. } k_1 = p^r\right\}.$$

$$\sum_{\substack{p_1,p_2,p_3\\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) = r(n) - \underbrace{\sum_{\substack{k_1,k_2,k_3\\ k_1+k_2+k_3=n\\ \{k_1,k_2,k_3\}\not\subset P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)}_{\textbf{(*) NEED TO BOUND!}}.$$

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$$\textbf{(*)} \leq 3 \sum_{\substack{k_1,k_2,k_3\\ k_1+k_2+k_3=n\\ k_1\notin P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3) = 3 \sum_{2\leq r\leq \log_2 n} \sum_{p\leq n^{1/r}} (\log p) \sum_{\substack{k_2,k_3\\ k_2+k_3=n-p^r}} \Lambda(k_2)\Lambda(k_3),$$

$$\sum_{\substack{p_1,p_2,p_3\\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) = r(n) - \underbrace{\sum_{\substack{k_1,k_2,k_3\\ k_1+k_2+k_3=n\\ \{k_1,k_2,k_3\}\not\subset P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)}_{\textbf{(*) NEED TO BOUND!}}\,.$$

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$$\le 3n(\log n)^2 \sum_{2\le r\le \log_2 n} \sum_{p\le n^{1/r}} (\log p)$$

$$\sum_{\substack{p_1,p_2,p_3\\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) = r(n) - \underbrace{\sum_{\substack{k_1,k_2,k_3\\ k_1+k_2+k_3=n\\ \{k_1,k_2,k_3\}\not\subset P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)}_{\text{(*) NEED TO BOUND!}}\,.$$

Here

$$\textbf{(*)} \le 3 \sum_{\substack{k_1,k_2,k_3\\ k_1+k_2+k_3=n\\ k_1\notin P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3) = 3 \sum_{2\le r\le \log_2 n} \sum_{p\le n^{1/r}} (\log p) \sum_{\substack{k_2,k_3\\ k_2+k_3=n-p^r}} \Lambda(k_2)\Lambda(k_3),$$

$$\le 3n(\log n)^2 \sum_{2\le r\le \log_2 n} \sum_{p\le n^{1/r}} (\log p)$$

$$\le 3n(\log n)^2 \sum_{2\le r\le \log_2 n} \vartheta(n^{1/r})$$

$$\sum_{\substack{p_1,p_2,p_3\\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) = r(n) - \underbrace{\sum_{\substack{k_1,k_2,k_3\\ k_1+k_2+k_3=n\\ \{k_1,k_2,k_3\}\not\subset P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)}_{\textbf{(*) NEED TO BOUND!}} .$$

Here

$$\begin{aligned}
\textbf{(*)} \le 3 \sum_{\substack{k_1,k_2,k_3\\ k_1+k_2+k_3=n\\ k_1\notin P}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3) &= 3 \sum_{2\le r\le \log_2 n} \sum_{p\le n^{1/r}} (\log p) \sum_{\substack{k_2,k_3\\ k_2+k_3=n-p^r}} \Lambda(k_2)\Lambda(k_3), \\
&\le 3n(\log n)^2 \sum_{2\le r\le \log_2 n} \sum_{p\le n^{1/r}} (\log p) \\
&\le 3n(\log n)^2 \sum_{2\le r\le \log_2 n} \vartheta(n^{1/r}) \\
&\ll n(\log n)^2 \sum_{2\le r\le \log_2 n} n^{1/r} \\
&\le n(\log n)^2 \left(n^{\frac{1}{2}} + (\log_2 n) n^{\frac{1}{3}}\right) \\
&\ll n^{\frac{3}{2}} (\log n)^2.
\end{aligned}$$

$$\Longrightarrow \sum_{\substack{p_1, p_2, p_3 \\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) \gg n^2 \qquad \text{for } n \text{ sufficiently large \& odd.}$$

$$\implies \sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n}} (\log p_1)(\log p_2)(\log p_3) \gg n^2 \qquad \text{for } n \text{ sufficiently large \& odd.}$$

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$$\Longrightarrow \sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n}} (\log n)(\log n)(\log n) \gg n^2.$$

$$\Longrightarrow \sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n}} 1 \gg n^2(\log n)^{-3}.$$

QED! (The number of cases with some of p_1, p_2, p_3 *even* are ≤ 3.)

