NOTES FOR THE COURSE "ANALYSIS FOR PHD STUDENTS"

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Contents

1. Lecture 1: Sums and integrals

1.1. **Introductory examples.** Integration and summation are very closely related. Indeed, integrals are *defined* using sums. Furthermore, the general integral (cf., e.g., Folland Ch. 2) is a *generalization* of the concept of a sum; the latter is obtained from the former when the measure of integration is taken to be a counting measure. However in this first lecture I'd like to focus on some explicit connections between sums and the "elementary, first-year-calculus integral $\int f(x)\,dx$". Our focus will be on transforming sums into integrals, since the latter are often easier to work with.

A well-known explicit connection between sums and integrals is the following:

Example 1.1. Let $M < N$ be integers and let f be any *increasing* function $[M-1, N+1] \to \mathbb{R}$. Then

$$\int_{M-1}^{N} f(x)\,dx \le \sum_{n=M}^{N} f(n) \le \int_{M}^{N+1} f(x)\,dx.$$

Indeed, "draw a picture"! A similar example: Suppose that f is any *convex* function $[M-\frac{1}{2}, N+\frac{1}{2}] \to \mathbb{R}$. Then

$$\sum_{n=M}^{N} f(n) \le \int_{M-\frac{1}{2}}^{N+\frac{1}{2}} f(x)\,dx.$$

Indeed, again "draw a picture"!

Another familiar way in which integrals can sometimes be used to estimate sums is if the sum can be recognized as a *Riemann sum* (we will recall the definition of a Riemann integral using Riemann sums below; see Section 1.2). For example this method can be applied to the following question:

Example 1.2. Given a fixed number $\alpha > -1$, what is the asymptotic behavior of the sum $\sum_{n=1}^{N} n^\alpha$ as $N \to \infty$?

One solution is to rewrite the sum as

$$\sum_{n=1}^{N} n^\alpha = N^{\alpha+1} \sum_{n=1}^{N} \Big(\frac{n}{N}\Big)^\alpha \frac{1}{N} = N^{\alpha+1} \sum_{n=1}^{N} \Big(\frac{n}{N}\Big)^\alpha \Big(\frac{n}{N} - \frac{n-1}{N}\Big).$$

Here the last sum, $\sum_{n=1}^{N} (\frac{n}{N})^\alpha (\frac{n}{N} - \frac{n-1}{N})$, can be recognized as a Riemann sum for the integral $\int_0^1 x^\alpha\,dx$, and from this we conclude that the sum tends to the value of $\int_0^1 x^\alpha\,dx$ as $N \to \infty$. Hence:

$$N^{-\alpha-1} \sum_{n=1}^{N} n^\alpha \to \int_0^1 x^\alpha\,dx = \frac{1}{\alpha+1}, \quad \text{as } N \to \infty.$$

The answer can be expressed:

$$\sum_{n=1}^{N} n^\alpha \sim \frac{N^{\alpha+1}}{\alpha+1} \qquad \text{as } N \to \infty. \tag{1.1}$$

Here we used the relation "$\sim$", which is defined as follows:

Definition 1.1. We write "$f(x) \sim g(x)$ as $x \to a$" to denote that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. Here a can be any real number, or $\pm\infty$. Note that this notation can only be used when $g(x) \neq 0$ for all x sufficiently near a.

Note that the same answer (1.1) could also be obtained, actually in a more precise form, using the technique of Example 1.1. Namely, let's *assume* $\alpha \geq 0$ so that the function $f(x) = x^\alpha$ is increasing (the other case $-1 < \alpha < 0$ can be treated similarly). Then

$$\int_0^N x^\alpha \, dx \leq \sum_{n=1}^{N} n^\alpha \leq \int_1^{N+1} x^\alpha \, dx,$$

i.e. (for $\alpha \geq 0$)

$$\frac{N^{\alpha+1}}{\alpha+1} \leq \sum_{n=1}^{N} n^\alpha \leq \frac{(N+1)^{\alpha+1} - 1}{\alpha+1} \qquad (\forall N \in \mathbb{N}). \tag{1.2}$$

This is clearly a more precise result than (1.1). We can deduce from (1.2) that $\sum_{n=1}^{N} n^\alpha$ equals $\frac{N^{\alpha+1}}{\alpha+1}$ plus a "lower order error", namely:

$$\sum_{n=1}^{N} n^\alpha = \frac{N^{\alpha+1}}{\alpha+1} + O(N^\alpha), \quad \forall N \in \mathbb{N} \quad (\text{for fixed } \alpha \geq 0). \tag{1.3}$$

Here the symbol "$O(\cdots)$" ("Big O") is defined as follows:

Definition 1.2. If a is a *non-negative* number, the symbol "$O(a)$" is used to denote any number b for which $|b| \leq Ca$, where C is a positive "constant", called *the implied constant.* We write "constant" within quotation marks since C is often allowed to depend on certain parameters.

We will discuss the "Big O" symbol and the implied constant more thoroughly in later lectures; for now we just give an exercise:

Exercise 1.1. Deduce (1.3) from (1.2); note that we then have to allow the implied constant in (1.3) to depend on α. But of course the implied constant is independent of N — this is the whole point of the statement (1.3)!

We now turn to a slightly different example:

Example 1.3. Assume that we are given an increasing sequence of positive numbers, $0 < \omega_1 \leq \omega_2 \leq \cdots$, which satisfy

$$\#\{n \in \mathbb{N} : \omega_n \leq T\} \sim cT^2 \qquad \text{as } T \to \infty, \tag{1.4}$$

where $c > 0$ is some constant. Then for which real numbers α do the series $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ converge? When convergence holds, can we estimate $\sum_{\omega_n > T} \omega_n^{-\alpha}$ as a function of T as $T \to \infty$?

(Notation: “$\sum_{\omega_n > T}$” means that we add over all n which satisfy the condition $\omega_n > T$.)

(To motivate the example, let us point out that the ω_n’s may e.g. be the square roots of the non-zero eigenvalues of the Dirichlet problem for some bounded domain $\Omega \subset \mathbb{R}^2$ — in other words the eigenfrequencies of vibration of a given idealized “drum” in the plane. then (1.4) is known to hold, with $c = (4\pi)^{-1}$, by the famous *Weyl’s law.* In the study of such systems, sums like $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ are often of interest.)

Note that the sum $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ is a *positive* sum; each term is positive. If we only care about “order of magnitude”, viz. if we are willing to sacrifice a numerical constant in our bounds, then questions about the asymptotic size of positive sums can often be answered using *dyadic decomposition.* We illustrate this for the first question in Example 1.4:

Note that $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ clearly diverges when $\alpha \leq 0$; hence from now on we may assume $\alpha > 0$. Our sum can be decomposed as:

$$\sum_{n=1}^{\infty} \omega_n^{-\alpha} = \sum_{\omega_n \leq 1} \omega_n^{-\alpha} + \sum_{m=0}^{\infty} \Big(\sum_{2^m < \omega_n \leq 2^{m+1}} \omega_n^{-\alpha} \Big). \tag{1.5}$$

(This is the so-called *dyadic decomposition.*) Using $\alpha > 0$ we see that (1.5) is

$$\geq \sum_{m=0}^{\infty} \#\{2^m < \omega_n \leq 2^{m+1}\} \cdot 2^{-(m+1)\alpha}$$

and[1]

$$\leq \sum_{\omega_n \leq 1} \omega_n^{-\alpha} + \sum_{m=0}^{\infty} \#\{2^m < \omega_n \leq 2^{m+1}\} \cdot 2^{-m\alpha}.$$

The cardinalities appearing in these two bounds are precisely the cardinalities which (1.4) gives us information about! Namely, if we set

$$A(T) = \#\{\omega_n \leq T\} \qquad \text{for } T > 0,$$

[1] We here use the shorthand notation “$\{a < \omega_n \leq b\}$” for “$\{n \in \mathbb{N} : a < \omega_n \leq b\}$”.

then the bounds which we pointed out above read:

$$\sum_{m=0}^{\infty}\big(A(2^{m+1})-A(2^m)\big)\cdot 2^{-(m+1)\alpha}\leq\sum_{n=1}^{\infty}\omega_n^{-\alpha}$$
$$\leq\sum_{\omega_n\leq 1}\omega_n^{-\alpha}+\sum_{m=0}^{\infty}\big(A(2^{m+1})-A(2^m)\big)\cdot 2^{-m\alpha}, \tag{1.6}$$

and (1.4) says that $A(T)\sim cT^2$ as $T\to\infty$.

Let us note that apart from the sum $\sum_{\omega_n\leq 1}\omega_n^{-\alpha}$ (which is finite since $A(1)$ is finite, by (1.4)), the lower and the upper bound in (1.6) only differ by the constant factor $2^{-\alpha}$. This is the central point of dyadic decomposition: In favorable situations the total contribution from each individual "dyadic interval" can be estimated from above and below by some simple expressions which only differ up to a multiplicative constant! (We could get rid of the sum $\sum_{\omega_n\leq 1}\omega_n^{-\alpha}$ by applying dyadic decomposition also to the interval $0<\lambda\leq 1$, i.e. writing the sum in (1.5) as $\sum_{m=-\infty}^{\infty}\sum_{2^m<\omega_n\leq 2^{m+1}}\omega_n^{-\alpha}$; however we won't need this in the present discussion.)

Continuing, we note that $A(2^{m+1})-A(2^m)\geq 0$ for each $m\geq 0$, and from (1.4) it follows that

$$\#\big(A(2^{m+1})-A(2^m)\big)\sim 3c\cdot 2^{2m}\qquad\text{as }\ m\to\infty.$$

Using this and the bounds in (1.6), the convergence/divergence of $\sum_{n=1}^{\infty}\omega_n^{-\alpha}$ is seen to be equivalent to the convergence/divergence of the sum $\sum_{m=0}^{\infty}2^{2m}\cdot 2^{-\alpha m}$, and we thus conclude that $\sum_{n=1}^{\infty}\omega_n^{-\alpha}$ *converges when $\alpha>2$, and diverges when $\alpha\leq 2$.*

Remark 1.3. Another quick way to get this answer goes via noticing that (1.4) actually implies $\omega_n\sim\sqrt{n/c}$ as $n\to\infty$.

We now move on to the second question in Example 1.3: For $\alpha>2$ we know that $\sum_{n=1}^{\infty}\omega_n^{-\alpha}$ converges, and hence $\sum_{\omega_n>T}\omega_n^{-\alpha}$ is a well-defined function of T (which is clearly positive, and decreasing). We now wish to give an asymptotic estimate of this sum as $T\to\infty$. For this we will use another very important method for the asymptotic study of sums: Consider the following way to rewrite $\sum_{\omega_n>T}\omega_n^{-\alpha}$ as an integral over the counting function $A(x)$ for $x\geq T$. Using $\omega_n^{-\alpha}=\int_{\omega_n}^{\infty}\alpha x^{-\alpha-1}\,dx$ we have

$$\sum_{\omega_n>T}\omega_n^{-\alpha}=\sum_{\omega_n>T}\int_{\omega_n}^{\infty}\alpha x^{-\alpha-1}\,dx=\int_T^{\infty}\sum_{\substack{n=1\\(T<\omega_n\leq x)}}^{\infty}\alpha x^{-\alpha-1}\,dx$$
$$=\int_T^{\infty}\big(A(x)-A(T)\big)\alpha x^{-\alpha-1}\,dx. \tag{1.7}$$

(The change of order of summation here is permitted since all functions involved are nonnegative; indeed, write $\sum_{\omega_n>T}\int_{\omega_n}^{\infty}\alpha x^{-\alpha-1}\,dx$ as $\sum_{\omega_n>T}\int_T^{\infty} I(x>\omega_n)\,\alpha x^{-\alpha-1}\,dx$ and apply Folland's Theorem 2.15. [2])

Continuing from (1.7) we get:

$$\sum_{\omega_n>T}\omega_n^{-\alpha}=\int_T^{\infty}A(x)\alpha x^{-\alpha-1}\,dx-A(T)T^{-\alpha}. \tag{1.8}$$

Using here (1.4) we have:

$$\int_T^{\infty}A(x)\alpha x^{-\alpha-1}\,dx\sim\int_T^{\infty}cx^2\cdot\alpha x^{-\alpha-1}\,dx=c\frac{\alpha T^{2-\alpha}}{\alpha-2}\qquad\text{as }T\to\infty.$$

[Detailed proof of the last "$\sim$" relation: We know that $A(T)\sim cT^2$; hence given any $\varepsilon>0$ there exists some $T_0>0$ such that $(c-\varepsilon)T^2<A(T)<(c+\varepsilon)T^2$ for all $T\geq T_0$; hence for all $T\geq T_0$ we have $\int_T^{\infty}A(x)\alpha x^{-\alpha-1}\,dx\leq\int_T^{\infty}(c+\varepsilon)x^2\cdot\alpha x^{-\alpha-1}\,dx=(c+\varepsilon)\frac{\alpha T^{2-\alpha}}{\alpha-2}$ and similarly $\int_T^{\infty}A(x)\alpha x^{-\alpha-1}\,dx\geq(c-\varepsilon)\frac{\alpha T^{2-\alpha}}{\alpha-2}$. The fact that this can be achieved for each $\varepsilon>0$ leads to the desired conclusion.]

