Hints / short solution sketches to problems

2.1. (b). By Proposition 2.7,

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)\chi(p)}{p^s} + \frac{\mu(p^2)\chi(p^2)}{p^{2s}} + \frac{\mu(p^3)\chi(p^3)}{p^{3s}} + \ldots\right)$$
$$= \prod_p \left(1 - \frac{\chi(p)}{p^s}\right) = \left(\prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}\right)^{-1} = L(s,\chi)^{-1}.$$

□

2.2. By Proposition 2.7,

$$\sum_{n=1}^{\infty} \phi(n) n^{-s} = \prod_p (1 + \phi(p)p^{-s} + \phi(p^2)p^{-2s} + \ldots) = \prod_p \left(1 + \sum_{k=1}^{\infty} (p-1)p^{k-1} \cdot p^{-ks}\right)$$
$$= \prod_p \left(1 + (p-1)p^{-s} \sum_{m=0}^{\infty} p^{m(1-s)}\right) = \prod_p \left(1 + \frac{(p-1)p^{-s}}{1-p^{1-s}}\right) = \prod_p \frac{1-p^{-s}}{1-p^{1-s}} = \frac{\zeta(s-1)}{\zeta(s)}.$$

2.7.

(a) For example one may take any real numbers $a_1, a_2, \ldots > 1$, satisfying $a_n \to 1$ as $n \to \infty$, and then set $u_{2n-1} = a_n - 1$ and $u_{2n} = a_n^{-1} - 1$ for $n = 1, 2, \ldots$. Then

$$\prod_{n=1}^{N} (1 + u_n) = \begin{cases} 1 & \text{if } 2 \mid N, \\ a_{(N+1)/2} & \text{if } 2 \nmid N, \end{cases}$$

and hence $\prod_{n=1}^{\infty}(1 + u_n)$ converges. On the other hand

$$\sum_{n=1}^{2N} u_n = \sum_{n=1}^{N} \left(a_n + a_n^{-1} - 2\right) = \sum_{n=1}^{N} \frac{(a_n - 1)^2}{a_n},$$

and hence if we take, for example, $a_n := 1 + n^{-1/3}$, then $\sum_{n=1}^{\infty} u_n$ diverges. □

(b) For example one may take any positive numbers $a_1, a_2, \ldots$ with $a_n \to 0$ as $n \to \infty$, and set $u_{2n-1} = ia_n$ and $u_{2n} = -ia_n$ for $n = 1, 2, 3, \ldots$. Then

$$\sum_{n=1}^{N} u_n = \begin{cases} 0 & \text{if } 2 \mid N, \\ u_N & \text{if } 2 \nmid N, \end{cases}$$

and hence the sum $\sum_{n=1}^{\infty} u_n$ converges. On the other hand we have $\prod_{n=1}^{2N}(1 + u_n) = \prod_{n=1}^{N}(1 + a_n^2) \geq \sum_{n=1}^{N} a_n^2$, and hence if we let, e.g., $a_n = n^{-1/3}$ for all n, then $\sum_{n=1}^{N} a_n^2 \to +\infty$ as $N \to \infty$, and hence $\prod_{n=1}^{2N}(1 + u_n) \to +\infty$ as $N \to \infty$, i.e. $\prod_{n=1}^{\infty}(1 + u_n)$ does not converge. □

2.8. We have

$$\prod_{n=2}^{2N+1}\Big(1+\frac{(-1)^n}{\sqrt{n}}\Big)=\prod_{k=1}^{N}\Big(\Big(1+\frac{1}{\sqrt{2k}}\Big)\Big(1-\frac{1}{\sqrt{2k+1}}\Big)\Big)$$
$$=\prod_{k=1}^{N}\Big(1+\frac{1}{\sqrt{2k}}-\frac{1}{\sqrt{2k+1}}-\frac{1}{\sqrt{2k(2k+1)}}\Big),$$