□

Theorem 19.3.
There exist some positive constants X and c such that every odd integer $N > X$ can be expressed as a sum of three odd primes in $> cN^2(\log N)^{-3}$ ways.

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In fact, Problem 19.2 MISPRINT in problem: "n" $\to$ "N".

If $t(N)$ is the number of ways to write N as a sum of three odd primes, i.e.

$$t(N) = \#\big\{\langle p_1, p_2, p_3\rangle \ : \ p_1 + p_2 + p_3 = N\big\},$$

then

$$t(N) \sim \frac{\mathfrak{S}(N)N^2}{2(\log N)^3} \qquad \text{as } N \to \infty.$$

(Both sides $= 0$ when N even.)

Proof of Theorem 19.2.

Recall

$$r(N) = \int_0^1 S(\alpha)^3 e(-N\alpha)\, d\alpha; \qquad S(\alpha) = \sum_{k \le N} \Lambda(k) e(k\alpha).$$

Split the range of integration, [0, 1], into subintervals!

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$$

Recall

$$r(N) = \int_0^1 S(\alpha)^3 e(-N\alpha)\,d\alpha; \qquad S(\alpha) = \sum_{k \le N} \Lambda(k) e(k\alpha).$$

Split the range of integration, $[0, 1]$, into subintervals!

Keep N large!

For $1 \le q \le P,\ 1 \le a \le q,\ (a, q) = 1:$

$$\mathfrak{M}(q, a) := \left[\frac{a}{q} - \frac{1}{Q}, \frac{a}{q} + \frac{1}{Q}\right].$$

— The *major arcs*.

Disjoint, since

$$\frac{a}{q} \ne \frac{a'}{q'} \Longrightarrow \left|\frac{a}{q} - \frac{a'}{q'}\right| \ge \frac{1}{qq'} \ge \frac{1}{P^2} > \frac{2}{Q}.$$

Complement: The *minor arcs*.

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

For $\alpha \in \mathfrak{M}(q,a)$: $S(\alpha) =$???

Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \le Q^{-1}$;

$$S(\alpha) = \sum_{k \le N} \Lambda(k) e\left(\frac{ka}{q}\right) e(k\beta).$$

$k \mapsto e\left(\frac{ka}{q}\right)$ is periodic modulo q! Express as lin comb of χ's! $(\chi \in X_q)$

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

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$k \mapsto e\left(\frac{ka}{q}\right)$ is periodic modulo q! Express as lin comb of χ's! $(\chi \in X_q)$

Only works for $(k,q) = 1$; get

$$\begin{Bmatrix} e(ka/q) & \text{if } (k,q)=1 \\ 0 & \text{if } (k,q)>1 \end{Bmatrix} = \sum_{m \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{ma}{q}\right) \begin{Bmatrix} 1 & \text{if } m \equiv k \pmod q \\ 0 & \text{if } m \not\equiv k \pmod q \end{Bmatrix}$$

$$= \frac{1}{\phi(q)} \sum_{m \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{ma}{q}\right) \sum_{\chi \in X_q} \overline{\chi(m)}\chi(k) = \frac{1}{\phi(q)} \sum_{\chi \in X_q} \left(\sum_{m \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{ma}{q}\right) \overline{\chi(m)} \right) \chi(k)$$

