Analytic Number Theory 2023; Assignment 3

Problem 1. Show that any positive definite binary quadratic form of discriminant -19 is equivalent to

$$Q_0(x, y) = x^2 + xy + 5y^2.$$

Show also that for every positive integer n with $19 \nmid n$, we have

$$R'(n, Q_0) = 0$$

if n is divisible by some prime p which is congruent to -9 or -7 or -6 or -5 or -4 or -1 of 2 or 3 or 8 modulo 19, and otherwise

$$R'(n, Q_0) = 2^{s+1},$$

where s is the number of distinct primes dividing n.

(12p)

Problem 2. a) Let $R(n)$ denote the number of ways of writing n as a sum of a prime and a square-free number. Prove that

$$R(n) = \sum_{d \le \sqrt{n}} \mu(d) \pi(n-1; d^2, n), \qquad \forall n \in \mathbb{Z}^+.$$

b) Using the formula in a), prove that for every $A > 0$ we have

$$R(n) = \mathrm{Li}(n) \cdot \prod_{p \nmid n} \Big(1 - \frac{1}{p(p-1)}\Big) + O_A\Big(\frac{n}{(\log n)^A}\Big), \qquad \forall n \ge 2.$$

[Hint: Estimate $\pi(n-1; d^2, n)$ using Siegel-Walfisz when it is applicable, and using trivial bounds in the remaining cases.]

(12p)

Problem 3. Use the product formula for Θ to prove:

(a) The "triangular number" identity

$$\prod_{n=0}^{\infty} (1 + x^n)(1 - x^{2n+2}) = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2},$$

which holds for $|x| < 1$.

(b) The "septagonal number" identity

$$\prod_{n=0}^{\infty} (1 - x^{5n+1})(1 - x^{5n+4})(1 - x^{5n+5}) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(5n+3)/2},$$

which holds for $|x| < 1$.

(12p)

Problem 4. For each odd prime p, let η_p be the smallest positive integer which is not a quadratic residue mod p. For any real numbers $x \geq y \geq 1$, let $\mathcal{P}_{x,y}$ be the set of odd primes $p \leq x$ such that $\eta_p > y$, and let $\mathcal{A}_y$ be the set of all positive integers which contain only primes $\leq y$ in their prime factorizations.

(a) Prove that for every $p \in \mathcal{P}_{x,y}$, all elements in $\mathcal{A}_y$ are quadratic residues mod p.

(b) Using part (a) and the large sieve, prove that

$$\#\big(\mathcal{A}_y \cap (0, x^2]\big) \ll \frac{x^2}{\#\mathcal{P}_{x,y}},$$

where the implied constant is absolute.

(c) Fix $\varepsilon > 0$. Using part (b) and homework problem 1:5, prove that for any $x > 0$, the number of primes $p \leq x$ satisfying $\eta_p > x^\varepsilon$ is bounded above by a constant which only depends on ε.

(d) Using part (c), prove that the number of primes $p \leq x$ which satisfy $\eta_p > p^\varepsilon$, is $\ll_\varepsilon \log\log x$.

(14p)

GOOD LUCK!