Furthermore in (1.8) we have $A(T)T^{-\alpha}\sim cT^{2-\alpha}$. Hence, since $\frac{\alpha}{\alpha-2}>1$ and $\frac{\alpha}{\alpha-2}-1=\frac{2}{\alpha-2}$, we conclude:

$$\sum_{\omega_n>T}\omega_n^{-\alpha}\sim\frac{2c}{\alpha-2}T^{2-\alpha}\qquad\text{as }T\to\infty.$$

This holds for any fixed $\alpha>2$, and we have thus answered the second question in Example 1.3.

The computation in (1.7), (1.8) is very reminiscent of *integration by parts*, and in the next section will show that it is indeed a special case of integration by parts when viewed in the framework of the *Riemann-Stieltjes integral.* Namely, $\sum_{\omega_n>T}\omega_n^{-\alpha}$ can be expressed as $\int_T^{\infty}x^{-\alpha}\,dA(x)$, and integrating by parts we get $[A(x)x^{-\alpha}]_{x=T}^{x=\infty}+\alpha\int_T^{\infty}x^{-\alpha-1}A(x)\,dx$, i.e. the formula in (1.8)!

1.2. **The Riemann-Stieltjes Integral.**

In this section we loosely follow [13, Appendix A].

Let us first recall the definition of the *Riemann integral* over a bounded interval:

[2]This is if we view the integrals as Lebesgue integrals; it is of course also possible to justify the present computation using only the Riemann integral.

Definition 1.4. Let real numbers $A < B$ and a function $g : [A, B] \to \mathbb{C}$ be given. We call a finite sequence $\{x_n\}_{n=0}^N$ is a *partition*[3] of $[A, B]$ if

$$A = x_0 \leq x_1 \leq \ldots \leq x_N = B. \tag{1.9}$$

For any partition $\{x_n\}_{n=0}^N$ of $[A, B]$ and any choice of numbers $\xi_n \in [x_{n-1}, x_n]$ for $n = 1, 2, \ldots, N$, we form the sum

$$S(\{x_n\}, \{\xi_n\}) = \sum_{n=1}^{N} g(\xi_n)(x_n - x_{n-1}). \tag{1.10}$$

We say that *the Riemann integral* $\int_A^B g(x)\,dx$ exists if there is some $I \in \mathbb{C}$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\big|S(\{x_n\}, \{\xi_n\}) - I\big| < \varepsilon \tag{1.11}$$

holds whenever $\{x_n\}$ and $\{\xi_n\}$ are as above and

$$\text{mesh}\{x_n\} = \max_{1 \leq n \leq N} (x_n - x_{n-1}) \leq \delta. \tag{1.12}$$

If this holds, then we also say that $\int_A^B g(x)\,dx$ *equals* I, and the function g is said to be *Riemann-integrable* on $[A, B]$.

It will be convenient in our discussion to call any pair of finite sequences $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ such that $\{x_n\}_{n=0}^N$ is a partition of $[A, B]$ and $\xi_n \in [x_{n-1}, x_n]$ for $n = 1, 2, \ldots, N$ a *"tagged partition of* $[A, B]$*";* we also agree that the mesh of $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ equals the mesh of $\{x_n\}$.

We will later give a precise criterion for which functions are Riemann-integrable; however let us already now point out the following fundamental result. We write $C([A, B])$ for the space of continuous functions $[A, B] \to \mathbb{C}$.

Theorem 1.5. *If* $g \in C([A, B])$ *then* g *is Riemann-integrable on* $[A, B]$.

This is a special case of Theorem 1.10 which we will prove below.

Let us also note:

Proposition 1.6. *If* $g : [A, B] \to \mathbb{C}$ *is Riemann integrable then* g *is bounded (that is, there exists some number* $M > 0$ *such that* $|g(x)| \leq M$ *for all* $x \in [A, B]$*).*

[3]Of course, this is *not* the standard notion of partition! Recall that the standard notion of a partition of a set X is: A family of nonempty subsets of X such that every element $x \in X$ belongs to exactly one of these subsets. However the two different usages of the word "partition" will not cause any confusion. Note also that the two concepts are related: If $\{x_n\}_{n=0}^N$ is a partition of $[A, B]$ in the sense of (1.9), then (e.g.) $\{[x_0, x_1), [x_1, x_2), \ldots, [x_{N-1}, X_N]\}$ is a partition of $[A, B]$ in the more standard sense.

Proof. Assume that g is *not* bounded. We will then prove that for *every* tagged partition $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ of $[A,B]$ there exists another sequence $\{\xi'_n\}_{n=1}^N$ such that also $\langle \{x_n\}_{n=0}^N, \{\xi'_n\}_{n=1}^N \rangle$ (with the *same* $\{x_n\}$!) is a tagged partition of $[A,B]$, and

$$\big|S(\{x_n\},\{\xi_n\}) - S(\{x_n\},\{\xi'_n\})\big| \geq 1. \tag{1.13}$$

Clearly this implies that g is *not* Riemann-integrable on $[A,B]$.

To prove the above claim, let $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ be a given tagged partition of $[A,B]$. Note that since g is not bounded, there is some $m \in \{1,\ldots,N\}$ such that the restriction of g to $[x_{m-1}, x_m]$ is not bounded. This implies that $x_{m-1} < x_m$ and that there is some $y \in [x_{m-1}, x_m]$ such that

$$|g(y)| \geq |g(\xi_m)| + (x_m - x_{m-1})^{-1},$$

and therefore

$$|g(y) - g(\xi_m)|(x_m - x_{m-1}) \geq 1.$$

Now define $\{\xi'_n\}_{n=1}^N$ by $\xi'_n = \xi_n$ for $n \neq m$ and $\xi'_m = y$. Then clearly $\langle \{x_n\}_{n=0}^N, \{\xi'_n\}_{n=1}^N \rangle$ is a tagged partition of $[A,B]$, and

$$\big|S(\{x_n\},\{\xi_n\}) - S(\{x_n\},\{\xi'_n\})\big| = \big|(g(\xi_m) - g(y))(x_m - x_{m-1})\big| \geq 1,$$

i.e. (1.13) holds. □

We next turn to the *Riemann-Stieltjes Integral* $\int_A^B g(x)\,df(x)$, which is a generalization of the Riemann integral. Intuitively, this integral is meant to give "$\int_A^B g(x) f'(x)\,dx$" (see Theorem 1.13 below for an aposteriori justification), but the integral exists also in many cases when $f'(x)$ does not exist for all x.

Definition 1.7. Let real numbers $A < B$ and two functions $f, g : [A,B] \to \mathbb{C}$ be given. For any tagged partition $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ of $[A,B]$, we form the sum

$$S(\{x_n\},\{\xi_n\}) = \sum_{n=1}^N g(\xi_n)\big(f(x_n) - f(x_{n-1})\big). \tag{1.14}$$

We say that *the Riemann-Stieltjes integral* $\int_A^B g\,df = \int_A^B g(x)\,df(x)$ exists and has the value I if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\big|S(\{x_n\},\{\xi_n\}) - I\big| < \varepsilon \tag{1.15}$$

whenever $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ is a tagged partition of $[A,B]$ of mesh $\leq \delta$.

Note that in the special case $f(x) = x$, Definition 1.7 specializes to Definition 1.4; hence the Riemann-Stieltjes integral is indeed a generalization of the Riemann integral!

Example 1.4. For any $A < B$ and any function $f : [A, B] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_A^B df(x)$ (viz. "$g \equiv 1$" in Definition 1.7) exists and equals $f(B) - f(A)$. This is trivial, since in this case $S(\{x_n\}, \{\xi_n\}) = f(B) - f(A)$ holds for *all* tagged partitions $\langle\{x_n\}, \{\xi_n\}\rangle$ of $[A, B]$.

Example 1.5. Let $A < B$, $g \in C([A, B])$, and assume that $f : [A, B] \to \mathbb{C}$ is piecewise constant, that is, there are numbers $A = x_0 < x_1 < x_2 < \ldots < x_n = B$ such that f is constant on each open interval (x_j, x_{j+1}), $j = 0, 1, \ldots, n-1$. Then

$$\int_A^B g\,df = \big(f(A+) - f(A)\big)g(A) + \sum_{j=1}^{n-1} \big(f(x_j+) - f(x_j-)\big)g(x_j) + \big(f(B) - f(B-)\big)g(B), \tag{1.16}$$

where

$$f(x+) = \lim_{t \to x^+} f(t) \qquad \text{and} \qquad f(x-) = \lim_{t \to x^-} f(t). \tag{1.17}$$

The proof is a simple exercise.

Example 1.6. One has to be careful when working with the general Riemann-Stieltjes integral, since some rules which are familiar from ordinary integrals may fail to hold in general. For example, it is *not* always true that if $A < C < B$ then $\int_A^B g(x)\,df(x) = \int_A^C g(x)\,df(x) + \int_C^B g(x)\,df(x)$! An example of this is the following: Suppose that

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise;} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.18}$$

Then $\int_{-1}^0 g\,df$ and $\int_0^1 g\,df$ both exist, but $\int_{-1}^1 g\,df$ does not exist! We leave the proof as an exercise. On the positive side, note that $\int_A^B g(x)\,df(x) = \int_A^C g(x)\,df(x) + \int_C^B g(x)\,df(x)$ holds whenever $\int_A^B g(x)\,df(x)$ exists (this can for example be easily proved using Lemma 1.11).

Unpleasant behavior such as in Example 1.6 typically arises in cases when both f and g have a common point of discontinuity.

A natural assumption when working with the Riemann-Stieltjes integral $\int_A^B g(x)\,df(x)$ is that f is of *bounded variation.* This concept is defined as follows:

Definition 1.8. If f is a function $f : [A, B] \to \mathbb{C}$, then the *variation of f over* $[A, B]$, $\mathrm{Var}_{[A,B]}(f)$, is defined by

$$\mathrm{Var}_{[A,B]}(f) = \sup \sum_{n=1}^{N} |f(x_n) - f(x_{n-1})|, \tag{1.19}$$

where the supremum is taken over all partitions $\{x_n\}_{n=0}^N$ of $[A,B]$. Thus $\mathrm{Var}_{[A,B]}(f)$ is a well-defined number in $[0,\infty]$ (cf. Folland, Sec. 0.5). The function f is said to be of *bounded variation* if $\mathrm{Var}_{[a,b]}(f) < \infty$. The space of all functions $f : [A,B] \to \mathbb{C}$ of bounded variation is denoted $BV([A,B])$.

The reader should check that the above definition agrees with that in Folland [4, p. 102]. Let us give an intuitive motivation of the above definition of the variation of f, following Folland [4, p. 101]: If $f(t)$ represents the position of a particle moving along the real line (or more generally in the complex plane) at time t, the “total variation” of f over the interval $[A,B]$ is the total distance traveled from time A to time B, as shown on an odometer. If f has a continuous derivative, this is just the integral of the “speed”, $\int_A^B |f'(t)|\,dt$. The above definition of $\mathrm{Var}_{[A,B]}(f)$ is simply the natural extension of “$\int_A^B |f'(t)|\,dt$” to the case when we have no smoothness hypothesis on f.

The assertion of the last sentence can be proved rigorously.

Proposition 1.9. *If $f \in C^1([A,B])$ [4] then*

$$\mathrm{Var}_{[A,B]}(f) = \int_A^B |f'(x)|\,dx. \tag{1.20}$$

In particular every function in $C^1([A,B])$ is of bounded variation, i.e. $C^1([A,B]) \subset BV([A,B])$.

We will prove Proposition 1.9 after the proof of our first main theorem:

Theorem 1.10. *Let $g \in C([A,B])$ and $f \in BV([A,B])$. Then the Riemann-Stieltjes integral $\int_A^B g\,df$ exists.*

To prepare for the proof, let us note a simple reformulation of the criterion for existence of $\int_A^B g(x)\,df(x)$:

Lemma 1.11. *The Riemann-Stieltjes integral $\int_A^B g\,df$ exists if and only if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for any two tagged partitions $\langle\{x_n\},\{\xi_n\}\rangle$ and $\langle\{x'_n\},\{\xi'_n\}\rangle$ of $[A,B]$, both having mesh $\leq \delta$, we have $|S(\{x_n\},\{\xi_n\}) - S(\{x'_n\},\{\xi'_n\})| < \varepsilon$.*

Proof. One direction is trivial: Namely, assume that $\int_A^B g\,df$ exists and equals I. Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that $|S(\{x_n\},\{\xi_n\}) - I| < \varepsilon/2$ holds for any tagged partition $\langle\{x_n\},\{\xi_n\}\rangle$

[4] As usual, $C^k([A,B])$ denotes the space of functions $f : [A,B] \to \mathbb{C}$ which are k times continuously differentiable, where at $x = A$ we only consider the *right* derivative(s), and at $x = B$ we only consider the *left* derivative(s).

