#20. The Jacobi Theta Function

Def: $\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n z} e^{\pi i n^2 \tau}$ $(z \in \mathbb{C},\ \tau \in \mathbb{H})$

capital... But I'll often be sloppy and write "θ"

$\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$

Notation: $e(w) := e^{2\pi i w}$ $(w \in \mathbb{R} \rightsquigarrow w \in \mathbb{C})$

$e(w+1) = e(w)$. Then $\theta(z|\tau) = \sum_{n \in \mathbb{Z}} e(nz)\, e(\tfrac{1}{2} n^2 \tau)$

Prop 1.1: (i) $\theta(z|\tau)$ is holomorphic in $\mathbb{C} \times \mathbb{H}$. Discuss! Several $\mathbb{C}$-variables...

(ii) $\theta(z+1\,|\,\tau) = \theta(z|\tau)$

(iii) $\theta(z+\tau\,|\,\tau) = \theta(z|\tau)\, e^{-\pi i \tau} e^{-2\pi i z}$

(iv) $\theta(z|\tau) = 0$ when $z = \frac{1+\tau}{2} + m + n\tau$ (any $m, n \in \mathbb{Z}$)

proof: (i) Use $\left| e^{2\pi i n z} e^{\pi i n^2 \tau} \right| = e^{-2\pi n \mathrm{Im}(z)} \cdot e^{-\pi n^2 \mathrm{Im}(\tau)}$

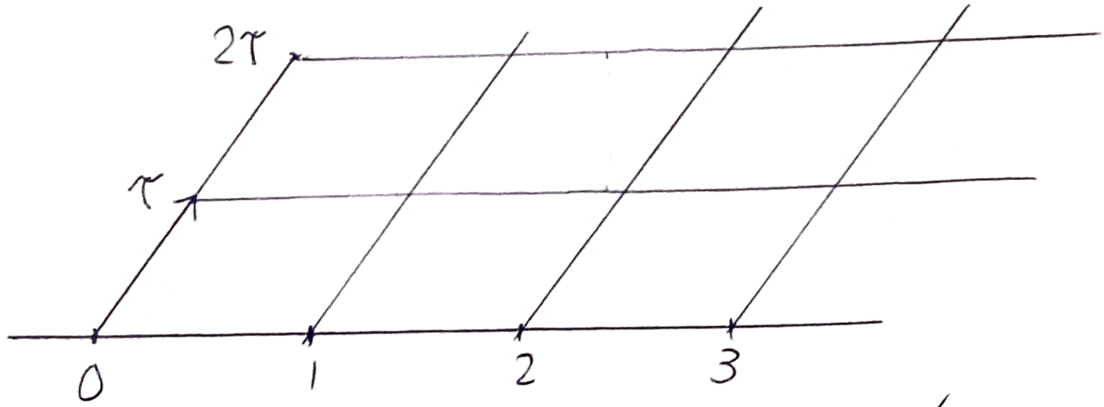
and Weierstrass Theorem.

(ii) Clear (since $e^{2\pi i n(z+1)} = e^{2\pi i n z}$)

(iii) $$\theta(z+\tau\,|\,\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n z} e^{\pi i \tau \overbrace{(n^2+2n)}^{(n+1)^2 - 1}}$$

$$= e^{-\pi i \tau} e^{-2\pi i z} \cdot \sum_{n \in \mathbb{Z}} e^{2\pi i (n+1) z} e^{\pi i (n+1)^2 \tau} = \text{(JA)}$$

(ii) & (iii) $\Rightarrow$



$$\theta(z+m+n\tau \mid \tau) = \left(e^{-\pi i \tau} e^{-2\pi i z}\right)^n \cdot \theta(z \mid \tau), \quad (\forall m, n \in \mathbb{Z})$$

If "1" : Elliptic function

outside the scope of our course. See SS Ch. 9 for intro!

if holo the constant but meromorphic!