$$= \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi}) \chi(a) \chi(k)$$

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0),\ 1 \le q \le P$$

For $\alpha \in \mathfrak{M}(q,a)$: $S(\alpha) = ???$

Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \le Q^{-1}$; *HENCE:*

$$\begin{aligned}
S(\alpha) &= \sum_{k \le N} \Lambda(k) e\Big(\frac{ka}{q}\Big) e(k\beta) \\
&= \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi})\chi(a) \sum_{k \le N} \Lambda(k)\chi(k) e(k\beta) + O\Big(\Big| \sum_{\substack{k \le N \\ (k,q)>1}} \Lambda(k) e\Big(\frac{ka}{q}\Big) e(k\beta)\Big|\Big) \\
&= \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi})\chi(a) \int_1^N e(x\beta)\, d\psi(x,\chi) + O\Big(\sum_{p|q} \sum_{r \le \log_p N} \Lambda(p^r)\Big) \\
&= \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi})\chi(a) \Big(e(N\beta)\psi(N,\chi) - 2\pi i \beta \int_1^N e(x\beta)\psi(x,\chi)\, dx \Big) + O\Big(\log^2 N\Big).
\end{aligned}$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

For $\alpha \in \mathfrak{M}(q,a)$: $S(\alpha) =$??? Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \le Q^{-1}$;

$$S(\alpha) = \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi})\chi(a)\Big(e(N\beta)\psi(N,\chi) - 2\pi i\beta \int_1^N e(x\beta)\psi(x,\chi)\,dx\Big) + O\Big(\log^2 N\Big).$$

By Theorem 16.5:

$$\psi(x,\chi) = O\Big(xe^{-c_1\sqrt{\log x}}\Big), \qquad \forall \chi \in X_q \setminus \{\chi_0\},\ \underbrace{x \ge \exp\Big(q^{\frac{1}{2B}}\Big)}_{\Leftrightarrow (\log x)^{2B} \ge q}.$$

$c_1 > 0$ abs const (effective).
Impl const depends only on B but is *noneffective!*

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

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$c_1 > 0$ abs const (effective).
Impl const depends only on B but is *noneffective!*

Combined with easy bounds:

$$\Longrightarrow \psi(x,\chi) = O\Big(N e^{-c_1\sqrt{\log N}}\Big), \qquad \forall \chi \in X_q \setminus \{\chi_0\},\ x \in [1,N].$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

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$$S(\alpha) = \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi})\chi(a) \underbrace{\left(e(N\beta)\psi(N,\chi) - 2\pi i \beta \int_1^N e(x\beta)\psi(x,\chi)\,dx \right)}_{\textbf{(*)}} + O\left(\log^2 N\right).$$

By Theorem 16.5:

$$\psi(x,\chi) = O\left(x e^{-c_1\sqrt{\log x}}\right), \qquad \forall \chi \in X_q \setminus \{\chi_0\},\ \underbrace{x \ge \exp\left(q^{\frac{1}{2B}}\right)}_{\Leftrightarrow (\log x)^{2B} \ge q}.$$

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Combined with easy bounds:

$$\Longrightarrow \psi(x,\chi) = O\left(N e^{-c_1\sqrt{\log N}}\right), \qquad \forall \chi \in X_q \setminus \{\chi_0\},\ x \in [1,N].$$

Hence:

$$\textbf{(*)} = O\left((1 + |\beta|N) N e^{-c_1\sqrt{\log N}}\right), \qquad \forall \chi \in X_q \setminus \{\chi_0\}.$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \leq q \leq P$

For $\alpha \in \mathfrak{M}(q, a)$: $S(\alpha) =$??? Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \leq Q^{-1}$;

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For $\chi = \chi_0 \in X_q$ *principal*, write:

$$\psi(x, \chi_0) = \lfloor x \rfloor + R(x).$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

For $\alpha \in \mathfrak{M}(q,a)$: $S(\alpha) =$??? Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \le Q^{-1}$;

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For $\chi = \chi_0 \in X_q$ *principal*, write:

$$\psi(x,\chi_0) = \lfloor x \rfloor + R(x).$$

For all $x \in [1, N]$:

$$\psi(x,\chi_0) = \psi(x) - \sum_{p|q} \sum_{r \le \log_p x} \Lambda(p^r) = \psi(x) - O\left(\log^2 N\right).$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