and here (e.g. by the Mean Value Theorem) $0<\frac{1}{\sqrt{2k}}-\frac{1}{\sqrt{2k+1}}\leq\frac{1}{2(2k)^{3/2}}$; hence if we set

$$u_k:=-\frac{1}{\sqrt{2k}}+\frac{1}{\sqrt{2k+1}}+\frac{1}{\sqrt{2k(2k+1)}},$$

then for all sufficiently large k we have $1>u_k\gg k^{-1}$. Hence by Proposition 2.6, for all sufficiently large K we have $\prod_{k=K}^{\infty}(1-u_k)=0$. Hence the same also holds for $K=1$, i.e. $\prod_{k=1}^{\infty}(1-u_k)=0$, and it follows that the product $\prod_{n=2}^{\infty}\big(1+\frac{(-1)^n}{\sqrt{n}}\big)$ also converges to zero. $\square$

3.4. (a). Let A_1 be an arbitrary number $> A$. Then there is some $R > 0$ such that $\frac{\log N(r)}{\log r} < A_1$ for all $r \geq R$, hence by exponentiating: $N(r) < r^{A_1}$ for all $r \geq R$. It follows that

$$N(r) \ll (1+r)^{A_1}, \qquad \forall r \geq 0. \tag{1}$$

Now note that for every $\alpha > 0$:

$$\begin{aligned} \sum_{j=1}^{\infty} (1+|\rho_j|)^{-\alpha} &= \int_{0-}^{\infty} (1+r)^{-\alpha}\, dN(r) \\ &= \lim_{R\to\infty} \Big((1+R)^{-\alpha} N(R) + \alpha \int_0^R (1+r)^{-\alpha-1} N(r)\, dr \Big). \end{aligned} \tag{2}$$

Using here (1), it follows that $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha}$ converges for all $\alpha > A_1$. We have proved this for every $\alpha > A_1$ and every $A_1 > A$; hence $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha}$ converges for every $\alpha > A$, and thus by the definition of τ we have $\tau \leq A$. $\square$

(b). Assume that $\alpha > \tau$. Then $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha}$ converges, and since $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha} \geq N(R)\cdot(1+R)^{-\alpha}$ for all $R > 0$, it follows that there is a constant $C > 0$ such that $N(R)\cdot(1+R)^{-\alpha} \leq C$ for all R. Hence (by taking the logarithm)

$$\log N(R) \leq \log C + \alpha \log(1+R), \qquad \forall R > 0,$$

and dividing by $\log R$ (assuming $R > 1$) and then letting $R \to +\infty$, it follows that

$$\limsup_{R\to\infty} \frac{\log N(R)}{\log R} \leq 0 + \alpha, \tag{3}$$

i.e. $A \leq \alpha$. Since this is true for all $\alpha > \tau$ we conclude that $A \leq \tau$. $\square$

3.5. Set $A(x) = \sum_{1 \le n \le x} a_n$; then the assumption says that $A(x) \sim x^2$ as $x \to \infty$, and hence for any given $\varepsilon > 0$ there is some $X > 1$ such that

$$\left|A(x) - x^2\right| < \varepsilon x^2, \qquad \forall x \ge X. \tag{4}$$

Now for each $N \in \mathbb{Z}^+$ we have

$$\sum_{n=1}^{N} a_n (N-n)^2 = \int_{1-}^{N} (N-x)^2 \, dA(x) = 0 + 2\int_1^N (N-x)A(x)\,dx$$

If $A(x) \equiv x^2$ then the last expression equals

$$2\int_1^N (N-x)x^2\,dx = 2\Big[\frac{N}{3}x^3 - \frac{1}{4}x^4\Big]_{x=1}^{x=N} = \frac{1}{6}N^4 - \frac{2}{3}N + \frac{1}{2}.$$