of $[A,B]$ with mesh $\leq \delta$. Then if $\langle\{x_n\},\{\xi_n\}\rangle$ and $\langle\{x_n'\},\{\xi_n'\}\rangle$ are any two tagged partitions of $[A,B]$ both having mesh $\leq \delta$, we have

$$|S(\{x_n\},\{\xi_n\}) - S(\{x_n'\},\{\xi_n'\})|$$
$$\leq |S(\{x_n\},\{\xi_n\}) - I| + |S(\{x_n'\},\{\xi_n'\}) - I| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, now assume that the condition given in the lemma holds. For each $j \in \mathbb{N}$, let us fix once and for all a tagged partition $\langle\{x_n^{(j)}\},\{\xi_n^{(j)}\}\rangle$ of $[A,B]$ having mesh $\leq j^{-1}$, and set

$$I_j = S(\{x_n^{(j)}\},\{\xi_n^{(j)}\}).$$

Then our assumption implies that $\{I_j\}_{j=1}^{\infty}$ is a Cauchy sequence! Hence

$$I = \lim_{j\to\infty} I_j \in \mathbb{R}$$

exists. Now let $\varepsilon > 0$ be given. Because of our assumption there exists some $\delta > 0$ such that $|S(\{x_n\},\{\xi_n\}) - S(\{x_n'\},\{\xi_n'\})| < \varepsilon/2$ holds whenever $\langle\{x_n\},\{\xi_n\}\rangle$ and $\langle\{x_n'\},\{\xi_n'\}\rangle$ are tagged partitions of $[A,B]$ having mesh $\leq \delta$. Now fix j so large that both $j^{-1} \leq \delta$ and $|I_j - I| < \varepsilon/2$ hold, and take $\langle\{x_n'\},\{\xi_n'\}\rangle$ equal to $\langle\{x_n^{(j)}\},\{\xi_n^{(j)}\}\rangle$ in the previous statement. The conclusion is that $|S(\{x_n\},\{\xi_n\}) - I_j| < \varepsilon/2$ holds for any tagged partition $\langle\{x_n\},\{\xi_n\}\rangle$ of $[A,B]$ having mesh $\leq \delta$. Hence also

$$|S(\{x_n\},\{\xi_n\}) - I| \leq |S(\{x_n\},\{\xi_n\}) - I_j| + |I_j - I| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that $\int_A^B g\,df$ exists and equals I. □

Proof of Theorem 1.10. Let $\varepsilon > 0$ be given. Since g is continuous on the closed and bounded interval $[A,B]$, g is *uniformly* continuous on $[A,B]$; hence there exists $\delta > 0$ such that

$$|g(x) - g(x')| < \varepsilon \qquad \text{for all } x, x' \in [a,b] \text{ with } |x - x'| \leq \delta. \tag{1.21}$$

We now claim that for any two tagged partitions $\langle\{x_n\},\{\xi_n\}\rangle$ and $\langle\{x_n'\},\{\xi_n'\}\rangle$ of $[A,B]$, both having mesh $\leq \delta$, we have

$$\Big|S(\{x_n\},\{\xi_n\}) - S(\{x_n'\},\{\xi_n'\})\Big| \leq 2\varepsilon\,\mathrm{Var}_{[A,B]}(f). \tag{1.22}$$

This suffices to prove the existence of $\int_A^B g\,df$, by Lemma 1.11.

In order to prove (1.22), let us pick a tagged partition $\langle\{x_n''\},\{\xi_n''\}\rangle$ of $[A,B]$ such that both $\{x_n\}$ and $\{x_n'\}$ are subsequences of $\{x_n''\}$. We will then prove that

$$\Big|S(\{x_n\},\{\xi_n\}) - S(\{x_n''\},\{\xi_n''\})\Big| \leq \varepsilon\,\mathrm{Var}_{[A,B]}(f). \tag{1.23}$$

This will complete the proof, since exactly the same argument will also give (1.23) with $\langle\{x_n\},\{\xi_n\}\rangle$ replaced by $\langle\{x'_n\},\{\xi'_n\}\rangle$; and then (1.22) follows using the triangle inequality.

In order to prove (1.23), assume $\{x_n\} = \{x_n\}_{n=0}^N$ and $\{x''_n\} = \{x''_n\}_{n=0}^M$, and note that since $\{x_n\}_{n=0}^N$ is a subsequence of $\{x''_n\}_{n=0}^M$ there exist indices $0 = k_0 < k_1 < \ldots < k_N = M$ such that $x_n = x''_{k_n}$ for $n = 0,\ldots,N$. Now

$$\begin{aligned}
&S(\{x_n\},\{\xi_n\}) - S(\{x''_n\},\{\xi''_n\})\\
&= \sum_{n=1}^{N} g(\xi_n)\big(f(x_n) - f(x_{n-1})\big) - \sum_{n=1}^{M} g(\xi''_n)\big(f(x''_n) - f(x''_{n-1})\big)\\
&= \sum_{n=1}^{N}\Bigg(g(\xi_n)\big(f(x_n) - f(x_{n-1})\big) - \sum_{k=1+k_{n-1}}^{k_n} g(\xi''_k)\big(f(x''_k) - f(x''_{k-1})\big)\Bigg)
\end{aligned}$$

(1.24)

$$= \sum_{n=1}^{N}\sum_{k=1+k_{n-1}}^{k_n}\big(g(\xi_n) - g(\xi''_k)\big)\big(f(x''_k) - f(x''_{k-1})\big),$$

where in the last equality we used the fact that for every $n \in \{1,\ldots,N\}$ we have

$$\sum_{k=1+k_{n-1}}^{k_n}(f(x''_k) - f(x''_{k-1})) = f(x''_{k_n}) - f(x''_{k_{n-1}}) = f(x_n) - f(x_{n-1}).$$

It follows from (1.24) that

$$\begin{aligned}
&\Big|S(\{x_n\},\{\xi_n\}) - S(\{x''_n\},\{\xi''_n\})\Big|\\
&\le \sum_{n=1}^{N}\sum_{k=1+k_{n-1}}^{k_n}\big|g(\xi_n) - g(\xi''_k)\big|\big|f(x''_k) - f(x''_{k-1})\big|.
\end{aligned}$$

Here for any pair $\langle n,k\rangle$ appearing in the sum we have $\xi_n \in [x_{n-1},x_n]$ and $\xi''_k \in [x''_{k-1},x''_k] \subset [x_{n-1},x_n]$, and hence

$$|\xi_n - \xi''_k| \le |x_n - x_{n-1}| \le \mathrm{mesh}\{x_n\} \le \delta.$$

Hence by (1.21) we have $|g(\xi_n) - g(\xi''_k)| < \varepsilon$ for all $\langle n,k\rangle$ appearing in our sum, and we conclude

$$\begin{aligned}
\Big|S(\{x_n\},\{\xi_n\}) - S(\{x''_n\},\{\xi''_n\})\Big| &\le \varepsilon\sum_{n=1}^{N}\sum_{k=1+k_{n-1}}^{k_n}\big|f(x''_k) - f(x''_{k-1})\big|\\
&= \varepsilon\sum_{k=1}^{M}\big|f(x''_k) - f(x''_{k-1})\big| \le \varepsilon\,\mathrm{Var}_{[A,B]}(f).
\end{aligned}$$

We have thus proved (1.23), and the proof of the theorem is complete. □

Proof of Proposition 1.9. Since $f \in C^1([A,B])$, the function $x \to |f'(x)|$ is continuous, and thus the Riemann integral $\int_A^B |f'(x)|\,dx$ exists by Theorem 1.5. Hence for any $\varepsilon > 0$ there is some $\delta > 0$ such that for any tagged partition $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ of $[A,B]$ of mesh $\leq \delta$ we have

$$\left| \sum_{n=1}^N (x_n - x_{n-1})|f'(\xi_n)| - \int_A^B |f'(x)|\,dx \right| < \varepsilon. \tag{1.25}$$

Furthermore since f' is uniformly continuous on $[A,B]$, by taking δ sufficiently small we can ensure that for any numbers $x \leq \xi \leq y$ in $[A,B]$ satisfying $y - x \leq \delta$ we have

$$|f(y) - f(x) - (y-x)f'(\xi)| \leq (y-x)\varepsilon. \tag{1.26}$$

((Let us recall a proof of the last statement: Since $f \in C^1([A,B])$ and $[A,B]$ is a closed and bounded interval, f' is uniformly continuous on $[A,B]$; thus we can take $\delta > 0$ so small that $|f'(\xi) - f'(\eta)| < \varepsilon/2$ for any $\xi, \eta \in [A,B]$ with $|\xi - \eta| \leq \delta$. Now let $x \leq \xi \leq y$ be arbitrary numbers in $[A,B]$ with $y - x \leq \delta$; then by the mean-value theorem applied to $\Re f$ and $\Im f$ there exist $\eta_1, \eta_2 \in [x,y]$ such that $\Re f(y) - \Re f(x) = (y-x)\Re f'(\eta_1)$ and $\Im f(y) - \Im f(x) = (y-x)\Im f'(\eta_2)$. But $|\eta_1 - \xi| \leq y - x \leq \delta$; thus $|f'(\eta_1) - f'(\xi)| < \varepsilon/2$; hence also $|\Re f'(\eta_1) - \Re f'(\xi)| < \varepsilon/2$ and $|\Re f(y) - \Re f(x) - (y-x)\Re f'(\xi)| \leq \frac{\varepsilon}{2}(y-x)$; similarly $|\Im f(y) - \Im f(x) - (y-x)\Im f'(\xi)| \leq \frac{\varepsilon}{2}(y-x)$; adding these two we obtain (1.26).))

By taking $\delta > 0$ so small that both the statements around (1.25) and (1.26) hold, then for any partition $\{x_n\}_{n=0}^N$ of $[A,B]$ of mesh $\leq \delta$,

$$\begin{aligned}
&\left| \sum_{n=1}^N |f(x_n) - f(x_{n-1})| - \int_A^B |f'(x)|\,dx \right| \\
&\qquad < \varepsilon + \left| \sum_{n=1}^N |f(x_n) - f(x_{n-1})| - \sum_{n=1}^N (x_n - x_{n-1})|f'(x_n)| \right| \\
&\qquad \leq \varepsilon + \sum_{n=1}^N \Big| f(x_n) - f(x_{n-1}) - (x_n - x_{n-1})f'(x_n) \Big| \\
&\qquad \leq \varepsilon + \sum_{n=1}^N (x_n - x_{n-1})\varepsilon = (1 + B - A)\varepsilon. \qquad (1.27)
\end{aligned}$$

Such a $\delta > 0$ can be obtained for every $\varepsilon > 0$; this immediately implies that *the supremum in* (1.19) *is* $\geq \int_A^B |f'(x)|\,dx$. Now note also that if $\{x'_n\}_{n=0}^N$ is an arbitrary partition of $[A,B]$ then we can find another partition $\{x_n\}_{n=0}^M$ of $[A,B]$ of mesh $\leq \delta$ such that $\{x'_n\}_{n=0}^N$ is a subsequence of $\{x_n\}_{n=0}^M$. Then by the triangle inequality we have $\sum_{n=1}^N |f(x'_n) - f(x'_{n-1})| \leq \sum_{n=1}^M |f(x_n) - f(x_{n-1})|$, and also (1.27) holds. Using this we conclude also that *the supremum in* (1.19) *is* $\leq \int_A^B |f'(x)|\,dx$, and the proposition is proved. $\square$

We next prove a formula for *integration by parts:*

Theorem 1.12. *For arbitrary functions f and $g : [A,B] \to \mathbb{C}$, if $\int_A^B g(x)\,df(x)$ exists then $\int_A^B f(x)\,dg(x)$ also exists, and*

$$\int_A^B g(x)\,df(x) = \Big(f(B)g(B) - f(A)g(A)\Big) - \int_A^B f(x)\,dg(x). \tag{1.28}$$

Proof. For any tagged partition $\langle\{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N\rangle$ of $[A,B]$ we have the following identity, if we set $\xi_0 = A$ and $\xi_{N+1} = B$:

$$\sum_{n=1}^N g(\xi_n)\big(f(x_n) - f(x_{n-1})\big) = f(B)g(B) - f(A)g(A) - \sum_{n=1}^{N+1} f(x_{n-1})\big(g(\xi_n) - g(\xi_{n-1})\big).$$

Here note that $\langle\{\xi_n\}_{n=0}^{N+1}, \{x_{n-1}\}_{n=1}^{N+1}\rangle$ is also a tagged partition of $[A,B]$, since $x_{n-1} \in [\xi_{n-1}, \xi_n]$, and the sum on the right hand sum is a Riemann-Stieltjes sum $S(\{\xi_n\}, \{x_{n-1}\})$ approximating $\int_A^B f(x)\,dg(x)$, Moreover, $\text{mesh}\{\xi_n\} \leq 2\,\text{mesh}\{x_n\}$, so that the sum on the right tends to $\int_A^B f(x)\,dg(x)$ as $\text{mesh}\{x_n\}$ tends to 0. $\square$

Recall that the intuition behind the definition of the Riemann-Stieltjes integral is that $\int_A^B g\,df$ should equal $\int_A^B g(x)f'(x)\,dx$ when g and f are nice functions. The following theorem shows that this holds in quite some generality:

Theorem 1.13. *Let $f \in C^1([A,B])$ and let $g : [A,B] \to \mathbb{C}$ be Riemann-integrable. Then the Riemann-Stieltjes integral $\int_A^B g(x)\,df(x)$ exists, the function $x \mapsto g(x)f'(x)$ is Riemann-integrable, and we have*

$$\int_A^B g(x)\,df(x) = \int_A^B g(x)f'(x)\,dx. \tag{1.29}$$

In order to prepare for the proof of Theorem 1.13 we first prove two propositions – which are also useful in their own right.

Proposition 1.14. *Let $A < B$ and let g be an arbitrary function $[A,B] \to \mathbb{C}$. Then g is Riemann integrable if and only if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every partition $\{x_n\}_{n=0}^N$ of $[A,B]$ with* $\text{mesh}\{x_n\} \leq \delta$ *we have*

$$\sum_{n=1}^N (x_n - x_{n-1}) \cdot \sup\big\{|g(\xi) - g(\xi')| \;:\; \xi, \xi' \in [x_{n-1}, x_n]\big\} \leq \varepsilon. \tag{1.30}$$

(Note that for any every partition $\{x_n\}$ of $[A,B]$, the left hand side of (1.30) is a well-defined number in $[0,\infty]$; cf. Folland, Sec. 0.5.)