For $\alpha \in \mathfrak{M}(q,a)$: $S(\alpha) =???$ Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \le Q^{-1}$;

$$S(\alpha) = \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi})\chi(a) \underbrace{\left(e(N\beta)\psi(N,\chi) - 2\pi i\beta \int_1^N e(x\beta)\psi(x,\chi)\,dx \right)} + O\left(\log^2 N\right).$$

For $\chi = \chi_0 \in X_q$ *principal*, write:

$$\psi(x,\chi_0) = \lfloor x \rfloor + R(x).$$

$$\mathrm{R(x)=O}\left(Ne^{-c_2\sqrt{\log N}}\right), \qquad \forall x \in [1,N]$$

For all $x \in [1,N]$:

$$\psi(x,\chi_0) = \psi(x) - \sum_{p|q} \sum_{r \le \log_p x} \Lambda(p^r) = \psi(x) - O\left(\log^2 N\right).$$

Hence by Theorem 13.8, for all $x \in [1,N]$:

$$R(x) = \left(\psi(x) - \lfloor x \rfloor\right) - O(\log^2 N) = O\left(xe^{-c_2\sqrt{\log x}}\right) + O(\log^2 N) = O\left(Ne^{-c_2\sqrt{\log N}}\right).$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

For $\alpha \in \mathfrak{M}(q,a)$: $S(\alpha) =$??? Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \le Q^{-1}$;

$$S(\alpha) = \frac{1}{\phi(q)} \sum_{\chi \in X_q} \tau(\overline{\chi})\chi(a) \underbrace{\left(e(N\beta)\psi(N,\chi) - 2\pi i \beta \int_1^N e(x\beta)\psi(x,\chi)\,dx\right)}_{(*)} + O\left(\log^2 N\right).$$

For $\chi = \chi_0 \in X_q$ *principal*, write:

$$\psi(x,\chi_0) = \lfloor x \rfloor + R(x).$$

$R(x) = O\left(Ne^{-c_2\sqrt{\log N}}\right)$, $\forall x \in [1, N]$

Hence:

$$\begin{aligned}(*) &= \left(e(N\beta)\lfloor N \rfloor - 2\pi i \beta \int_1^N e(x\beta)\lfloor x \rfloor\,dx\right) \\ &\quad + \left(e(N\beta)R(N) - 2\pi i \beta \int_1^N e(x\beta)R(x)\,dx\right) \\ &= \int_{1-}^N e(x\beta)d\lfloor x \rfloor + O\left((1 + |\beta|N)Ne^{-c_2\sqrt{\log N}}\right).\end{aligned}$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \leq q \leq P$

For $\alpha \in \mathfrak{M}(q, a)$: $S(\alpha) =???$ Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \leq Q^{-1}$;

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For $\chi = \chi_0 \in X_q$ *principal:*

$$(*) = \underbrace{\int_{1-}^{N} e(x\beta)d\lfloor x \rfloor}_{\text{Call } =: T(\beta)} + O\left((1 + |\beta|N)Ne^{-c_2\sqrt{\log N}}\right).$$

$$T(\beta) := \int_{1-}^{N} e(x\beta)d\lfloor x \rfloor = \sum_{k=1}^{N} e(k\beta).$$

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For $\alpha \in \mathfrak{M}(q,a)$: $S(\alpha) = ???$ Write $\alpha = \frac{a}{q} + \beta$; $|\beta| \le Q^{-1}$;

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$$T(\beta) := \int_{1-}^{N} e(x\beta)d\lfloor x \rfloor = \sum_{k=1}^{N} e(k\beta).$$

Also $\tau(\chi_0) = \mu(q)$ and $\left|\tau(\chi)\right| \le \sqrt{q}$ for any $\chi \in X_q$ (cf. Problem 9.2).