Hence for our general $A(x) = \sum_{1 \le n \le x} a_n$ we have, for each integer $N > X$:

$$\Big|\sum_{n=1}^{N} a_n (N-n)^2 - \frac{1}{6}N^4\Big| = \left|2\int_1^N (N-x)A(x)\,dx - 2\int_1^N (N-x)x^2\,dx - \frac{2}{3}N + \frac{1}{2}\right|$$

$$\le 2\int_1^N (N-x)\big|A(x) - x^2\big|\,dx + \frac{2}{3}N + \frac{1}{2}$$

$$\le 2\int_1^X N\big|A(x) - x^2\big|\,dx + 2\int_X^N N \cdot \varepsilon x^2\,dx + \frac{2}{3}N + \frac{1}{2}$$

$$\le 2N\int_1^X \big|A(x) - x^2\big|\,dx + 2\varepsilon\int_X^N N^3\,dx + \frac{2}{3}N + \frac{1}{2}$$

$$\le 2\varepsilon N^4 + \left(2\int_1^X \big|A(x) - x^2\big|\,dx + \frac{2}{3}\right)N + \frac{1}{2}$$

The expression inside the last parenthesis does not depend on N, and hence for all sufficiently large N the above is $< 3\varepsilon N^4$, i.e. we have proved that for all sufficiently large N we have

$$\Big|\sum_{n=1}^{N} a_n (N-n)^2 - \frac{1}{6}N^4\Big| < 3\varepsilon N^4.$$

Since ε was arbitrarily small this implies that

$$\sum_{n=1}^{N} a_n (N-n)^2 \sim \frac{1}{6}N^4 \qquad \text{as } N \to \infty.$$

□

3.13.
(a). Writing $z = x + iy$ (with $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$), we have

$$|m + nz|^2 = (m + nx)^2 + (ny)^2.$$

Now let us note that

$$(m + nx)^2 + (ny)^2 \geq c \cdot (m^2 + n^2), \qquad \forall (m, n) \in \mathbb{R}^2, \tag{5}$$

where

$$c = c(x, y) = \frac{y^2}{x^2 + y^2 + 1} > 0. \tag{6}$$

One way to prove (5) is to note that the quadratic form in the left hand side of (5), which has the matrix $\begin{pmatrix} 1 & x \\ x & x^2 + y^2 \end{pmatrix}$, has the two eigenvalues[1]

$$\frac{1}{2}\Big(x^2 + y^2 + 1 \pm \sqrt{(x^2 + y^2 + 1)^2 - 4y^2}\Big),$$

and since both these eigenvalues are positive, the inequality in (5) holds with c being equal to the smallest eigenvalue, viz., with

$$\begin{aligned} c &= \frac{1}{2}\Big(x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4y^2}\Big) \\ &= \frac{2y^2}{x^2 + y^2 + 1 + \sqrt{(x^2 + y^2 + 1)^2 - 4y^2}} \qquad (> 0). \end{aligned} \tag{7}$$

Hence (5) is also valid for any *smaller* value of c; in particular (5) is valid for c as in (6).

A more elementary (but essentially equivalent) treatment: We wish to find some constant $c > 0$ such that (5) holds. viz.,

$$(1 - c)m^2 + 2xmn + (x^2 + y^2 - c)n^2 \geq 0, \qquad \forall (m, n) \in \mathbb{R}^2.$$

Completing the square, this is equivalent with

$$(1 - c)m^2 + 2xmn + (x^2 + y^2 - c)n^2 \geq 0, \qquad \forall (m, n) \in \mathbb{R}^2.$$

Clearly for this to hold we must have $1 - c \geq 0$. Assuming $1 - c > 0$, we can complete the square to see that the above is equivalent with

$$(1 - c)\Big(m + \frac{x}{1 - c}n\Big)^2 + \Big(x^2 + y^2 - c - \frac{x^2}{1 - c}\Big)n^2 \geq 0, \qquad \forall (m, n) \in \mathbb{R}^2,$$

and this is clearly true if and only if $x^2 + y^2 - c - \frac{x^2}{1-c} \geq 0$. Solving for c [2] we reach again the conclusion that the above is true for c as in (7), or any smaller c-value.

