Analytic Number Theory 2023; Assignment 1

Problem 1. a) Prove that

$$\sum_{n \le x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x), \qquad \forall x \ge 2.$$

(Hint: You may start by verifying that $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$, $\forall n \in \mathbb{Z}^+$.)

b) Using part a), prove that

$$\sum_{n \le x} \frac{\phi(n)}{n} \sim \frac{6}{\pi^2} x \qquad \text{as } x \to \infty.$$

(10p)

Problem 2. Later in the course we will prove the prime number theorem with a precise error term, namely: There exists an absolute constant $c > 0$ such that

$$\text{(A)} \qquad \pi(x) = \operatorname{Li} x + O\Big(x e^{-c\sqrt{\log x}}\Big) \qquad \text{as } x \to \infty.$$

We will also see that if the Riemann Hypothesis holds, then the following much more precise estimate is valid:

$$\text{(B)} \qquad \pi(x) = \operatorname{Li} x + O(x^{\frac{1}{2}} \log x) \qquad \text{as } x \to \infty.$$

Use (A) to prove that there exists a real constant A such that, for any $0 < c_1 < c$, we have:

$$\sum_{p \le x} \frac{1}{p} = \log\log x + A + O\Big(e^{-c_1\sqrt{\log x}}\Big) \qquad \text{as } x \to \infty.$$

Prove also that if (B) holds, then we even have:

$$\sum_{p \le x} \frac{1}{p} = \log\log x + A + O\Big(x^{-\frac{1}{2}} \log x\Big) \qquad \text{as } x \to \infty.$$

(10p)

Problem 3. Let $q \in \mathbb{Z}^+$. Prove that all Dirichlet characters modulo q are real if and only if $q \in \{1, 2, 3, 4, 6, 8, 12, 24\}$.

(10p)

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Problem 4. (a) Prove the following fact mentioned in a lecture: Given any complex numbers $u_1, u_2, \ldots$ with $\sum_{n=1}^{\infty} |u_n| < \infty$, we have

$$\prod_{n=1}^{\infty}(1+u_n) = \sum_{\substack{A \subset \mathbb{Z}^+ \\ (A \text{ finite})}} \prod_{n \in A} u_n,$$

where the sum in the right hand side is absolutely convergent. (The sum in the right hand side runs over all finite subsets A of $\mathbb{Z}^+$, including $A = \emptyset$.)

(b) Prove that for any $x \in \mathbb{C}$ with $|x| < 1$,

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=1}^{\infty} p(n) x^n$$

with

$$p(n) := \#\Big\{ (a_k) \in S \, : \, \sum_{k=1}^{\infty} k a_k = n \Big\},$$

where S denotes the family of all sequences $(a_k) = (a_1, a_2, \ldots)$ of non-negative integers. As part of the proof, show that both the product and the sum are absolutely convergent.

[Hint: Both in (a) and (b), ideas from the proof of LN Lemma 2.8 can be useful.]

(10p)

Problem 5. Prove that for every $\varepsilon > 0$, the set of all positive integers $n \leq N$ which have no prime divisors larger than N^{ε}, has cardinality $\gg_{\varepsilon} N$ as $N \to \infty$.

[Hint: One way to proceed is as follows. First prove that it is no restriction to assume that $\varepsilon = k^{-1}$ for some $k \in \mathbb{Z}^+$. Next prove that for every integer of the form $n = mp_1 \cdots p_k$ where $m \in \mathbb{Z}^+$ and $p_1, \ldots, p_k$ are primes in the interval $N^{\varepsilon - \varepsilon^2} < p_1, \ldots, p_k < N^{\varepsilon}$, if $n \leq N$ then n has no prime divisor larger than N^{ε}. It follows that, after sorting out certain issues of 'overrepresentation', the cardinality in question can be bounded from below by a sum of the form $\sum_{p_1, \ldots, p_k} \left[\frac{N}{p_1 \cdots p_k} \right]$, and such a sum can be bounded from below using Theorem 13.6 in Baker's book.]

(10p)

GOOD LUCK!