$$\Longrightarrow S(\alpha) = \frac{\mu(q)}{\phi(q)} T(\beta) + O\left(q^{\frac{1}{2}}(1+|\beta|N)Ne^{-c_3\sqrt{\log N}}\right).$$

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

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For $\chi = \chi_0 \in X_q$ *principal:*

$$(*) = \underbrace{\int_{1-}^{N} e(x\beta)\,d\lfloor x \rfloor}_{\text{Call } =: T(\beta)} + O\left((1 + |\beta|N)Ne^{-c_2\sqrt{\log N}}\right).$$

$$T(\beta) := \int_{1-}^{N} e(x\beta)\,d\lfloor x \rfloor = \sum_{k=1}^{N} e(k\beta).$$

Also $\tau(\chi_0) = \mu(q)$ and $\left|\tau(\chi)\right| \le \sqrt{q}$ for any $\chi \in X_q$ (cf. Problem 9.2).

$$\Longrightarrow S(\alpha) = \frac{\mu(q)}{\phi(q)} T(\beta) + \underbrace{O\left(q^{\frac{1}{2}}(1 + |\beta|N)Ne^{-c_3\sqrt{\log N}}\right)}_{= O\left(Ne^{-c_4\sqrt{\log N}}\right)}.$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0),\ 1 \le q \le P$

For all $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}(q,a)$:

$$S(\alpha) = \frac{\mu(q)}{\phi(q)} T(\beta) + O\left(N e^{-c_4\sqrt{\log N}}\right); \qquad T(\beta) = \sum_{k=1}^{N} e(k\beta).$$

Intuitively nice!

$$S(\alpha) = \sum_{n \le N} \Lambda(n) e(\alpha n) = \sum_{n \le N} \Lambda(n) e\left(n\frac{a}{q}\right) e(n\beta)$$

$$\text{“} \approx \frac{q}{\phi(q)} \sum_{\substack{k \le N \\ (k,q)=1}} e\left(k\frac{a}{q}\right) e(k\beta)\text{”}.$$

(would give same main term).

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0),\ 1 \le q \le P$$

For all $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}(q, a)$:

$$S(\alpha) = \frac{\mu(q)}{\phi(q)} T(\beta) + O\Big(N e^{-c_4\sqrt{\log N}}\Big); \qquad T(\beta) = \sum_{k=1}^{N} e(k\beta).$$

Using also $\left|\frac{\mu(q)}{\phi(q)} T(\beta)\right| \le |T(\beta)| \le N$:

$$S(\alpha)^3 = \frac{\mu(q)}{\phi(q)^3} T(\beta)^3 + O\Big(N^3 e^{-c_4\sqrt{\log N}}\Big).$$

Hence

$$\left[\text{Contribution from } \alpha \in \mathfrak{M}(q, a) \text{ in } r(N) = \int_0^1 S(\alpha)^3 e(-N\alpha)\, d\alpha\right]$$

$$= \frac{\mu(q)}{\phi(q)^3} e\Big(-\frac{aN}{q}\Big) \int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\, d\beta + O\Big(N^2 e^{-c_5\sqrt{\log N}}\Big).$$

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$, $1 \le q \le P$

$$\left[\text{Contribution from } \alpha \in \mathfrak{M}(q,a) \text{ in } r(N) = \int_0^1 S(\alpha)^3 e(-N\alpha)\, d\alpha\right]$$

$$= \frac{\mu(q)}{\phi(q)^3} e\Big(-\frac{aN}{q}\Big) \int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\, d\beta + O\Big(N^2 e^{-c_5\sqrt{\log N}}\Big).$$

Now add over all $\mathfrak{M}(q,a)$!

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\, d\beta + O\Big(N^2 e^{-c_6\sqrt{\log N}}\Big),$$

where

$$c_q(N) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\Big(\frac{an}{q}\Big)$$

(Ramanujan's sum, which we introduced in Problem 9.3).

