SOLUTION SUGGESTIONS FOR THREE PROBLEMS

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2.2. It follows from the formula $\phi(n) = n\prod_{p|n}\Big(1 - \frac{1}{p}\Big)$ that $\phi(n)$ is a multiplicative function. Hence also the function $f(n) = \phi(n)n^{-s}$ is multiplicative for any fixed s. Next note that if $\sigma > 2$ then the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, since

$$\sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} \phi(n)n^{-\sigma} \leq \sum_{n=1}^{\infty} n^{1-\sigma} < \infty. \tag{1}$$

Hence Proposition 2.7 applies when $\sigma > 2$ and we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \phi(n)n^{-s} &= \prod_p \big(1 + \phi(p)p^{-s} + \phi(p^2)p^{-2s} + \ldots\big) \\
&= \prod_p \Big(1 + \sum_{k=1}^{\infty} \Big(1 - \frac{1}{p}\Big)p^k \cdot p^{-ks}\Big) \\
&= \prod_p \Big(1 + \Big(1 - \frac{1}{p}\Big) \sum_{k=1}^{\infty} p^{k(1-s)}\Big) \\
&= \prod_p \Big(1 + \frac{p-1}{p} \cdot \frac{p^{1-s}}{1 - p^{1-s}}\Big) \\
&= \prod_p \frac{p(1 - p^{1-s}) + (p-1)p^{1-s}}{p(1 - p^{1-s})} \\
&= \prod_p \frac{1 - p^{-s}}{1 - p^{1-s}} \\
&= \frac{\zeta(s-1)}{\zeta(s)}.
\end{aligned}$$

3.4 (final part of a solution).

I discussed this problem in class, but only gave a first part of a solution. I noted that *we may assume that* $N(r) < \infty$ *for all* $r > 0$ (since otherwise $\tau = A = \infty$ and the problem is solved). I proved that under this assumption, the following equivalence relation holds for every $\alpha > 0$:

$$\text{(2)} \quad \sum_{j=1}^{\infty} (1 + |\rho_j|)^{-\alpha} < \infty \quad \Leftrightarrow \quad \Big[\limsup_{r \to \infty} \frac{N(r)}{(1+r)^{\alpha}} < \infty \text{ and } \int_0^{\infty} \frac{N(r)}{(1+r)^{\alpha+1}} \, dr < \infty \Big].$$

Now the solution can be completed as follows:

(a). For every $\alpha > A$ we can argue as follows: Choose a number A_1 in the interval $A < A_1 < \alpha$. Then by the definition of A (and the definition of "$\limsup$"), we have $\frac{\log N(r)}{\log r} < A_1$ for all sufficiently large r. Equivalently: $N(r) < r^{A_1}$ for all sufficiently large r. In precise terms, this means that there exists some $R_0 > 0$ such that

$$\forall r \geq R_0 : \quad N(r) < r^{A_1}.$$

It follows that for all $r \geq R_0$ we have $\frac{N(r)}{(1+r)^{\alpha}} < \frac{r^{A_1}}{(1+r)^{\alpha}} < r^{A_1 - \alpha}$, and since $A_1 - \alpha < 0$ this implies that $\lim_{r \to \infty} \frac{N(r)}{(1+r)^{\alpha}} = 0$, and in particular $\limsup_{r \to \infty} \frac{N(r)}{(1+r)^{\alpha}} < \infty$. It also follows that

$$\int_0^{\infty} \frac{N(r)}{(1+r)^{\alpha+1}} \, dr \leq \int_0^{R_0} \frac{R_0^{A_1}}{(1+r)^{\alpha}} \, dr + \int_{R_0}^{\infty} \frac{r^{A_1}}{(1+r)^{\alpha+1}} \, dr$$
$$\leq \int_0^{R_0} R_0^{A_1} \, dr + \int_{R_0}^{\infty} r^{A_1 - \alpha - 1} \, dr < \infty,$$

where we used the fact that $A_1 - \alpha - 1 < -1$. Hence, using the equivalence in (2) (in the "$\Leftarrow$" direction), we conclude that $\sum_{j=1}^{\infty} (1 + |\rho_j|)^{-\alpha} < \infty$. By the definition of τ, this implies $\tau \leq \alpha$.

