Analysis for PhD students (2013)
Assignment 3

Problem 1. In Folland's book: p. 289, Problem 1.

(15p)

Problem 2. In Folland's book: p. 289, Problem 10.

(15p)

Problem 3. Prove that if $\psi \in C^\infty(\mathbb{R}^n)$ is a slowly increasing function then $\phi \mapsto \psi\phi$ is a continuous map of $\mathcal{S}$ into $\mathcal{S}$.

(15p)

Problem 4. Find the Fourier transform (in $\mathcal{S}'$) of the following functions on $\mathbb{R}$:
a) $x \mapsto x \sin x$. b) $x \mapsto x \sin^2 x$. c) $x \mapsto x^{-1} \sin x$.
d) $x \mapsto (\sin x)^k$ (for $k \in \mathbb{N}$). e) $x \mapsto \sin |x|$.

(15p)

Problem 5. Let m, n be even positive integers. Prove that the function $f(x) = \exp(x^n + i \exp(x^m))$ on $\mathbb{R}$ lies in $\mathcal{S}'$ if and only if $n \leq m$.

(15p)

Problem 6. In Folland's book:
a) p. 255, Problem 18(a). b) p. 308, Problem 31.

(15p)

Problem 7. Let $F_0 = 0, F_1 = 1, F_2 = 1, \ldots$ be the Fibonacci sequence. Fix $k \in \mathbb{N}$. Prove that for almost every $x \in (0, 1)$, the pattern $1, \ldots, 1$ (k digits 1) appears in the continued fraction expansion $x = [a_1, a_2, \ldots]$ with frequency $(-1)^k \log(1 + (-1)^k F_{k+2}^{-2}) / \log 2$, that is:

$$\lim_{n\to\infty} \frac{1}{n} \#\big\{ j \in \{1, \ldots, n\} \ : \ a_j = a_{j+1} = \ldots = a_{j+k-1} = 1 \big\} = (-1)^k \frac{\log(1 + (-1)^k F_{k+2}^{-2})}{\log 2}.$$

(15p)

Problem 8. Let π_1, π_2 be the coordinate projections from $\mathbb{R}^2$ to $\mathbb{R}$, i.e. $\pi_j((x_1, x_2)) = x_j$, and let $b^{(1)}, b^{(2)} \in \mathbb{R}^2$ be a basis of $\mathbb{R}^2$ such that the two real numbers $\pi_2(b^{(1)}), \pi_2(b^{(2)})$ are linearly independent over $\mathbb{Q}$. Let $\mathcal{L} \subset \mathbb{R}^2$ be the lattice spanned by $b^{(1)}, b^{(2)}$, i.e.

$$\mathcal{L} = \mathbb{Z}b^{(1)} + \mathbb{Z}b^{(2)} = \left\{ j_1 b^{(1)} + j_2 b^{(2)} \ : \ j_1, j_2 \in \mathbb{Z} \right\}.$$

Prove that for any fixed bounded open interval $J \subset \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{2T} \# \Big(([-T,T] \times J) \cap \mathcal{L} \Big) = m(J) \left| \det \begin{pmatrix} b_1^{(1)} & b_2^{(1)} \\ b_1^{(2)} & b_2^{(2)} \end{pmatrix} \right|^{-1}.$$

(15p)

Submission deadline: 29 April, 10.15 (i.e. before the problem discussion starts).