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \underbrace{\int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\, d\beta}\, + O\left(N^2 e^{-c_6\sqrt{\log N}}\right).$$

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$$

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Recall

$$T(\beta) = \sum_{k=1}^{N} e(k\beta) \qquad \text{— only depends on } \beta \text{ mod } 1.$$

$$T(\beta) = \frac{e((N+1)\beta) - e(\beta)}{e(\beta) - 1} = O\Big(\min(N, \|\beta\|^{-1})\Big).$$

Hence

$$\int_{1/Q}^{1-1/Q} |T(\beta)|^3\,d\beta = O\Big(\int_{1/Q}^{1/2} \beta^{-3}\,d\beta\Big) = O\big(Q^2\big) = O\big(N^2(\log N)^{-2B}\big),$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \underbrace{\int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\, d\beta}_{(*)} + O\Big(N^2 e^{-c_6\sqrt{\log N}}\Big).$$

Recall

$$T(\beta) = \sum_{k=1}^{N} e(k\beta) \qquad \text{— only depends on } \beta \bmod 1.$$

$$T(\beta) = \frac{e((N+1)\beta) - e(\beta)}{e(\beta) - 1} = O\Big(\min(N, \|\beta\|^{-1})\Big).$$

Hence

$$\int_{1/Q}^{1-1/Q} |T(\beta)|^3\, d\beta = O\Big(\int_{1/Q}^{1/2} \beta^{-3}\, d\beta\Big) = O\big(Q^2\big) = O\big(N^2(\log N)^{-2B}\big),$$

so that

$$(*) = \int_0^1 T(\beta)^3 e(-N\beta)\, d\beta + O\big(N^2(\log N)^{-2B}\big).$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \underbrace{\int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\, d\beta}_{(*)} + O\left(N^2 e^{-c_6\sqrt{\log N}}\right).$$

$$(*) = \int_0^1 T(\beta)^3 e(-N\beta)\, d\beta + O\left(N^2 (\log N)^{-2B}\right).$$

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \underbrace{\int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\, d\beta}_{(*)} + O\Big(N^2 e^{-c_6\sqrt{\log N}}\Big).$$

$$(*) = \underbrace{\int_0^1 T(\beta)^3 e(-N\beta)\, d\beta}_{(**)} + O\big(N^2 (\log N)^{-2B}\big).$$

$$(**) = \int_0^1 \Big(\sum_{1 \le k_1, k_2, k_3 \le N} e((k_1 + k_2 + k_3)\beta) \Big) e(-N\beta)\, d\beta$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\,d\alpha = \sum_{q\le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \underbrace{\int_{-1/Q}^{1/Q} T(\beta)^3 e(-N\beta)\,d\beta}_{(*)} + O\left(N^2 e^{-c_6\sqrt{\log N}}\right).$$

$$(*) = \underbrace{\int_0^1 T(\beta)^3 e(-N\beta)\,d\beta}_{(**)} + O\left(N^2 (\log N)^{-2B}\right).$$

$$\begin{aligned}(**) &= \int_0^1 \Bigg(\sum_{1\le k_1,k_2,k_3\le N} e((k_1+k_2+k_3)\beta) \Bigg) e(-N\beta)\,d\beta \\ &= \sum_{\substack{1\le k_1,k_2,k_3\le N \\ k_1+k_2+k_3=N}} 1, \\ &= \tfrac{1}{2}(N-1)(N-2) = \tfrac{1}{2}N^2 + O(N).\end{aligned}$$

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$$

Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\,d\alpha$$

$$= \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N)\Big(\tfrac{1}{2}N^2 + O\big(N^2(\log N)^{-2B}\big)\Big) + O\Big(N^2 e^{-c_6\sqrt{\log N}}\Big)$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$

Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha$$

$$= \underbrace{\sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \left(\tfrac{1}{2}N^2 + O\left(N^2(\log N)^{-2B}\right)\right) + O\left(N^2 e^{-c_6\sqrt{\log N}}\right)}_{(*)}$$