(iv) : This is now easy! By above, suffices to show $\theta(\frac{1+\tau}{2} \mid \tau) = 0$. This follows since the terms cancel in pairs! $\left(\sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i \tau (n+n^2)} \quad n \leftrightarrow -1-n; \quad (-1-n)+(-1-n)^2 = n^2+n \right)$

Next: Behaviour in τ

Recall: $\underline{\underline{\theta(\tau) := \theta(0|\tau)}}$; $\qquad \underline{\underline{\vartheta(x) := \theta(ix) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 x}}}$

$(x>0)$

For the proof of the functional equation for $\zeta(s)$,

we used $\quad \underline{\underline{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\cdot\frac{\vartheta(x)-1}{2}\,dx}}$

and $\quad \underline{\underline{\vartheta(x^{-1}) = \sqrt{x}\,\vartheta(x) \quad (\forall x>0)}}$

Follows from $\quad \underline{\underline{\theta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\theta(\tau), \quad \forall \tau\in\mathbb{H}}}$

"principal part"; use $\arg\left(\frac{\tau}{i}\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

More general:

Theorem 1.6:

$$\underline{\underline{\theta\left(z\,\middle|\,-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,e^{\pi i\tau z^2}\,\theta(z\tau\,|\,\tau)}} \quad \forall z\in\mathbb{C},\ \tau\in\mathbb{H}$$

proof: Suffices to prove for $\underline{z=a\in\mathbb{R}}$, $\underline{\tau=it\ (t>0)}$.

$\leadsto$ Want to prove

$$\underline{\underline{\sum_{n\in\mathbb{Z}} e^{2\pi i n a}\,e^{-\pi n^2/t} = \sqrt{t}\,e^{-\pi t a^2}\sum_{n\in\mathbb{Z}} e^{-2\pi n a t}\,e^{-\pi n^2 t}}}$$

$$\Leftrightarrow \underline{\underline{\sum_{n\in\mathbb{Z}} e^{-\pi t(n+a)^2} = t^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t + 2\pi i n a}}}$$

This is LN, Thm 9.2, with $x = \frac{1}{t}$.

The above follows by the Poisson summation formula,

$$\boxed{\underline{\underline{\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)}} \quad \text{for any nice function } \underline{f: \mathbb{R}\to\mathbb{C}}, \text{ with } \underline{\underline{\hat{f}(n) = \int_{-\infty}^{\infty} f(u) e^{-2\pi i n u} du}}}$$

Indeed, take $\underline{\underline{f(u) = e^{-\pi t(u+a)^2}}}$;

compute $\underline{\hat{f}(n) = t^{-\frac{1}{2}} e^{-\pi n^2/t + 2\pi i n a}}$ ...

□

Note that we also have, trivially:

$$\boxed{\theta(z|\tau+2) = \theta(z|\tau)}$$

Exercise: consequence of $\theta(\tau+2)=\theta(\tau)$ and

$\theta\left(-\frac{1}{\tau}\right)=\sqrt{\frac{\tau}{i}}\,\theta(\tau)$:

$\theta(\tau)$ is a modular form of weight $\frac{1}{2}$

Let Λ be the theta group:

$$\Lambda=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z}) \;:\; ab\equiv cd\equiv 0 \pmod 2\right\}$$

Let $R_8:=\{z\in\mathbb{C} : z^8=1\}$ (thus $|R_8|=8$)

Prove that there exists a function $r:\Lambda\to R_8$ such that

$$\boxed{\begin{gathered}\theta\left(\frac{a\tau+b}{c\tau+d}\right)=r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\cdot\underbrace{(c\tau+d)^{\frac{1}{2}}}_{\circledast}\cdot\,\theta(\tau)\\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Lambda,\quad \tau\in\mathbb{H}\end{gathered}}$$

$\circledast$ Say
$$\begin{cases} c>0 \Rightarrow 0<\arg(c\tau+d)<\pi \\ c<0 \Rightarrow -\pi<\arg(c\tau+d)<0 \\ c=0 \rightarrow \arg(c\tau+d)=0 \text{ or } \pi \end{cases}$$

Fact which you may use: Λ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The triple product formula

Theorem 1.3: $\forall z \in \mathbb{C},\ \tau \in \mathbb{H}$, writing $\boxed{q = e^{\pi i \tau}}$:

$$\theta(z \mid \tau) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z})$$

HW 3:3 – consequence of the above (& "playing")

Cor 1.4: $\theta(\tau) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1})^2$

proof: Immediate since $\theta(\tau) = \theta(0 \mid \tau)$.