To sum up, we have prove that $[\forall \alpha > A : \tau \leq \alpha]$. This implies that $\tau \leq A$. $\square$

(b). For every $\alpha > 0$ such that $\sum_{j=1}^{\infty} (1 + |\rho_j|)^{-\alpha} < \infty$, we have by (2): $\limsup_{r \to \infty} \frac{N(r)}{(1+r)^{\alpha}} < \infty$, and this implies that there exist constants $C > 0$ and $R_0 > 0$ such that for all $r \geq R_0$ we have $\frac{N(r)}{(1+r)^{\alpha}} < C$, i.e., $N(r) < C(1+r)^{\alpha}$. This implies that for all $r \geq R_0$ we have

$$\frac{\log N(r)}{\log r} \leq \frac{\log(C(1+r)^{\alpha})}{\log r}.$$

Hence

$$A = \limsup_{r\to\infty} \frac{\log N(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log(C(1+r)^{\alpha})}{\log r} = \limsup_{r\to\infty} \frac{\log C + \alpha \log(1+r)}{\log r} = \alpha.$$

To sum up, we have proved that for every $\alpha > 0$ satisfying $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha} < \infty$, we have $A \le \alpha$. In view of the definition of τ, this implies that $A \le \tau$. $\square$

3.5. Set $A(x) = \sum_{1\leq n\leq x} a_n$; then the assumption says that $A(x) \sim x^2$ as $x \to \infty$, and hence for any given $\varepsilon > 0$ there is some $X > 1$ such that

$$\big|A(x) - x^2\big| < \varepsilon x^2, \qquad \forall x \geq X. \tag{3}$$

Now for each $N \in \mathbb{Z}^+$ we have

$$\sum_{n=1}^{N} a_n (N-n)^2 = \int_0^N (N-x)^2\, dA(x) = 0 + 2\int_0^N (N-x)A(x)\, dx$$

If $A(x) \equiv x^2$ then the last expression equals

$$2\int_0^N (N-x)x^2\, dx = 2\Big[\frac{N}{3}x^3 - \frac{1}{4}x^4\Big]_{x=0}^{x=N} = \frac{1}{6}N^4.$$

Hence for our general $A(x) = \sum_{1\leq n\leq x} a_n$ we have, for each integer $N > X$:

$$\begin{aligned}
\Big|\sum_{n=1}^{N} a_n (N-n)^2 - \frac{1}{6}N^4\Big| &= \Big|2\int_1^N (N-x)A(x)\, dx - 2\int_0^N (N-x)x^2\, dx\Big| \\
&\leq 2\int_0^N (N-x)\big|A(x) - x^2\big|\, dx \\
&\leq 2\int_0^X N\big|A(x) - x^2\big|\, dx + 2\int_X^N (N-x)\varepsilon x^2\, dx \\
&\leq 2N\int_0^X \big|A(x) - x^2\big|\, dx + 2\varepsilon\int_X^N N^3\, dx \\
&\leq 2N\int_0^X \big|A(x) - x^2\big|\, dx + 2\varepsilon N^4.
\end{aligned}$$

Here the number $\int_0^X \big|A(x) - x^2\big|\, dx$ is independent of N; hence for all sufficiently large N the above is $< 3\varepsilon N^4$, i.e. we have proved that for all sufficiently large N we have

$$\Big|\sum_{n=1}^{N} a_n (N-n)^2 - \frac{1}{6}N^4\Big| < 3\varepsilon N^4,$$

or equivalently

$$\Big|\frac{\sum_{n=1}^{N} a_n (N-n)^2}{N^4} - \frac{1}{6}\Big| < 3\varepsilon.$$

Since ε was arbitrarily small, this implies that

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} a_n (N-n)^2}{N^4} = \frac{1}{6},$$

or equivalently:

$$\sum_{n=1}^{N} a_n (N-n)^2 \sim \frac{1}{6}N^4 \qquad \text{as } N \to \infty.$$

$\square$