Analytic Number Theory 2021; Assignment 2

Problem 1. For each $n \in \mathbb{Z}^+$, let $\lambda(n) = (-1)^r$ where r is the number of prime factors of n, counting multiplicity. Thus, e.g., $\lambda(1) = 1$, $\lambda(8) = -1$ and $\lambda(10) = 1$. Set $S(x) = \sum_{1 \le n \le x} \lambda(n)$. The goal of the following problem is to prove, by mimicking the proof of the prime number theorem, that $S(x)$ satisfies the bound $S(x) = o(x)$ as $x \to \infty$. This can be interpreted as saying that the asymptotic probability for a "random" large integer to have an odd number of primes in its prime factorization is 50%.

(a). Set $S_1(x) = \int_0^x S(u)\,du$ $(x > 0)$. Prove that

$$S_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{\zeta(2s)}{\zeta(s)}\,ds$$

for any $x > 0$ and any $c > 1$.

(b). Using (a), prove that $S_1(x) = o(x^2)$ as $x \to \infty$.

(c). Using (b), prove that $S(x) = o(x)$ as $x \to \infty$.

[Hint for (c): Trying to imitate the proof of Theorem 7.10 in the lecture notes we run into the problem that $S(u)$ is not increasing, as opposed to $\psi(u)$. However $S(u)$ has the property that $|S(u_1) - S(u_2)| \le 1 + |u_1 - u_2|$ for any $u_1, u_2 > 0$ (proof?), and this can be used as a substitute for monotonicity.]

(20p)

Problem 2. (a) Prove that for any $a, b \in \mathbb{R}_{>0}$:

$$\prod_{n=1}^{\infty} \frac{n(n+a+b)}{(n+a)(n+b)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}$$

(b) Use the fact that $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ to prove that for all $t \in \mathbb{R}$:

$$\left|\Gamma\Big(\frac{1}{2} + it\Big)\right| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}}.$$

(10p)

Problem 3. Prove from the Siegel-Walfisz theorem that for any $\varepsilon > 0$, $q \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$ with $(a, q) = 1$, the smallest prime $p \equiv a \bmod q$ satisfies $p \ll e^{q^{\varepsilon}}$, where the implied constant depends only on ε.

(10p)

Problem 4. Let $d(n) := \#\{a \in \mathbb{Z}^+ \,:\, a \,|\, n\}$ (the divisor function). For any $\delta > 0$, show that $d(n) < 2^{(1+\delta)\log n / \log\log n}$ for all n sufficiently large.

[Hint: One approach is to prove that for every $\varepsilon > 0$ we have $d(n) \le C(\varepsilon) \cdot n^{\varepsilon}$ $(\forall n \in \mathbb{Z}^+)$, with an explicit constant $C(\varepsilon) > 0$. If your $C(\varepsilon)$ is not too wasteful, the desired inequality can then be obtained by choosing ε depending on n appropriately.]

(10p)

GOOD LUCK!