Proof. Assume first that the stated condition holds. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that (1.30) holds for all partitions $\{x_n\}_{n=0}^N$

of $[A,B]$ with $\text{mesh}\{x_n\} \leq \delta$. We then claim that $|S(\{x_n\},\{\xi_n\}) - S(\{x''_n\},\{\xi''_n\})| \leq \varepsilon$ holds whenever $\langle\{x_n\},\{\xi_n\}\rangle$ and $\langle\{x''_n\},\{\xi''_n\}\rangle$ are tagged partitions of $[A,B]$ with mesh $\leq \delta$ such that $\{x_n\}$ is a subsequence of $\{x''_n\}$. Note that this suffices to show that g is Riemann integrable, by the same argument as in the proof of Theorem 1.10.

To prove the claim, note that if $\langle\{x_n\},\{\xi_n\}\rangle$ and $\langle\{x''_n\},\{\xi''_n\}\rangle$ are as above then we have, using the same notation as in the proof of Theorem 1.10 (see (1.24), but now with $f(x) \equiv x$):

$$\begin{aligned}
&\big|S(\{x_n\},\{\xi_n\}) - S(\{x''_n\},\{\xi''_n\})\big| \\
&= \Big|\sum_{n=1}^{N} \sum_{k=1+k_{n-1}}^{k_n} \big(g(\xi_n) - g(\xi''_k)\big)\big(x''_k - x''_{k-1}\big)\Big| \\
&\leq \sum_{n=1}^{N} \sum_{k=1+k_{n-1}}^{k_n} \big(x''_k - x''_{k-1}\big) \cdot \sup\big\{|g(\xi) - g(\xi')| \,:\, \xi,\xi' \in [x_{n-1},x_n]\big\} \\
&= \sum_{n=1}^{N} \big(x_n - x_{n-1}\big) \cdot \sup\big\{|g(\xi) - g(\xi')| \,:\, \xi,\xi' \in [x_{n-1},x_n]\big\} \leq \varepsilon,
\end{aligned}$$

and the claim is proved.

Conversely, assume that g is Riemann-integrable. Let $\varepsilon > 0$ be given. Then by Lemma 1.11 there is some $\delta > 0$ such that for any two tagged partitions $\langle\{x_n\},\{\xi_n\}\rangle$ and $\langle\{x'_n\},\{\xi'_n\}\rangle$ of $[A,B]$, both having mesh $\leq \delta$, we have $|S(\{x_n\},\{\xi_n\}) - S(\{x'_n\},\{\xi'_n\})| < \varepsilon/2$. Applying this in particular when $\{x_n\} = \{x'_n\}$ and considering the real part, it follows that if $\{x_n\}_{n=0}^N$ is any partition of $[A,B]$ with mesh $\leq \delta$, then

$$\sum_{n=1}^{N} (x_n - x_{n-1}) \cdot \Re(g(\xi_n) - g(\xi'_n)) < \frac{\varepsilon}{2}$$

for all choices of $\{\xi_n\}_{n=1}^N$ and $\{\xi'_n\}_{n=1}^N$ with $\xi_n, \xi'_n \in [x_{n-1},x_n]$. Hence also

$$\sum_{n=1}^{N} (x_n - x_{n-1}) \cdot \sup\big\{\Re(g(\xi) - g(\xi')) \,:\, \xi,\xi' \in [x_{n-1},x_n]\big\} \leq \frac{\varepsilon}{2} \tag{1.31}$$

Similarly one proves

$$\sum_{n=1}^{N} (x_n - x_{n-1}) \cdot \sup\big\{\Im(g(\xi) - g(\xi')) \,:\, \xi,\xi' \in [x_{n-1},x_n]\big\} \leq \frac{\varepsilon}{2} \tag{1.32}$$

Note also that if F is any set of complex numbers satisfying $z \in F \Rightarrow -z \in F$ then $\sup\{|z| \,:\, z \in F\} \leq \sup\{\Re z \,:\, z \in F\} + \sup\{\Im z \,:\, z \in F\}$. Applying this with $F = \{g(\xi) - g(\xi') \,:\, \xi,\xi' \in [x_{n-1},x_n]\}$ for each n and using (1.31) and (1.32) we conclude that (1.30) holds. $\square$

Proposition 1.15. *Let $A < B$ and let $f, g : [A, B] \to \mathbb{C}$ be two Riemann-integrable functions. Then also the (pointwise) product function fg is Riemann-integrable on $[A, B]$.*

Proof. By Proposition 1.6 both f and g are bounded, i.e. there exists some $M > 0$ such that $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in [A, B]$. Now the Riemann-integrability of fg follows by using the criterion in Proposition 1.14 and the inequality

$$\begin{aligned} \big|f(\xi)g(\xi) - f(\xi')g(\xi')\big| &\leq \big|f(\xi) - f(\xi')\big||g(\xi)| + |f(\xi')|\big|g(\xi) - g(\xi')\big| \\ &\leq M\Big(\big|f(\xi) - f(\xi')\big| + \big|g(\xi) - g(\xi')\big|\Big). \end{aligned}$$

□

Proof of Theorem 1.13. Take $M > 0$ such that $|g(x)| \leq M$ for all $x \in [A, B]$ (this is possible by Proposition 1.6). The fact that $x \mapsto g(x)f'(x)$ is Riemann integrable follows from Theorem 1.5 and Proposition 1.15. Let us write $S_1(\{x_n\}, \{\xi_n\})$ for the Riemann sum corresponding to $\int_A^B g(x)\, df(x)$, and $S_2(\{x_n\}, \{\xi_n\})$ for the Riemann sum corresponding to $\int_A^B g(x)f'(x)\, dx$.

Let $\varepsilon > 0$ be given. We can now choose $\delta > 0$ so small that $|S_2(\{x_n\}, \{\xi_n\}) - \int_A^B g(x)f'(x)\, dx| < \varepsilon$ holds for any tagged partition $\langle\{x_n\}, \{\xi_n\}\rangle$ of $[A, B]$ of mesh $\leq \delta$, and also

$$|f(y) - f(x) - (y - x)f'(\xi)| \leq (y - x)\varepsilon$$

for any numbers $x \leq \xi \leq y$ in $[A, B]$ satisfying $y - x \leq \delta$ (see below (1.26) for a proof of the latter.) Now let $\langle\{x_n\}, \{\xi_n\}\rangle$ be an arbitrary tagged partition of $[A, B]$ of mesh $\leq \delta$. Then

$$\begin{aligned} &\big|S_1(\{x_n\}, \{\xi_n\}) - \textstyle\int_A^B g(x)f'(x)\, dx\big| \\ &\leq \Big|S_2(\{x_n\}, \{\xi_n\}) - \textstyle\int_A^B g(x)f'(x)\, dx\Big| + \Big|S_1(\{x_n\}, \{\xi_n\}) - S_2(\{x_n\}, \{\xi_n\})\Big| \\ &< \varepsilon + \left|\sum_{n=1}^{N}\Big(g(\xi_n)(f(x_n) - f(x_{n-1})) - g(\xi_n)f'(\xi_n)(x_n - x_{n-1})\Big)\right| \\ &\leq \varepsilon + \sum_{n=1}^{N}\big|g(\xi_n)\big| \cdot \Big|f(x_n) - f(x_{n-1}) - (x_n - x_{n-1})f'(\xi_n)\Big| \\ &< \varepsilon + \sum_{n=1}^{N} M(x_n - x_{n-1})\varepsilon = (1 + M(B - A))\varepsilon. \end{aligned}$$

This proves that the Riemann-Stieltjes integral $\int_A^B g(x)\, df(x)$ exists and equals $\int_A^B g(x)f'(x)\, dx$. □

Example 1.7. Assume $A < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq B$; let $c_1, c_2, \ldots, c_m \in \mathbb{C}$, and set

$$f(x) = \sum_{\lambda_n \leq x} c_n$$

(the notation indicates a summation over the finite set of those $n \in \mathbb{N}$ for which $\lambda_n \leq x$). Then, if $g \in C^1([A, B])$, we have

$$\sum_{n=1}^{m} c_n g(\lambda_n) = f(B)g(B) - \int_A^B f(x)g'(x)\,dx.$$

Indeed,

$$\sum_{n=1}^{m} c_n g(\lambda_n) = \int_A^B g\,df = f(B)g(B) - f(A)g(A) - \int_A^B f\,dg$$
$$= f(B)g(B) - \int_A^B f(x)g'(x)\,dx,$$

where the first equality holds by Example 1.5, the second by Theorem 1.12, and the last equality holds by Theorem 1.13 (using also $f(A) = 0$).

In order to make the notation really flexible we also need the following definition of generalized Riemann-Stieltjes integrals.

Definition 1.16. We define the *generalized Riemann-Stieltjes integral*

$$\int_{A+}^{B} g(x)\,df(x) := \lim_{a \to A^+} \int_a^B g(x)\,df(x), \tag{1.33}$$

provided that $\int_a^B g(x)\,df(x)$ exists for all $a > A$ sufficiently near A.

Similarly we define

$$\int_{A-}^{B} g(x)\,df(x) := \lim_{a \to A^-} \int_a^B g(x)\,df(x); \tag{1.34}$$

$$\int_{-\infty}^{B} g(x)\,df(x) := \lim_{a \to -\infty} \int_a^B g(x)\,df(x), \tag{1.35}$$

Also, the generalized Riemann-Stieltjes integrals $\int_A^{B-} g(x)\,df(x)$, $\int_A^{B+} g(x)\,df(x)$ and $\int_A^{\infty} g(x)\,df(x)$ are defined in the analogous way.

Finally generalized Riemann-Stieltjes integrals with limits on both end-points are defined in the natural way, i.e.

$$\int_{A-}^{B+} g(x)\,df(x) := \lim_{b \to B^+} \lim_{a \to A^-} \int_a^b g(x)\,df(x); \tag{1.36}$$

$$\int_{-\infty}^{B-} g(x)\,df(x) := \lim_{b \to B^-} \lim_{a \to -\infty} \int_a^b g(x)\,df(x), \tag{1.37}$$

etc.

Remark 1.17. In (1.36) (and similarly in any of the other cases with limits on both end-points) it does not matter if the limit is considered as an iterated limit (in either order) or as a simultanous limit in a, b; if one of these limits exist (as a finite real number) then so do the other ones. This follows by fixing an arbitrary number $C \in (A, B)$ and using $\int_a^b g(x)\,df(x) = \int_a^C g(x)\,df(x) + \int_C^b g(x)\,df(x)$ inside the limit.

Example 1.8. Let $a_1, a_2, \ldots$ be any sequence of complex numbers, and set $f(x) = \sum_{1 \leq n < x} a_n$. Also let $g \in C(\mathbb{R}^+)$. We then have, for any integers $1 \leq M \leq N$:

$$\sum_{n=M}^{N} a_n g(n) = \int_M^{N+} g(x)\,df(x) = \int_M^{N+\frac{1}{2}} g(x)\,df(x). \tag{1.38}$$

Hence also

$$\sum_{n=M}^{\infty} a_n g(n) = \int_M^{\infty} g(x)\,df(x). \tag{1.39}$$

On the other hand, if we set $f_1(x) = \sum_{1 \leq n \leq x} a_n$ (thus $f_1(x) = f(x)$ except when x is an integer) then

$$\sum_{n=M}^{N} a_n g(n) = \int_{M-}^{N} g(x)\,df_1(x) = \int_{M-\frac{1}{2}}^{N} g(x)\,df_1(x) \tag{1.40}$$

and

$$\sum_{n=M}^{\infty} a_n g(n) = \int_{M-}^{\infty} g(x)\,df_1(x). \tag{1.41}$$

1.3. **Example: Euler-MacLaurin summation.** (The following presentation is partly influenced by Olver, [16, Ch. 8].) We will now discuss how the Riemann-Stieltjes integral can be used together with integration by parts to give increasingly precise estimates of a sum $\sum_{n=M}^{N} f(n)$ where M and N are integers, $M < N$, and f is a given (nice, not wildly oscillating) function on $[M, N]$. Actually let us consider instead the sum

$$\sum_{A < n \leq B} f(n),$$

where $A < B$ are arbitrary *real* numbers (and it is understood that the sum is taken over all *integers* n satisfying $A < n \leq B$).[5]

[5]This sum is of course not at all more general than "$\sum_{n=M}^{N} f(n)$" but the treatment becomes, in my opinion, slightly clearer when we do not impose the unnecessary assumption of the integration limits of the Riemann Stieltjes integral being integers.

Referring to Example 1.5 we see that this sum can be expressed as

$$\int_A^B f(x)\,d\lfloor x\rfloor, \tag{1.42}$$

where $\lfloor x\rfloor$ is the "floor function", i.e. $\lfloor x\rfloor$ is the largest integer $\leq x$. (Make sure to think this through; in particular check that we do get the correct contributions at $x = A$ and $x = B$, if A or B happen to be an integer.)

