Analytic Number Theory 2023; Assignment 2

Problem 1. For each $n \in \mathbb{Z}^+$, let $\omega(n)$ be the number of prime factors of n, not counting multiplicity. Thus, e.g., $\omega(1) = 0$, $\omega(8) = 1$ and $\omega(10) = 2$. Set $S(x) = \sum_{1 \le n \le x} 2^{\omega(n)}$. The goal of the following problem is to prove an asymptotic formula for $S(x)$ as $x \to \infty$, by mimicking the proof of the prime number theorem.

(a). Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$.

(b). Set $S_1(x) = \int_0^x S(u)\,du$ $(x > 0)$. Prove that

$$S_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{\zeta(s)^2}{\zeta(2s)}\,ds$$

for any $x > 0$ and any $c > 1$.

(c). Using (b), prove that $S_1(x) \sim \frac{x^2 \log x}{2\zeta(2)}$ as $x \to \infty$.

(d). Using (c), prove that $S(x) \sim \frac{x \log x}{\zeta(2)}$ as $x \to \infty$.

[Comments/hints: In this case, one can use a simpler contour of integration than in the proof of Theorem 7.9 in LN; namely, one can move the contour to the vertical line $\operatorname{Re} s = \delta$ for any fixed $\frac{1}{2} < \delta < 1$. We also note that in the present case, by working a bit more carefully, one can prove the more precise formula $S(x) = \frac{x \log x}{\zeta(2)} + cx + O(x^{\delta_1})$ as $x \to \infty$, for certain fixed constants $c \in \mathbb{R}$ and $\delta_1 < 1$. You may like to try to do this, although it won't earn you any extra score on the problem.]

(20p)

Problem 2. (a) Prove that for any $a, b \in \mathbb{R}_{>0}$:

$$\prod_{n=1}^{\infty} \frac{n(n+a+b)}{(n+a)(n+b)} = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

(b) Use the fact that $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ to prove that for all $t \in \mathbb{R}$:

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}}.$$

(10p)

Problem 3. Prove from the Siegel-Walfisz theorem that for any $\varepsilon > 0$, $q \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$ with $(a, q) = 1$, the smallest prime $p \equiv a \bmod q$ satisfies $p \ll e^{q^{\varepsilon}}$, where the implied constant depends only on ε.

(10p)

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Problem 4. Let $d(n) := \#\{a \in \mathbb{Z}^+ \ : \ a \mid n\}$ (the divisor function). For any $\delta > 0$, show that $d(n) < 2^{(1+\delta)\log n/\log\log n}$ for all n sufficiently large.

[Hint: One approach is to prove that for every $\varepsilon > 0$ we have $d(n) \leq C(\varepsilon) \cdot n^{\varepsilon}$ ($\forall n \in \mathbb{Z}^+$), with an explicit constant $C(\varepsilon) > 0$. If your $C(\varepsilon)$ is not too wasteful, the desired inequality can then be obtained by choosing ε depending on n appropriately.] (10p)

GOOD LUCK!