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[1] These eigenvalues are real, since $(x^2 + y^2 + 1)^2 - 4y^2 \geq (y^2 + 1)^2 - 4y^2 = (y^2 - 1)^2 \geq 0$.

[2] Encountering again the characteristic polynomial of the matrix $\begin{pmatrix} 1 & x \\ x & x^2 + y^2 \end{pmatrix}$.

Using (5) we have

$$\left|\frac{1}{(m+nz)^{2k}}\right| \leq c^{-k}\cdot(m^2+n^2)^{-k},$$

and hence in order to prove the uniform absolute convergence required in the problem, it suffices to prove that the series

$$c^{-k}\sum_{(m,n)\neq(0,0)}\frac{1}{(m^2+n^2)^k}$$

is uniformly absolutely convergent for $z = x+iy$ in any compact subset of $\mathbf{H}$. But when $z = x+iy$ ranges over a given compact subset of $\mathbf{H}$, the number $c = c(x,y)$ (see (6)) is bounded from below by a positive number; hence c^{-k} is bounded from above by a finite number. Hence it now suffices to prove that the series

$$\sum_{(m,n)\neq(0,0)}\frac{1}{(m^2+n^2)^k}$$

converges!

This can be done e.g. using dyadic decomposition: Let

$$A(R) = \{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\} \;:\; m^2+n^2<R\}.$$

Then, $A(1)=\emptyset$ for $R<1$, and for $R\geq 1$ we have (as a quite crude bound):

$$\begin{aligned}\#A(R) &\leq \#\{(m,n)\in\mathbb{Z}^2 \;:\; |m|<\sqrt{R} \text{ and } |n|<\sqrt{R}\}\\ &\leq \big(1+2\sqrt{R}\big)^2 \leq \big(3\sqrt{R}\big)^2 \leq 9R.\end{aligned}$$

Hence (using $A(1)=\emptyset$, i.e. $m^2+n^2\geq 1$ for all $(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}$):

$$\begin{aligned}\sum_{(m,n)\neq(0,0)}\frac{1}{(m^2+n^2)^k} &\leq \sum_{j=0}^{\infty}\sum_{(m,n)\in A(2^{j+1})\setminus A(2^j)}\frac{1}{(2^j)^k} \leq \sum_{j=0}^{\infty}\#A(2^{j+1})\cdot 2^{-jk}\\ &\leq \sum_{j=0}^{\infty}9\cdot 2^{j+1}\cdot 2^{-jk}<\infty,\end{aligned}$$

where in the last step we used the assumption that $k\geq 2$. $\square$

(b). First note that $\frac{az+b}{cz+d} \in \mathbf{H}$ for every $z \in \mathbf{H}$; indeed,

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)\overline{(cz+d)}}{|cz+d|^2}\right) = \frac{\mathrm{Im}\,(adz+bc\overline{z})}{|cz+d|^2} = \frac{(ad-bc)\mathrm{Im}\,z}{|cz+d|^2}$$
$$= \frac{\mathrm{Im}\,z}{|cz+d|^2} > 0.$$

Now we compute (using the absolute convergence proved in (a)):

$$\begin{aligned} E_k\left(\frac{az+b}{cz+d}\right) &= \sum_{(m,n)\neq(0,0)} \frac{1}{\left(m+n\frac{az+b}{cz+d}\right)^{2k}} \\ &= \sum_{(m,n)\neq(0,0)} \frac{(cz+d)^{2k}}{(m(cz+d)+n(az+b))^{2k}} \\ &= \sum_{(m,n)\neq(0,0)} \frac{(cz+d)^{2k}}{((md+nb)+(mc+na)z)^{2k}}. \end{aligned}$$

Now note that the map

$$(m,n) \mapsto \big(md+nb, mc+na\big) = (m,n)\begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

is a permutation of $\mathbb{Z}^2 \setminus \{(0,0)\}$, with inverse

$$(m',n') \mapsto (m',n')\begin{pmatrix} a & -c \\ -b & d \end{pmatrix}.$$

Hence we get

$$E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k}E_k(z),$$

qed. □