Applying also integration by parts (Theorem 1.12, and then Theorem 1.13) we get, assuming $f \in C^1([A, B])$:

$$\sum_{A<n\leq B} f(n) = \int_A^B f(x)\,d\lfloor x\rfloor = \Big[f(x)\lfloor x\rfloor\Big]_{x=A}^{x=B} - \int_A^B f'(x)\lfloor x\rfloor\,dx. \tag{1.43}$$

It is natural to compare $\sum_{A<n\leq B} f(n)$ with $\int_A^B f(x)\,dx$. Applying the analogous integration by parts for $\int_A^B f(x)\,dx$ we have

$$\int_A^B f(x)\,dx = \Big[f(x)(x-K)\Big]_A^B - \int_A^B f'(x)(x-K)\,dx, \tag{1.44}$$

for any constant K. (We used the fact that the most general primitive function of "1" is "$x - K$".) Combining (1.44) and (1.43) we conclude

$$\sum_{A<n\leq B} f(n) = \int_A^B f(x)\,dx - \Big[f(x)\big(x - \lfloor x\rfloor - K\big)\Big]_{x=A}^{x=B}$$

$$+\int_A^B f'(x)\big(x - \lfloor x\rfloor - K\big)\,dx. \tag{1.45}$$

In order to make use of this formula we have to understand the last term, $\int_A^B f'(x)(x - \lfloor x\rfloor - K)\,dx$. Note that the function $x \mapsto x - \lfloor x\rfloor - K$ is oscillating around the mean value $\frac{1}{2} - K$. Now there's a general principle that *when dealing with an integral $\int_A^B h(x)g(x)\,dx$, where $h(x)$ is "slowly varying" while $g(x)$ is oscillating with mean value 0, it is often advantageous to* ***integrate by parts****: $\int_A^B h(x)g(x)\,dx = [h(x)G(x)]_{x=A}^{x=B} - \int_A^B h'(x)G(x)\,dx$ (where G is a primitive function of x); the point is that $\int_A^B h'(x)G(x)\,dx$ can here typically be expected to be* ***comparatively small!***

Applying this principle to $\int_A^B f'(x)\big(x - \lfloor x\rfloor - K\big)\,dx$ we see that we should take $K = \frac{1}{2}$ and then integrate by parts, assuming now $f \in C^2([A, B])$. We then need to compute the primitive function of $x - \lfloor x\rfloor - \frac{1}{2}$. (This function is sometimes called the *saw-tooth* function; draw a picture!) It is convenient to set

$$\omega_1(x) = x - \tfrac{1}{2} \quad \text{and} \quad \widetilde{\omega}_1(x) = \omega_1(x - \lfloor x\rfloor) = x - \lfloor x\rfloor - \tfrac{1}{2}.$$

(Thus $\widetilde{\omega}_1(x)$ is the periodic function with period one which agrees with $\omega_1(x)$ for $x \in [0,1)$.) We note that for $x \in [0,1]$ we have $\int_0^x \widetilde{\omega}_1(x_1)\,dx_1 = \int_0^x \omega_1(x_1)\,dx_1 = \frac{1}{2}(x^2-x)$. Since $\widetilde{\omega}_1(x_1)$ is periodic with period one and $\int_0^1 \widetilde{\omega}_1(x_1)\,dx_1 = 0$, it follows that for general $x \in \mathbb{R}$, $\int_0^x \widetilde{\omega}_1(x_1)\,dx_1$ equals the periodic function with period one which agrees with $\frac{1}{2}(x^2-x)$ for $x \in [0,1)$; thus $\int_0^x \widetilde{\omega}_1(x_1)\,dx_1 = \widetilde{\tau}(x)$, where

$$\tau(x) = \tfrac{1}{2}(x^2-x) \quad \text{and} \quad \widetilde{\tau}(x) = \tau(x - \lfloor x \rfloor).$$

Hence we have

$$\begin{aligned}\int_A^B f'(x)\big(x - \lfloor x \rfloor - \tfrac{1}{2}\big)\,dx &= \int_A^B \widetilde{\omega}_1(x) f'(x)\,dx \\ &= \Big[\big(\widetilde{\tau}(x) - K\big) f'(x)\Big]_{x=A}^{x=B} - \int_A^B (\widetilde{\tau}(x) - K) f''(x)\,dx, \qquad (1.46)\end{aligned}$$

where K is an arbitrary constant (it does not have to be the same as our previous K).

This procedure can now be repeated: In order for the periodic function $\widetilde{\tau}(x) - K$ to have mean-value zero we should take $K = \int_0^1 \tau(x)\,dx = -\frac{1}{12}$; thus we set

$$\omega_2(x) = \tfrac{1}{2}(x^2 - x + \tfrac{1}{6}) \quad \text{and} \quad \widetilde{\omega}_2(x) = \omega_2(x - \lfloor x \rfloor).$$

Then $\widetilde{\omega}_2(x)$ is periodic with period one and $\int_0^1 \widetilde{\omega}_2(x)\,dx = 0$; the above formula reads

$$\int_A^B \widetilde{\omega}_1(x) f'(x)\,dx = \Big[\widetilde{\omega}_2(x) f'(x)\Big]_{x=A}^{x=B} - \int_A^B \widetilde{\omega}_2(x) f''(x)\,dx.$$

The rth step of this procedure ($r \in \mathbb{N}$) is to let $\omega_{r+1}(x)$ be that primitive function of $\omega_r(x)$ which satisfies $\int_0^1 \omega_{r+1}(x)\,dx = 0$; then set $\widetilde{\omega}_{r+1}(x) = \omega_{r+1}(x - \lfloor x \rfloor)$, and note that (if $f \in C^{r+1}([A,B])$):

$$\int_A^B \widetilde{\omega}_r(x) f^{(r)}(x)\,dx = \Big[\widetilde{\omega}_{r+1}(x) f^{(r)}(x)\Big]_{x=A}^{x=B} - \int_A^B \widetilde{\omega}_{r+1}(x) f^{(r+1)}(x)\,dx.$$

The result can be collected as follows: If $h \in \mathbb{N}$ and $f \in C^h([A,B])$, then

$$\begin{aligned}\sum_{A<n\leq B} f(n) = \int_A^B f(x)\,dx &+ \sum_{r=1}^{h} (-1)^r \Big[\widetilde{\omega}_r(x) f^{(r-1)}(x)\Big]_{x=A}^{x=B} \\ &+ (-1)^{h-1} \int_A^B \widetilde{\omega}_h(x) f^{(h)}(x)\,dx. \qquad (1.47)\end{aligned}$$

Here from the above recursion formula we see that $\omega_r(x)$ is a polynomial of degree r (with x^r-coefficient $= r!^{-1}$). We compute:

$$\begin{aligned}
\omega_1(x) &= x - \tfrac{1}{2}\\
\omega_2(x) &= \tfrac{1}{2}(x^2 - x + \tfrac{1}{6})\\
\omega_3(x) &= \tfrac{1}{6}(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x).\\
\omega_4(x) &= \tfrac{1}{24}(x^4 - 2x^3 + x^2 - \tfrac{1}{30}).
\end{aligned}$$

It is customary to use a slightly different normalization: The *rth Bernoulli polynomial* is given by $B_r(x) = r! \cdot \omega_r(x)$. Thus from the above discussion we see that we can define $B_r(x)$ as follows (we extend to the case $r = 0$ in a natural way).

Definition 1.18. The *Bernoulli polynomials* $B_0(x)$, $B_1(x)$, $B_2(x)$, ..., are defined by $B_0(x) = 1$ and recursively by the relations $B_r'(x) = rB_{r-1}(x)$ and $\int_0^1 B_r(x)\,dx = 0$ for $r = 1, 2, 3, \ldots$. The rth *Bernoulli number* is defined by $B_r = B_r(0)$.

We have now proved (see (1.47)):

Theorem 1.19. ***The Euler-MacLaurin summation formula.*** *Let $A < B$ be real numbers, $h \in \mathbb{N}$ and $f \in C^h([A, B])$. Then*

$$\sum_{A<n\leq B} f(n) = \int_A^B f(x)\,dx + \sum_{r=1}^{h} \frac{(-1)^r}{r!}\Big[\widetilde{B}_r(x) f^{(r-1)}(x)\Big]_{x=A}^{x=B} + (-1)^{h-1}\int_A^B \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx, \tag{1.48}$$

where $\widetilde{B}_r(x) = B_r(x - \lfloor x \rfloor)$.

The first Bernoulli polynomials are:

$$\begin{aligned}
B_0(x) &= 1\\
B_1(x) &= x - \tfrac{1}{2}\\
B_2(x) &= x^2 - x + \tfrac{1}{6}\\
B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x.\\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30}.
\end{aligned}$$

It follows immediately from the recursion formula that $B_r(1 - x) = B_r(x)$ for all even r and $B_r(1-x) = -B_r(x)$ for all odd r; also $B_r(0) = B_r(1) = 0$ for all odd $r \geq 3$. Furthermore, the periodized function $\widetilde{B}_r(x)$ is continuous for all $r \geq 2$.

The Euler-MacLaurin summation formula is very useful for obtaining asymptotic expansions of sums. For example we will see later how it is used to derive Stirling's formula for the Γ-function $\Gamma(z)$, with an error term with arbitrary power rate decay as $|z| \to \infty$. At present we content ourselves by giving a single example:

Example 1.9. Recall Example 1.2, the question about the asymptotic behavior of the sum $\sum_{n=1}^{N} n^\alpha$ for fixed $\alpha > -1$. We can use the Euler-MacLaurin summation formula to attack this question for an *arbitrary complex* α.

Indeed, by Theorem 1.19 applied with $f(x) = x^\alpha$, $A < 1$ tending to 1 and $B = N$, we have (using $\widetilde{B}_r(N) = B_r$ for $r \geq 1$ and $\widetilde{B}_r(1-) = B_r$ for $r \geq 2$ while $\widetilde{B}_1(1-) = \frac{1}{2} = 1 + B_1$):

$$\sum_{n=1}^{N} n^\alpha = \int_1^N x^\alpha \, dx + 1 + \sum_{r=1}^{h} \frac{(-1)^r B_r}{r!} \Big(f^{(r-1)}(N) - f^{(r-1)}(1) \Big) + (-1)^{h-1} \int_1^N \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x) \, dx.$$

Using $f^{(r)}(x) = \alpha(\alpha-1)\cdots(\alpha-r+1)x^{\alpha-r}$ we see that for $\alpha \neq -1$ the above can be expressed as

$$\sum_{n=1}^{N} n^\alpha = \frac{1}{\alpha+1}\Big(N^{\alpha+1} - 1\Big) + 1 + \frac{1}{\alpha+1} \sum_{r=1}^{h} (-1)^r B_r \binom{\alpha+1}{r} \Big(N^{\alpha+1-r} - 1\Big) + (-1)^{h-1} \binom{\alpha}{h} \int_1^N \widetilde{B}_h(x) x^{\alpha-h} \, dx. \tag{1.49}$$

Here since $|\widetilde{B}_h(x)|$ is bounded from above by a constant which only depends on h, we see that the last integral is $O(N^{\Re\alpha - h+1})$ if $\Re\alpha > h-1$, $O(\log N)$ if $\Re\alpha = h-1$, and $O(1)$ if $\Re\alpha < h-1$ (the implied constant may depend on α and h but not on N). In particular if $\Re\alpha > 0$ then this leads to a more precise asymptotic formula than (1.3)! For a concrete example, say $\alpha = \frac{3}{2}$; then taking $h = 3$ above we get:

$$\sum_{n=1}^{N} n^{\frac{3}{2}} = \frac{2}{5} N^{\frac{5}{2}} + \frac{1}{2} N^{\frac{3}{2}} + \frac{1}{8} N^{\frac{1}{2}} + O(1).$$

(Numerical example: For $N = 1000$ the left hand side equals $S = 12664925.95633\ldots$ and we find that $S - \frac{2}{5}N^{\frac{5}{2}} = 15815.3\ldots$, $S - (\frac{2}{5}N^{\frac{5}{2}} + \frac{1}{2}N^{\frac{3}{2}}) = 3.927\ldots$ and $S - (\frac{2}{5}N^{\frac{5}{2}} + \frac{1}{2}N^{\frac{3}{2}} + \frac{1}{8}N^{\frac{1}{2}}) = -0.0254\ldots$. In fact trying also $N = 10^4, 10^5, 10^6, \ldots$ it seems as if the difference $\sum_{n=1}^{N} n^\alpha - (\frac{2}{5}N^{\frac{5}{2}} + \frac{1}{2}N^{\frac{3}{2}} + \frac{1}{8}N^{\frac{1}{2}})$ tends to a number $-0.025485\ldots$ as $N \to \infty$. This will be explained below.)

Note that once $h > \Re\alpha + 1$, we do not get any better power of N in the error term by increasing h further! This is easy to fix: If $h > \Re\alpha + 1$ then the integral $\int_1^\infty \widetilde{B}_h(x) x^{\alpha-h} \, dx$ is absolutely convergent and hence we can express the last term in (1.49) as

$$(-1)^{h-1}\int_1^\infty \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx - (-1)^{h-1}\int_N^\infty \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx.$$

Here the first integral is a constant independent of N, and to the second integral we can now apply repeated integration by parts just as in the proof of the Euler-MacLaurin formula to obtain, for an arbitrary integer $k \geq h$:

$$= (-1)^{h-1}\int_1^\infty \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx + \sum_{r=h+1}^{k} \frac{(-1)^r B_r}{r!} f^{(r-1)}(N) + (-1)^k \int_N^\infty \frac{\widetilde{B}_k(x)}{k!} f^{(k)}(x)\,dx,$$

and the good thing is that the last integral is $O(N^{\Re\alpha-k+1})$! (This is since $|\widetilde{B}_k(x)| = O(1)$ and $|f^{(k)}(x)| = O(x^{\Re\alpha-k})$ for all $x \geq 1$, and $k \geq h > \Re\alpha + 1$.)