This also gives $\theta(\tau+1) = \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2$

$\tau_{new} = \tau + 1 \Rightarrow q_{new} = -q$

Note also $\theta(\tau+1) = \theta(\tfrac{1}{2} \mid \tau)$ ← easy exercise!

Proof of Theorem 1.3

Call the right hand side $\Pi(z \mid \tau)$.

Prop 1.2: (as Prop 1.1 but for $\Pi(z|\tau)$!)

(i) $\Pi(z|\tau)$ is holomorphic in $\mathbb{C} \times \mathbb{H}$.

(ii) $\Pi(z+1 \mid \tau) = \Pi(z \mid \tau)$

(iii) $\Pi(z+\tau \mid \tau) = \Pi(z|\tau)\, e^{-\pi i \tau} e^{-2\pi i z}$

(iv) $\Pi(z \mid \tau) = 0$ for $z = \frac{1+\tau}{2} + m + n\tau \quad (m, n \in \mathbb{Z})$,

and these points are simple zeros, and the only zeros of $\Pi(\cdot, \tau)$.

proof (outline):

(i) Use $\sum_{n=1}^{\infty} \left(|q^{2n}| + |q^{2n-1} e^{2\pi i z}| + |q^{2n-1} e^{-2\pi i z}| \right) < \infty$.

(for $|q| < 1$)

(ii) Clear.

(iii) (This motivates the formula!)

$$\theta(z+\tau \mid \tau) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1} e^{2\pi i(z+\tau)})(1+q^{2n-1} e^{-2\pi i(z+\tau)})$$

$$= \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n+1}e^{2\pi i z})(1+q^{2n-3}e^{-2\pi i z})$$

$$= \frac{1+q^{-1}e^{-2\pi i z}}{1+q\,e^{2\pi i z}} \cdot \Pi(z|\tau) = \underline{\underline{q^{-1}e^{-2\pi i z} \cdot \Pi(z|\tau)}}.$$

(iv) Abs. conv. prod $\Rightarrow$ can <u>"read off zeros formally"</u>!

LN, Cor 2.3

For $n \geq 1$:

$$\underline{1+q^{2n-1}e^{2\pi i z} = 0} \iff e^{\pi i(\tau(2n-1)+2z)} = -1$$

$$\iff \tau(2n-1)+2z = 1+2m \quad (\exists m \in \mathbb{Z})$$

$$\iff \underline{z = \frac{1+\tau}{2} + m - n\tau \quad (\exists m \in \mathbb{Z})}$$

Similarly,

$$\underline{1+q^{2n-1}e^{-2\pi i z} = 0} \iff \underline{z = \frac{-1-\tau}{2} + m + n\tau} \ (\exists m \in \mathbb{Z})$$

Done ... !

□

Using now Prop 1.2:

$$c(z,\tau) := \frac{\theta(z\,|\,\tau)}{\Pi(z\,|\,\tau)}$$ is entire in z for each fixed τ; and $c(z+1,\tau) = c(z+\tau,\tau) = c(z,\tau)$

("doubly periodic") $\Rightarrow$ bounded $\Rightarrow$ constant wrt z.

$\therefore$ $c(z,\tau) = c(\tau)$, independent of z!

Finally, some further, – beautiful! – reasoning gives $c(\tau) \equiv 1$!

Details: We have $$\frac{\theta(z\,|\,\tau)}{\Pi(z\,|\,\tau)} = c(\tau), \quad \forall z \in \mathbb{C},\ \tau \in H.$$

Use for $z = \frac{1}{2}$ $\Rightarrow$ $$c(\tau) = \frac{\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}}{\prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2}$$

Use for $z = \frac{1}{4}$ $\Rightarrow$?

$$\theta\left(\tfrac{1}{4}\,\middle|\,\tau\right) = \sum_{n\in\mathbb{Z}} i^n q^{n^2} \underset{n=2m}{=} \sum_{m\in\mathbb{Z}} (-1)^m q^{4m^2}$$

$$\Pi\left(\tfrac{1}{4}\,\middle|\,\tau\right) = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1} i)(1+q^{2n-1}(-i))$$

$$= \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{4n-2})$$

Combine every "$1+q^{4n-2}$" with a "$1-q^{4n-2}$" ...
The factors $1-q^{2n}$ for <u>n even</u> remain ...