Using $|c_q(N)| \le \phi(q)$ and $\phi(q) \gg q/\log q$ for all $q \ge 2$ (cf. Problem 15.1).

$$\sum_{q \le P} \frac{|c_q(N)|}{\phi(q)^3} \le \sum_{q=1}^{\infty} \frac{1}{\phi(q)^2} = O(1).$$

Thus:

$$(*) = \frac{1}{2}N^2 \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) + O\left(N^2(\log N)^{-2B}\right).$$

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$$

Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{1}{2}N^2 \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) + O\Big(N^2(\log N)^{-2B}\Big).$$

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$

Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{1}{2} N^2 \sum_{q \le P} \frac{\mu(q)}{\phi(q)^3} c_q(N) + O\Big(N^2 (\log N)^{-2B} \Big).$$

Again using $|c_q(N)| \le \phi(q)$ and $\phi(q) \gg q/\log q$:

$$\Big| \sum_{q > P} \frac{\mu(q)}{\phi(q)^3} c_q(N) \Big| \le \sum_{q > P} \frac{1}{\phi(q)^2} \ll_\varepsilon \sum_{q > P} q^{-2+2\varepsilon}$$

Proof of Theorem 19.2.

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$$\ll_\varepsilon (P-1)^{-1+2\varepsilon} \ll P^{-1+2\varepsilon} = (\log N)^{-B+2\varepsilon B},$$

Proof of Theorem 19.2.

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Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\,d\alpha = \frac{1}{2}N^2 \underbrace{\sum_{q\le P} \frac{\mu(q)}{\phi(q)^3} c_q(N)}_{(*)} + O\Big(N^2(\log N)^{-2B}\Big).$$

Again using $|c_q(N)| \le \phi(q)$ and $\phi(q) \gg q/\log q$:

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$$\ll_\varepsilon (P-1)^{-1+2\varepsilon} \ll P^{-1+2\varepsilon} = (\log N)^{-B+2\varepsilon B},$$

and thus if we choose $\varepsilon = \frac{1}{2B}$ we have

$$\Big|\sum_{q>P} \frac{\mu(q)}{\phi(q)^3} c_q(N)\Big| \ll (\log N)^{-B+1}$$

Hence

$$(*) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)^3} c_q(N) + O\big((\log N)^{-B+1}\big).$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$

Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{1}{2} N^2 \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)^3} c_q(N) + O\Big(N^2 (\log N)^{-B+1}\Big).$$

Can *factor!*

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$$

Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\,d\alpha = \frac{1}{2}N^2 \underbrace{\sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)^3} c_q(N)}_{(*)} + O\Big(N^2(\log N)^{-B+1}\Big).$$

Can *factor!*

Recall from Problem 9.3:

$$c_q(N) = \frac{\phi(q)}{\phi(\frac{q}{(q,N)})}\mu\left(\frac{q}{(q,N)}\right) \qquad \text{– multiplicative w.r.t. } q.$$

Hence, by Prop. 2.7, since $\mu(p^\alpha) = 0$, $\forall \alpha \geq 2$:

$$(*) = \prod_p \left(1 + \frac{\mu(p)}{\phi(p)^3} c_p(N)\right) = \prod_p \left(1 - \frac{c_p(N)}{(p-1)^3}\right)$$

$$= \prod_{p \mid N} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \nmid N} \left(1 + \frac{1}{(p-1)^3}\right) = \mathfrak{S}(N),$$

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$

Hence

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{1}{2}\mathfrak{S}(N)N^2 + O\Big(N^2(\log N)^{-B+1}\Big).$$

Proof of Theorem 19.2.

$$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$$

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$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{1}{2}\mathfrak{S}(N)N^2 + O\Big(N^2(\log N)^{-B+1}\Big).$$

Finally: Bound the contribution from the minor arcs, $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$.

Proof of Theorem 19.2. $P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$

Finally: Bound the contribution from the minor arcs, $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$.