Using this in (1.49), we obtain:

$$\sum_{n=1}^{N} n^\alpha = \frac{1}{\alpha+1}\sum_{r=0}^{k}(-1)^r B_r \binom{\alpha+1}{r} N^{\alpha+1-r} + C(\alpha) + (-1)^k \binom{\alpha}{k} \int_N^\infty \widetilde{B}_k(x) x^{\alpha-k}\,dx, \tag{1.50}$$

where

$$C(\alpha) = 1 - \frac{1}{\alpha+1}\sum_{r=0}^{h}(-1)^r B_r \binom{\alpha+1}{r} + (-1)^{h-1}\binom{\alpha}{h}\int_1^\infty \widetilde{B}_h(x) x^{\alpha-h}\,dx. \tag{1.51}$$

Recall that this formula is valid for any complex $\alpha \neq -1$ and any $k, h \in \mathbb{N}$ satisfying $k \geq h > \Re\alpha+1$. The point of separating out the term $C(\alpha)$ is that this term does not depend on N, i.e. it appears as a *constant* in our asymptotic expansion as $N \to \infty$ for fixed α! Note that $C(\alpha)$ is independent of h since all the other terms in (1.50) are independent of h; this can of course also be seen easily by using integration by parts in (1.51). Note also that in (1.50) we have incorporated the term $\frac{1}{\alpha+1}N^{\alpha+1}$ in the r-sum by letting it start at $r = 0$.

The constant $C(\alpha)$ can easily be computed in practice (with rigorous error bounds) by evaluating the two sums in (1.50) for a modest value of N and an appropriate k, and bounding the last integral using simply $|\widetilde{B}_k(x)| \leq \sup_{x\in[0,1]} |B_k(x)|$. As a concrete example, for $\alpha = \frac{3}{2}$ and

$k = 5$, (1.50) gives

$$\sum_{n=1}^{N} n^{\frac{3}{2}} = C(\tfrac{3}{2}) + \tfrac{2}{5}N^{\frac{5}{2}} + \tfrac{1}{2}N^{\frac{3}{2}} + \tfrac{1}{8}N^{\frac{1}{2}} + \tfrac{1}{1920}N^{-\frac{3}{2}} + O(N^{-\frac{5}{2}}),$$

and numerical evaluation for $N = 10^4, 10^5, 10^6$ strongly suggests that $C(\alpha) = -0.02548520188983303\ldots$.

Analytically, we can relate $C(\alpha)$ to the *Riemann zeta function!*[6] Namely, (recall that) the Riemann zeta function is defined by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } s \in \mathbb{C},\ \Re s > 1.$$

This sum is absolutely convergent, uniformly on compact subsets of $\{s \in \mathbb{C} : \Re s > 1\}$; hence $\zeta(s)$ is an analytic function in the region $\{s \in \mathbb{C} : \Re s > 1\}$. To see the connection, take $N \to \infty$ in (1.50) to conclude

$$C(\alpha) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} n^{\alpha} - \frac{1}{\alpha+1}\sum_{r=0}^{k}(-1)^r B_r \binom{\alpha+1}{r} N^{\alpha+1-r}\right),$$

since the last term in (1.50) tends to zero. If $\Re\alpha < -1$ then each term in $\sum_{r=0}^{k}(-1)^r B_r \binom{\alpha+1}{r} N^{\alpha+1-r}$ tends to zero as $N \to \infty$ and therefore $C(\alpha) = \lim_{N\to\infty}\sum_{n=1}^{N} n^{\alpha} = \sum_{n=1}^{\infty} n^{\alpha} = \zeta(-\alpha)$. On the other hand, for an arbitrary fixed $h \in \mathbb{N}$, the formula (1.51) can be seen to define $C(\alpha)$ as an analytic function of α in the *larger* region $\{\alpha \in \mathbb{C} : \alpha \neq -1,\ \Re\alpha < h-1\}$ (because of uniform convergence on compacta). Hence, by uniqueness of analytic continuation, our formula for $C(\alpha)$ provides *the* analytic continuation of $\zeta(-\alpha)$ to this region! In particular, since h is arbitrary, we have proved that $\zeta(s)$ has an analytic continuation to all of $s \in \mathbb{C} \setminus \{1\}$!

(Connecting with our previous example: In Maple, typing `Digits:=30:` and then `evalf(Zeta(-3/2));` indeed gives "$-0.02548520188983303\ldots$".)

Finally, it is interesting to consider the special case of α being a nonnegative integer: $\alpha \in \mathbb{Z}_{\geq 0}$. In this case we have $\binom{\alpha}{k} = \binom{\alpha}{h} = 0$ (since $k \geq h > \alpha + 1$) and $\binom{\alpha+1}{r} = 0$ for all integers $r \geq \alpha + 2$; hence (1.50) says

$$\sum_{n=1}^{N} n^{\alpha} = 1 + \frac{1}{\alpha+1}\sum_{r=0}^{\alpha+1}(-1)^r \binom{\alpha+1}{r} B_r \big(N^{\alpha+1-r} - 1\big) \tag{1.52}$$

[6]In this paragraph we assume knowledge of some complex analysis, and we don't give as many details as mostly elsewhere.

This has been proved for all $N \in \mathbb{N}$, but one easily convinces oneself that this identity must also be valid at $N = 0$, [7] and this implies

$$\sum_{r=0}^{\alpha}(-1)^r\binom{\alpha+1}{r}B_r = \alpha+1. \tag{1.53}$$

Using this, (1.52) can be simplified somewhat, into

$$\sum_{n=1}^{N} n^\alpha = \frac{1}{\alpha+1}\sum_{r=0}^{\alpha}(-1)^r\binom{\alpha+1}{r}B_r N^{\alpha+1-r}.$$

This is the so-called *Faulhaber's formula.* Furthermore, from (1.53) and (1.51) we obtain:

$$\zeta(-\alpha) = C(\alpha) = \frac{(-1)^\alpha B_{\alpha+1}}{\alpha+1}.$$

In particular $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, $\zeta(-2) = 0$, $\zeta(-3) = \frac{1}{120}$.

1.4. **Some more examples.**

Example 1.10. A counting function of fundamental importance in number theory is

$$\pi(x) = \#\{p \,:\, p \text{ is a prime number } \le x\}.$$

The *Prime Number Theorem* (PNT) gives an asymptotic formula for $\pi(x)$:

$$\pi(x) \sim \frac{x}{\log x} \qquad \text{as } x \to \infty.$$

The PNT was proved independently by Hadamard and de la Vallée-Poussin (1896); much of the work was based on a celebrated memoir by Riemann 1859. The starting point for the proof is the *Euler product formula* for the Riemann zeta function:

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}, \qquad \forall s \in \mathbb{C} \text{ with } \Re s > 1, \tag{1.54}$$

where the product is taken over all primes p. This formula is simply a rephrasing of the fundamental theorem of arithmetic (the fact that each positive integer has a unique factorization into primes) in terms

[7]A completely elementary way of seeing this without going back and generalizing the earlier discussion is to note that the right hand equals $P(N)$, where $P(X)$ is a polynomial of degree $\le \alpha+1$; and the identity implies that $P(X+1) - P(X) - (X+1)^\alpha = 0$ for all $X \in \mathbb{N}$; but both sides of the last relations are polynomials, hence the last identity in fact holds for all $X \in \mathbb{R}$, and taking $X = 0$ and using $P(1) = 1$ we get the desired claim.

of generating functions. Indeed, on a formal level unique factorization implies that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right) = \prod_p \frac{1}{1 - p^{-s}},$$

and this calculation can easily be made rigorous for all s with $\Re s > 1$. (Similarly in the remainder of this example we will present calculations in a rather formal style, but they can all be made rigorous.)

The starting idea for the (standard) proof of the PNT is to try to *invert* the formula (1.54) to extract information about the primes, or more specifically about $\pi(x)$! An obvious first step is to take the logarithm in (1.54) so as to transform the product into a sum; in fact it turns out to be slightly more convenient to deal with the *derivative* of the logarithm; i.e.

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds} \log \prod_p \frac{1}{1 - p^{-s}} = \sum_p \frac{d}{ds} \log \frac{1}{1 - p^{-s}} = -\sum_p \frac{p^{-s} \log p}{1 - p^{-s}} \\ &= -\sum_p \left(p^{-s} + p^{-2s} + p^{-3s} + \ldots\right) \log p = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \end{aligned}$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } \ n = p^r \text{ for some prime } p \text{ and } r \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Writing

$$\Psi(x) = \sum_{1 \le n \le x} \Lambda(n),$$

the last formula can be expressed as

$$\frac{\zeta'}{\zeta}(s) = -\int_1^{\infty} x^{-s} \, d\Psi(x) = -s \int_1^{\infty} x^{-s-1} \Psi(x) \, dx,$$

where we integrated by parts in the last step. This means that $\frac{\zeta'}{\zeta}(s)$ *is a kind of Fourier transform of* $\Psi(x)$*!* Indeed, writing $s = \sigma + it$ $(\sigma, t \in \mathbb{R})$ and substituting $x = e^u$ we get $\frac{\zeta'}{\zeta}(\sigma + it) = -s \int_0^{\infty} e^{-\sigma u} \Psi(e^u) e^{-itu} \, du$, i.e. the function $t \mapsto \frac{\zeta'}{\zeta}(\sigma + it)$ is the Fourier transform of the function $u \mapsto -s e^{-\sigma u} \Psi(e^u)$. We will see in a later lecture, as an example on the inverse Fourier transformation and methods of asymptotic expansions, how the last formula can be inverted and used, in combination with the very important fact that $\zeta(s)$ has no zeros for $\Re s \ge 1$, to deduce

$$\Psi(x) \sim x \qquad \text{as } \ x \to \infty,$$

which can be elementarily seen to imply the PNT!

Example 1.11. The *Gauss circle problem* is about estimating the number of integer points in a circle of radius r centered at the origin, for large r, that is

$$A(r) = \#\big\{n = (n_1, n_2) \in \mathbb{Z}^2 \,:\, |n|^2 = n_1^2 + n_2^2 \leq r^2\big\}.$$

Gauss made the first progress on this problem by proving

$$A(r) = \pi r^2 + O(r), \qquad \forall r \geq 1. \tag{1.55}$$

This can be proved by estimating $A(r)$ from above and below using circles of slightly larger/smaller radius. In precise terms: Let us write $B_r \subset \mathbb{R}^2$ for the open disc with center at the origin and radius r. Let $M_r \subset \mathbb{R}^2$ be the union of all squares $n + [-\frac{1}{2}, \frac{1}{2}]^2$ for $n \in \mathbb{Z}^2$, $|n| \leq r$; then the area of M_r equals $A(r)$. Now $B_{r-\sqrt{1/2}} \subset M_r \subset \overline{B_{r+\sqrt{1/2}}}$ for all $r \geq 1$; this is easily seen by drawing a picture! (The detailed proof uses the triangle inequality and the fact that every point in a square $n+[-\frac{1}{2}, \frac{1}{2}]^2$ has distance $\leq \sqrt{1/2}$ to its center n.) Hence by comparing areas we conclude:

$$\pi\big(r - \sqrt{1/2}\big)^2 \leq A(r) \leq \pi\big(r + \sqrt{1/2}\big)^2, \qquad \forall r \geq 1,$$

and this implies (1.55).

The error bound in (1.55) has been successively improved over the years; Voronoi (1903) improved it to $O(r^{2/3})$, and the best known bound today is due to Huxley (2003) [10] who proved $A(r) = \pi r^2 + O(r^{\frac{131}{208}+\varepsilon})$. [8] (Note that $\frac{131}{208} \approx 0.6298\ldots$.) It has been conjectured that $A(r) = \pi r^2 + O(r^{\frac{1}{2}+\varepsilon})$. This bound would be optimal; it is know that $A(r) = \pi r^2 + O(r^\theta)$ *cannot* hold with any $\theta \leq \frac{1}{2}$.

One way to attack the the Gauss circle problem, which we will discuss in a later lecture, is by using the *Poisson summation formula.* This formula says that for any sufficiently nice function $f : \mathbb{R}^m \to \mathbb{C}$, if $\widehat{f}(\xi) = \int_{\mathbb{R}^m} f(x) e^{-2\pi i \xi \cdot x}\, dx$ is the Fourier transform of f then

$$\sum_{n \in \mathbb{Z}^m} f(n) = \sum_{\xi \in \mathbb{Z}^m} \widehat{f}(\xi). \tag{1.56}$$

"Sufficiently nice" here means that f has to be both sufficiently smooth and decay sufficiently fast at infinity; cf., e.g., Folland Theorem 8.32 for one precise statement. For the circle problem, one *would like to* take $f(x) = I(|x| \leq r)$, i.e. $f(x) = 1$ when $|x| \leq r$ and $f(x) = 0$ when $|x| > r$. With this choice the left hand side of (1.56) would equal $A(r)$ exactly! The problem is that this function f is far fram smooth; it is even discontinuous, and correspondingly (1.56) is not absolutely convergent and one has to do some work before one can make sense

[8] This statement should be understood as: For any fixed $\varepsilon > 0$ one has $A(r) = \pi r^2 + O(r^{\frac{131}{208}+\varepsilon})$ as $r \to \infty$ (or equivalently: for all $r \geq 1$). The implied constant is allowed to depend on ε but not on r.

out of the right hand side in (1.56). In a later lecture we will see how to modify this approach and use it to prove the Voronoi estimate $A(r) = \pi r^2 + O(r^{\frac{2}{3}})$.

The Gauss circle problem is only one very special case of the general problem of *counting the lattice points in a given (large) region.* This general problem has applications in many areas of mathematics and we will come back to it several times.