$$= \prod_{m=1}^{\infty} (1-q^{4m})(1-q^{8m-4})$$

$$\therefore \quad c(\tau) = \frac{\sum_{m\in\mathbb{Z}} (-1)^m q^{4m^2}}{\prod_{m=1}^{\infty} (1-q^{4m})(1-q^{8m-4})}$$

Combine $\Rightarrow$ $\boxed{c(\tau) = c(4\tau), \quad \forall \tau \in \mathbb{H}}$

Also $\boxed{c(\tau) \to 1 \text{ as } \operatorname{Im}\tau \to \infty}$,

e.g. using $$c(\tau) = \frac{\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}}{\prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2} \to \frac{1}{1}.$$

Hence $c(\tau) = c(4\tau) = c(4^2\tau) = \dots \to 1$, i.e. $\boxed{c(\tau)=1}$.

#21. Sums of squares

GENERATING FUNCTIONS

Study $\underline{F(x) = \sum_{n=0}^{\infty} F_n x^n}$ $\left(\text{or } \sum_{n=1}^{\infty} \frac{F_n}{n^s}\right)$,

in order to understand a given sequence (F_n).

Ex 1 (very basic)

If (F_n) is the Fibonacci sequence,

F_0	F_1	F_2	F_3	F_4	F_5	F_6	...
0	1	1	2	3	5	8	...

then $\underline{F(x) = x^2 F(x) + x F(x) + x}$;

thus $F(x) = \frac{x}{1-x-x^2} = \frac{1/\sqrt{5}}{1-\varphi x} - \frac{1/\sqrt{5}}{1+\varphi^{-1}x}$

where $\varphi = \frac{1+\sqrt{5}}{2}$; thus $\boxed{F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (-\varphi)^{-n}\right)}$.

See SS p. 310, Exc. 2

Ex 2

$$\zeta(n) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Ex 3: The partition function. (SS Ch. 10.2)

$p(n) :=$ the number of ways to write n as a sum of positive integers.

n	0	1	2	3	4	5	6	7	...
$p(n)$	1	1	2	3	5	7	11	15	...

namely $4 = 1+1+1+1 = 1+1+2$
$= 1+3 = 2+2 = 4$

Then $$\boxed{\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}}$$ (when $|x|<1$)

Indeed, $$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} \left(1 + x^k + x^{2k} + x^{3k} + \dots\right) = \dots$$

Can use this to prove combinatorial identities involving $p(n)$ (see SS Ch. 10.2) and also:

$$p(n) \sim \frac{1}{4\sqrt{3}\,n} \cdot e^{K\sqrt{n}} \qquad \text{as } n \to \infty,$$

with $K = \pi\sqrt{\frac{2}{3}}$

Hardy & Ramanujan 1918

Uspensky 1920

(see SS, App. A.4)

Ex 4: $r_k(n)$:= the number of ways to write n as a sum of k squares.

ordered

$$= \# \{ \langle m_1, \ldots, m_k \rangle \in \mathbb{Z}^k : m_1^2 + m_2^2 + \ldots + m_k^2 = n \}$$

Generating function, for $|q| < 1$:

$$\sum_{n=0}^{\infty} r_k(n) q^n = \left(\sum_{m_1 \in \mathbb{Z}} q^{m_1^2} \right) \left(\sum_{m_2 \in \mathbb{Z}} q^{m_2^2} \right) \cdots \left(\sum_{m_k \in \mathbb{Z}} q^{m_k^2} \right)$$

$$= \left(\sum_{m \in \mathbb{Z}} q^{m^2} \right)^k = \theta(\tau)^k \qquad q = e^{\pi i \tau}$$

We will use this to reprove $r_2(n) = 4(d_1(n) - d_3(n))$

and also prove $r_4(n) = 8\sigma_1^*(n) := 8 \sum_{\substack{d|n \\ (4 \nmid d)}} d$