Note

$$\left|\int_{\mathfrak{m}} S(\alpha)^3 e(-N\alpha)\, d\alpha\right| \leq \int_{\mathfrak{m}} |S(\alpha)|^3\, d\alpha \leq \left(\sup_{\alpha\in\mathfrak{m}} |S(\alpha)|\right) \int_{\mathfrak{m}} |S(\alpha)|^2\, d\alpha$$
$$\leq \left(\sup_{\alpha\in\mathfrak{m}} |S(\alpha)|\right) \int_0^1 |S(\alpha)|^2\, d\alpha$$

and here the last integral is

$$\int_0^1 |S(\alpha)|^2\, d\alpha = \int_0^1 \sum_{k_1\leq N} \sum_{k_2 \leq N} \Lambda(k_1)\Lambda(k_2) e\big((k_1-k_2)\alpha\big)\, d\alpha$$
$$= \sum_{k_1\leq N} \Lambda(k_1) \sum_{k_2\leq N} \Lambda(k_2) \int_0^1 e\big((k_1-k_2)\alpha\big)\, d\alpha$$
$$= \sum_{k\leq N} \Lambda(k)^2 \leq (\log N) \sum_{k\leq N} \Lambda(k) \ll N \log N.$$

Proof of Theorem 19.2.

$P = (\log N)^B$, $Q = N(\log N)^{-B}$, $(B > 0)$

Finally: Bound the contribution from the minor arcs, $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$.

Hence

$$\left|\int_{\mathfrak{m}} S(\alpha)^3 e(-N\alpha)\, d\alpha\right| \ll N(\log N)\Big(\sup_{\alpha\in\mathfrak{m}}\big|S(\alpha)\big|\Big).$$

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$$\left|\int_{\mathfrak{m}} S(\alpha)^3 e(-N\alpha)\, d\alpha\right| \ll N(\log N)\Big(\sup_{\alpha\in\mathfrak{m}}\big|S(\alpha)\big|\Big).$$

For every $\alpha \in \mathfrak{m}$, by Dirichlet's theorem (Lemma 18.4):

$$\exists q, a \in \mathbb{Z}: \qquad 1 \le q \le Q, \quad (a,q) = 1, \quad \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qQ}.$$

$$\alpha \notin \mathfrak{M}(a,q) \Longrightarrow P < q \le Q.$$

Hence by Proposition 18.3 (for all $\alpha \in \mathfrak{m}$):

$$\big|S(\alpha)\big| \ll (Nq^{-\frac{1}{2}} + N^{\frac{4}{5}} + N^{\frac{1}{2}}q^{\frac{1}{2}})(\log N)^4 \ll \big(NP^{-\frac{1}{2}} + N^{\frac{4}{5}} + N^{\frac{1}{2}}Q^{\frac{1}{2}}\big)(\log N)^4$$
$$\ll N(\log N)^{-(B/2)+4}.$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$

Finally: Bound the contribution from the minor arcs, $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$.

Hence

$$\left| \int_{\mathfrak{m}} S(\alpha)^3 e(-N\alpha)\, d\alpha \right| \ll N^2 (\log N)^{-(B/2)+5}.$$

Proof of Theorem 19.2.

$P = (\log N)^B,\ Q = N(\log N)^{-B},\ (B > 0)$

Finally: Bound the contribution from the minor arcs, $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$.

Hence

$$\left| \int_{\mathfrak{m}} S(\alpha)^3 e(-N\alpha)\, d\alpha \right| \ll N^2 (\log N)^{-(B/2)+5}.$$

Take $B = 2A + 10$.

Also use

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{1}{2}\mathfrak{S}(N) N^2 + O\Big(N^2 (\log N)^{-B+1}\Big).$$

Hence

$$r(N) = \int_0^1 S(\alpha)^3 e(-N\alpha)\, d\alpha = \frac{1}{2}\mathfrak{S}(N) N^2 + O\Big(N^2 (\log N)^{-A}\Big).$$

□ □ □