2. Lecture 2: Measure and integration theory

I spent the first half of Lecture 2 on stuff from the previous section which I didn't get time to finish in Lecture 1: Euler-MacLaurin summation (cf. Sec. 1.3), and some quick words on the Prime Number Theorem and the Gauss circle problem (cf. Sec. 1.4).

In the second half of Lecture 2 I went through the basic definitions of measurable spaces and measure spaces, cf. Folland Ch. 1.1-1.3.

2.1. **Some notes.** On an example which I mentioned in class. I did (or intended to do...) the following: Let $q_1, q_2, \ldots$ be an enumeration of all the rational numbers in $[0, 1]$. Let $f_n : \mathbb{R} \to [0, \infty]$ be the piecewise linear function which equals 0 for $|x - q_n| \geq n^{-2}$ and equals 1 at $x = q_n$, and is linear in between these, i.e.

$$f_n(x) = \begin{cases} 1 - n^2|x - q_n| & \text{if } |x - q_n| \leq n^{-2} \\ 0 & \text{if } |x - q_n| \geq n^{-2}. \end{cases}$$

Now set

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

This is certainly a well-defined function $f : \mathbb{R} \to [0, \infty]$, and Folland's Prop. 2.7 and Theorem 2.15 imply that f is Borel measurable (thus $f \in L^+(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$) and that its Lebesgue integral is:

$$\int_{\mathbb{R}} f(x)\, dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)\, dx = \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}. \tag{2.1}$$

Note also that f has support $\subset [-1, 2]$, since each f_n has support $\subset [-1, 2]$.

However f *is not Riemann integrable* over the interval $[-1, 2]$; in fact f is not Riemann integrable over *any* interval $[A, B]$ with $0 \leq A < B \leq 1$. The most direct reason for this is the following:

Lemma 2.1. *f takes the value ∞ in any interval $[A, B]$ with $0 \leq A < B \leq 1$.*

(We give the proof below.) This lemma implies that f does not fit into Definition 1.4 since f is not even a function into $\mathbb{C}$! Now one might argue that it is natural, for functions $g : [A, B] \to [0, \infty]$, to extend the notion of Riemann-integrability by saying that the Riemann integral $\int_A^B g(x)\,dx$ exists if *either* $\int_A^B g(x)\,dx$ exists in the usual sense of Definition 1.4 *or* if for every $K > 0$ there is some $\delta > 0$ such that $S(\{x_n\}, \{\xi_n\}) > K$ holds whenever $\langle\{x_n\}, \{\xi_n\}\rangle$ is a tagged partition of $[A, B]$ with mesh $\leq \delta$; and in this latter case we would say that $\int_A^B g(x)\,dx$ *equals* ∞. However, it turns out that *even with this extended definition, our function f is not Riemann integrable over any interval $[A, B]$ with $0 \leq A < B \leq 1$!* Indeed, on the one hand side, by Lemma 2.1, *if* the Riemann integral $\int_A^B f(x)\,dx$ exists (according to our extended definition) then the only possibility is $\int_A^B f(x)\,dx = \infty$; and on the other hand the following lemma proves that $\int_A^B f(x)\,dx = \infty$ *doesn't* hold.

Lemma 2.2. *Let $0 \leq A < B \leq 1$. Given any partition $\{x_n\}_{n=0}^N$ of $[A, B]$, then for any $\varepsilon > 0$ we can find numbers $\xi_n \in [x_{n-1}, x_n]$ such that*

$$S(\{x_n\}, \{\xi_n\}) = \sum_{n=1}^{N} f(\xi_n)(x_n - x_{n-1}) \leq \int_{[A,B]} f(x)\,dx + \varepsilon,$$

where $\int_{[A,B]} f(x)\,dx$ denotes the Lebesgue integral, thus $\int_{[A,B]} f(x)\,dx \leq \int_{\mathbb{R}} f(x)\,dx = \frac{\pi^2}{6}$ by (2.1).

(I'm not sure if we may even take $\varepsilon = 0$ in the above statement.)

We conclude by giving the proofs of the two lemmas.

Proof of Lemma 2.1. Let $0 \leq A < B \leq 1$ be given. Then we can take some $n_1 \in \mathbb{N}$ such that $q_{n_1} \in (A, B)$. We now construct $n_2, n_3, \ldots$ recursively as follows: Given $n_1, \ldots, n_j$, we choose $n_{j+1} \in \mathbb{N}$ so that $n_{j+1} > n_j$ and

$$q_{n_{j+1}} \in (A, B) \cap \bigcap_{k=1}^{j} \left(q_{n_k} - \frac{1}{2n_k^2}, q_{n_k} + \frac{1}{2n_k^2} \right). \tag{2.2}$$

This is possible since at each step the intersection in the right hand side of (2.2) is a nonempty open interval and hence contains infinitely many rational points. The property that the intersection is a nonempty open interval is preserved when going from j to $j + 1$ since the intersection of two nonempty open intervals which have one point in common (in our case: the point $q_{n_{j+1}}$) is a new nonempty open interval.

In this way we obtain an infinite sequence $1 \leq n_1 < n_2 < n_3 < \ldots$ such that (2.2) holds for each $j \geq 0$. In particular then $|q_{n_k} - q_{n_\ell}| <$

$\frac{1}{2}n_k^{-2}$ whenever $1 \leq k \leq \ell$; hence the sequence $\{q_{n_k}\}_{k=1}^{\infty}$ is Cauchy and the limit $x_0 = \lim_{k\to\infty} q_{n_k} \in \mathbb{R}$ exists. Using (2.2) again gives

$$x_0 \in \overline{(A,B) \cap \bigcap_{k=1}^{j} \Big(q_{n_k} - \frac{1}{2n_k^2}, q_{n_k} + \frac{1}{2n_k^2} \Big)}, \qquad \forall j \in \mathbb{N},$$

and thus $x_0 \in [A,B]$ and $|x_0 - q_{n_j}| \leq \frac{1}{2n_j^2}$ for all $j \in \mathbb{N}$. Hence $f_{n_j}(x_0) \geq \frac{1}{2}$ for each $j \in \mathbb{N}$ and thus

$$f(x_0) = \sum_{n=1}^{\infty} f_n(x_0) \geq \sum_{j=1}^{\infty} f_{n_j}(x_0) = \sum_{j=1}^{\infty} \frac{1}{2} = \infty.$$

Hence: $[A,B]$ contains a point at which f takes the value ∞. □

Proof of Lemma 2.2. We prove more generally that for any real $A < B$, if f is *any* Lebesgue measurable function $f : [A,B] \to [0,\infty]$ with $\int_{[A,B]} f(x)\,dx < \infty$, then for any $\varepsilon > 0$ and any partition $\{x_n\}_{n=0}^{N}$ of $[A,B]$, we can find numbers $\xi_n \in [x_{n-1}, x_n]$ such that

$$S(\{x_n\},\{\xi_n\}) = \sum_{n=1}^{N} f(\xi_n)(x_n - x_{n-1}) \leq \int_{[A,B]} f(x)\,dx + \varepsilon. \tag{2.3}$$

To see this, let us first throw away any repetitions from the sequence $\{x_n\}$ (this does not affect $S(\{x_n\},\{\xi_n\})$ since $0 \cdot \infty = 0$ by definition); after this we may assume that $A = x_0 < x_1 < \ldots < x_N = B$. Now note that the function

$$\phi = \sum_{n=1}^{N-1} \inf\{f(x) \,:\, x \in [x_{n-1}, x_n)\} \cdot \chi_{[x_{n-1},x_n)} + \inf\{f(x) \,:\, x \in [x_{N-1}, x_N]\} \cdot \chi_{[x_{N-1},x_N]}$$

is a simple function given in its standard representation and which satisfies $0 \leq \phi \leq f$. (Note that each infimum appearing in the above sum is $< \infty$, indeed otherwise we would trivially have $\int_{[A,B]} f(x)\,dx = \infty$ contradicting our assumption.) Hence by the definition of the Lebesgue integral $\int_{[A,B]} f(x)\,dx$ we have

$$\begin{aligned}\int_{[A,B]} f(x)\,dx &\geq \int_{[A,B]} \phi(x)\,dx \\ &= \sum_{n=1}^{N-1} \inf\{f(x) \,:\, x \in [x_{n-1}, x_n)\} \cdot (x_n - x_{n-1}) \\ &\qquad + \inf\{f(x) \,:\, x \in [x_{N-1}, x_N]\} \cdot (x_N - x_{N-1}).\end{aligned}$$

Finally for each $n \in \{1,\ldots,N\}$ we can find some $\xi_n \in [x_{n-1}, x_n]$ such that the corresponding infimum appearing in the above sum is

$> f(\xi_n) - \frac{\varepsilon}{N(x_n - x_{n-1})}$, and it then follows that the last sum is

$$> \sum_{n=1}^{N} \Big(f(\xi_n)(x_n - x_{n-1}) - \frac{\varepsilon}{N} \Big) \geq S(\{x_n\}, \{\xi_n\}) - \varepsilon.$$

Hence (2.3) holds, and the proof is complete. $\square$

3. Lecture 3: Measure and integration theory

In this lecture I plan to discuss:

* Measurable functions and integration of functions in L^+: Folland Ch. 2.1-2.2..
(Please look through Ch. 2.3; I will basically just say: "now it is easy to generalize to *complex* functions...)
* Product measures: Folland Ch. 2.5.
* Lebesgue measure on $\mathbb{R}^n$: Folland Ch. 2.6.
* Jordan content; Folland Ch. 2.6, but I will highlight some other facts, see Theorem 3.7 below.

3.1. **A misprint.** Note that in Folland's book, Proposition 2.13(d); "$A \mapsto \int_A d\mu$" should read "$A \mapsto \int_A \phi \, d\mu$". (For some other misprints, see www.math.washington.edu/~folland/Homepage/reals.pdf.)

3.2. **"Monotone Convergence Theorem ⇔ Fatou's lemma".** Note that the Monotone Convergence Theorem and Fatou's lemma are "almost equivalent". Indeed Fatou's lemma is an immediate consequence of the Monotone Convergence Theorem as seen in Folland, p. 52. In the other direction, let us give a proof of the Monotone Convergence Theorem *assuming* Fatou's lemma (this is Folland's Exercise 17):

Let $\{f_n\}$ be a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and set $f = \lim_{n\to\infty} f_n \in L^+$. We wish to prove that $\int f = \lim_{n\to\infty} \int f_n$. Note that $\{\int f_n\}$ is an increasing sequence (by Folland's Prop. 2.13(c)), so $\lim_{n\to\infty} \int f_n$ exists in $[0, \infty]$. Whenever a sequence in $[0, \infty]$ has a limit as $n \to \infty$, this limit equals the $\liminf$ as $n \to \infty$; hence $\lim_{n\to\infty} \int f_n = \liminf_{n\to\infty} \int f_n$, and also $f = \liminf_{n\to\infty} f_n$. Thus Fatou's lemma gives $\int f \, d\mu \leq \lim_{n\to\infty} \int f_n$. But the opposite inequality is *trivial:* For each n we have $f_n \leq f$ and thus $\int f_n \leq \int f$; thus also $\lim_{n\to\infty} \int f_n \leq \int f$. Hence $\lim_{n\to\infty} \int f_n = \int f$. $\square$

3.3. **A remark on the Dominated Convergence Theorem.** In the Dominated Convergence Theorem (Folland's Theorem 2.24), one might *add the following conclusion:* $\lim_{n\to\infty} \int |f - f_n| = 0$.

Proof. (Cf. Rudin, [18, Thm. 1.34].) After modifying g by setting $g(x) = \infty$ for x in a set of measure zero, we may assume that $g \in L^+$, still $\int_X |g| \, d\mu < \infty$, and $|f_n(x)| \leq g(x)$ for *all* x and n. As noted in Folland's Theorem 2.24 we have $|f| \leq g$ and $f \in L^1$. Now note that $|f_n - f| \leq |f_n| + |f| \leq 2g$ for every n; hence $2g - |f_n - f| \in L^+$ for every n and we may apply Fatou's lemma to this sequence of functions. This gives, since $\liminf_{n\to\infty}(2g - |f_n - f|) = 2g$ a.e.:

$$\int_X 2g \, d\mu \leq \liminf_{n\to\infty} \int_X (2g - |f_n - f|) \, d\mu$$
$$= \int_X 2g \, d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu.$$

Since $\int_X 2g \, d\mu < \infty$, the above implies

$$\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0.$$

But also $\int_X |f_n - f| \, d\mu \geq 0$ for each n; hence we conclude

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

□

Note that the new conclusion is stronger than the conclusion $\int f = \lim_{n\to\infty} \int f_n$, in the sense that the latter follows immediately from the former using the triangle inequality: $|\int f - \int f_n| = |\int (f - f_n)| \leq \int |f - f_n| \to 0$.

3.4. **A remark on the definition of $\int_X f \, d\mu$.** Regarding the definition of $\int_X f \, d\mu$ for $f \in L^+$ (cf. Folland pp. 49-50), an alternative definition is as follows: For any $f \in L^+$, set

$$\int_X f \, d\mu = \int_0^\infty \mu(\{x \in X \; : \; f(x) \geq t\}) \, dt, \tag{3.1}$$

where the integral on the right is a *generalized Riemann integral* (cf. Def. 1.16 above), i.e. $= \lim_{A\to 0^+} \lim_{B\to\infty} \int_A^B \mu(\{x \in X \; : \; f(x) > t\}) \, dt$. The definition (3.1) is used in Lieb and Loss, [12], and I think that this definition makes *crystal clear* the fact which Folland mentions on p.58: To compute the Lebesgue integral $\int_X f \, d\mu$, “one is in effect partitioning the *range* of f into subintervals I_j and approximating f by a constant on each of the sets $f^{-1}(I_j)$”!

Let us *prove* (3.1)! First of all we note that if the right hand side of (3.1) is instead understood as a *Lebesgue integral* with respect to Lebesgue measure on $\mathbb{R}_{\geq 0}$, then, at least if we assume that $(X, \mathcal{M}, \mu)$ is

σ-finite, [9] the relation follows easily from the Fubini-Tonelli theorem.[10] Indeed, consider the following subset of $X \times \mathbb{R}_{\geq 0}$:

$$G_f = \{(x,t) \in X \times \mathbb{R}_{\geq 0} \,:\, t \leq f(x)\}.$$

This set is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}_{\geq 0}}$-measurable. [Proof: The map $(x,t) \mapsto f(x)$ from $X \times \mathbb{R}_{\geq 0}$ to $\overline{\mathbb{R}}$ is measurable since it is the composition of the projection map $X \times \mathbb{R}_{\geq 0} \to X$, the map $f : X \to [0,\infty]$, and the inclusion map $[0,\infty] \to \overline{\mathbb{R}}$ (which is continuous); and the map $(x,t) \mapsto t$ from $X \times \mathbb{R}_{\geq 0}$ to $\overline{\mathbb{R}}$ is measurable since it is the composition of the projection map $X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and the inclusion map $\mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$. . Hence the map $(x,t) \mapsto f(x) - t$ from $X \times \mathbb{R}_{\geq 0}$ to $\overline{\mathbb{R}}$ is measurable, by Folland's Prop. 2.6 and the ensuing remark. Hence G_f, being the inverse image of $[0,\infty]$ under this map, is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}_{\geq 0}}$-measurable.] Hence by the Fubini-Tonelli Theorem applied to the characteristic function of G_f (or, equivalently, Theorem 2.36 applied to the set G_f), the function

$$t \mapsto \mu((G_f)^t) = \mu(\{x \in X \,:\, f(x) \geq t\})$$

from $\mathbb{R}_{\geq 0}$ to $[0,\infty]$ is measurable, and so is the map $x \mapsto m((G_f)_x) = f(x)$ from X to $[0,\infty]$ (but this we already knew), and we have

$$\mu \times m(G_f) = \int_X f(x)\, d\mu(x) = \int_{\mathbb{R}_{\geq 0}} \mu(\{x \in X \,:\, f(x) \geq t\})\, dt.$$

This proves (3.1). [And it also proves that $\mu \times m(G_f) = \int_X f\, d\mu$, which, as Folland points out in his Exercise 50, is the definitive statement of the familiar theorem from calculus, "the integral of a function is the area under its graph".] $\square$

Remark 3.1. Note that redefining G_f by replacing $\leq$ by $<$, i.e. setting $G_f = \{(x,t) \in X \times \mathbb{R}_{\geq 0} \,:\, t < f(x)\}$, gives exactly the same result:

$$\mu \times m(G_f) = \int_X f(x)\, d\mu(x) = \int_{\mathbb{R}_{\geq 0}} \mu(\{x \in X \,:\, f(x) > t\})\, dt.$$

This is because $m((G_f)_x) = f(x)$ for all $x \in X$ regardless of which definition we use. Certainly $\mu(\{x \in X \,:\, f(x) \geq t\})$ may be *strictly larger* than $\mu(\{x \in X \,:\, f(x) > t\})$ for certain t-values, but the

[9] Actually we do not need the assumption that $(X,\mathcal{M},\mu)$ is σ-finite. Indeed, for an arbitrary measure space $(X,\mathcal{M},\mu)$ and $f \in L^+$, if there is some $t > 0$ such that $\mu(\{x \in X \,:\, f(x) \geq t\}) = \infty$ then both sides of (3.1) are easily seen to be ∞ and thus the equality holds. In the remaining case we have $\mu(\{x \in X \,:\, f(x) \geq t\}) < \infty$ for each $t > 0$; then set $X' = \{x \in X \,:\, f(x) > 0\}$; this set is in $\mathcal{M}$, and the restriction of μ to X' (cf. Folland p.27, Exercise 10) is a σ-finite measure, since X' equals the union of $\{x \in X \,:\, f(x) \geq t\}$ for $t = 1^{-1}, 2^{-1}, 3^{-1}, \ldots$. Now both the left and the right hand side of (3.1) are invariant under replacing X by X'; we have thus reduced the problem to proving (3.1) in the σ-finite case.

[10] The following will more or less give a solution to Folland's Exercise 50, except that we prefer to work with $X \times \mathbb{R}_{\geq 0}$ in place of $X \times [0,\infty]$.

identities just proved imply that this can only hold for a set of t-values which has (Lebesgue) measure zero.

Let us next check that (3.1) holds also when we view the right hand side as a generalized Riemann integral. For this, let us first of all check carefully that this generalized Riemann integral is well-defined. Define the function $F : \mathbb{R}_{\geq 0} \to [0, \infty]$ through $F(t) = \mu(\{x \in X : f(x) \geq t\})$, so that we are studying the generalized Riemann integral $\int_0^\infty F(t)\,dt = \lim_{A\to 0^+} \lim_{B\to\infty} \int_A^B F(t)\,dt$. Note that F is *decreasing.* If $F(t) = \infty$ for some $t > 0$ then also $F(t') = \infty$ for all $t' \in (0, t]$ and in this case we should clearly understand $\int_0^\infty F(t)\,dt$ to be ∞ (note that this agrees with $\int_0^\infty F(t)\,dt$ viewed as a Lebesgue integral). Hence from now on we may assume that $F(t) < \infty$ for all $t > 0$. Then F is bounded on any interval $[A, B]$ with $0 < A < B$; and the fact that F is decreasing implies that F is Riemann-integrable on each such interval $[A, B]$. [Details: If $\{t_n\}_{n=0}^N$ is a partition of $[A, B]$ with mesh $\leq \delta$ then

$$\begin{aligned}
&\sum_{n=1}^{N} (t_n - t_{n-1}) \cdot \sup\big\{|F(\xi) - F(\xi')| \,:\, \xi, \xi' \in [t_{n-1}, t_n]\big\} \\
&= \sum_{n=1}^{N} (t_n - t_{n-1}) \cdot (F(t_{n-1}) - F(t_n)) \\
&\leq \delta \sum_{n=1}^{N} (F(t_{n-1}) - F(t_n)) = \delta(F(A) - F(B)),
\end{aligned}$$

and this tends to 0 as $\delta \to 0$; hence Proposition 1.14 implies that F is Riemann-integrable on $[A, B]$. ((Alternatively we may use Folland's Theorem 2.28(b) together with the fact that any monotonic function has at most a countable number of discontinuity points.))] But F is nonnegative; hence $\int_A^B F(t)\,dt$ is an increasing function of B for any fixed A, and $\lim_{B\to\infty} \int_A^B F(t)\,dt$ is a decreasing function of $A > 0$; this implies that the generalized Riemann integral $\int_0^\infty F(t)\,dt$ exists as a uniquely defined number in $[0, \infty]$ (cf. also Remark 1.17).

Using now Folland's Theorem 2.28 together with the Monotone Convergence Theorem it follows that the generalized Riemann integral $\int_0^\infty F(t)\,dt$ equals the corresponding Lebesgue integral over $\mathbb{R}_{\geq 0}$, and hence by what we have already proved the identity (3.1) holds! □

Note that the above proof uses the full force of the integration theory developed in Folland, Ch. 2.1-5; and our purpose was to illustrate some standard use of this theory. However it is also interesting to note that (3.1) can be proved fairly easily directly from the *definitions* of the

Riemann integral and the Lebesgue integral $\int_X f\,d\mu$ (Folland p. 49-50). Thus taking (3.1) as the definition of $\int_X f\,d\mu$ need not cause much differences in the development of the theory!

To give such a direct proof, we keep $F(t) = \mu(\{x \in X \,:\, f(x) \geq t\})$ as above. Let $\phi : X \to \mathbb{R}$ be any simple function satisfying $0 \leq \phi \leq f$, and let its standard representation be $\phi = \sum_{n=0}^{N} a_n \chi_{E_n}$. Then for every $t \geq 0$ we have

$$F(t) = \mu(\{x \in X \,:\, f(x) \geq t\}) \geq \mu(\{x \in X \,:\, \phi(x) \geq t\}),$$

and hence

$$\int_0^\infty F(t)\,dt \geq \int_0^\infty \mu(\{x \in X \,:\, \phi(x) \geq t\})\,dt = \int_0^\infty \sum_{\substack{n\in\{0,\ldots,N\} \\ (a_n \geq t)}} \mu(E_n)\,dt$$
$$= \sum_{n\in\{0,\ldots,N\}} \mu(E_n) \int_0^{a_n} dt = \int_X \phi\,d\mu.$$

This holds for every simple function ϕ satisfying $0 \leq \phi \leq f$; hence by the definition of $\int_X f\,d\mu$ (Folland p.50, middle) we have

$$\int_X f\,d\mu \leq \int_0^\infty F(t)\,dt. \tag{3.2}$$

On the other hand, if I is any real number strictly less than $\int_0^\infty F(t)\,dt$, then there are some $0 < A < B$ such that also $I < \int_A^B F(t)\,dt$, and then by Definition 1.4 there is some δ such that for any tagged partition $\langle \{t_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ of $[A,B]$ with mesh $\leq \delta$ we have $\sum_{n=1}^N F(\xi_n)(t_n - t_{n-1}) > I$. Let us now take $\{t_n\}_{n=0}^N$ to be an arbitrary partition of $[A,B]$ with mesh $\leq \delta$ and satisfying $t_{n-1} < t_n$ for each $n \in \{1,\ldots,N\}$. Set $\xi_n = t_n$; it follows that $\sum_{n=1}^N F(t_n)(t_n - t_{n-1}) > I$. Now set $t_{-1} = 0$,

$$E_n = \{x \in X \,:\, t_n \leq f(x) < t_{n+1}\} \qquad \text{for } n = -1, 0, 1, \ldots, N-1,$$

and

$$E_N = \{x \in X \,:\, t_N \leq f(x)\}.$$

Then $E_{-1}, E_0, \ldots, E_N$ are pairwise disjoint measurable subsets of X whose union is X; hence

$$\phi = \sum_{n=-1}^{N} t_n \chi_{E_n}$$

is the standard representation of a simple function on X. By construction we have $0 \leq \phi \leq f$; thus

$$\int_X f\, d\mu \geq \int_X \phi\, d\mu = \sum_{n=-1}^{N} t_n \mu(E_n) = \sum_{n=0}^{N} \bigg(\sum_{j=0}^{n} (t_j - t_{j-1}) \bigg) \mu(E_n)$$

$$= \sum_{j=0}^{N} (t_j - t_{j-1}) \sum_{n=j}^{N} \mu(E_n) = \sum_{j=0}^{N} (t_j - t_{j-1}) F(t_j) \geq \sum_{j=1}^{N} (t_j - t_{j-1}) F(t_j) > I.$$

We have thus proved that $I < \int_X f\, d\mu$ holds for every number I which is strictly less than $\int_0^\infty F(t)\, dt$. Hence:

$$\int_X f\, d\mu \geq \int_0^\infty F(t)\, dt. \tag{3.3}$$

By (3.2) and (3.3) together we have now proved (3.1). $\square$

3.5. **Push-forward of measures.** In invariance results such as Folland's Theorem 2.42 or Theorem 2.44, the statement about functions is “completely equivalent” to the statement about sets! That is, the (a) and (b) parts of Theorem 2.42 are “completely equivalent”, and so are the (a) and (b) parts of Theorem 2.44. In order to show this equivalence in general, let us first consider the following natural notion:

Definition 3.2. If $T : X \to Y$ is a measurable map from one measurable space $(X, \mathcal{M})$ to another measurable space $(Y, \mathcal{N})$, and μ is a measure on $(X, \mathcal{M})$, then the *push-forward* $T_*\mu : \mathcal{N} \to [0, +\infty]$ is defined by the formula $T_*\mu(E) = \mu(T^{-1}(E))$, $\forall E \in \mathcal{N}$. One checks immediately that $T_*\mu$ is a measure on $(Y, \mathcal{N})$.

Now we have the following natural integration formula:

Proposition 3.3. *Let T, $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N})$ be as above. Then for any $f \in L^+(Y, \mathcal{N})$ we have $f \circ T \in L^+(X, \mathcal{M})$ and $\int_X (f \circ T)\, d\mu = \int_Y f\, d(T_*\mu)$. Similarly for any $f \in L^1(Y, T_*\mu)$ we have $f \circ T \in L^1(X, \mu)$ and, again, $\int_X (f \circ T)\, d\mu = \int_Y f\, d(T_*\mu)$.*

The proof is completely standard:

Proof. If ϕ is a simple function in $L^+(Y, \mathcal{N})$ with standard representation $\phi = \sum_{j=1}^n z_j \chi_{E_j}$ (thus $E_1, \ldots, E_n \in \mathcal{N}$ and these sets form a partition of Y), then $\phi \circ T = \sum_{j=1}^n z_j \chi_{T^{-1}(E_j)}$; this is a simple function in $L^+(X, \mathcal{M})$ in its standard representation of, and

$$\int_X (\phi \circ T)\, d\mu = \sum_{j=1}^{n} z_j \mu(T^{-1}(E_j)) = \sum_{j=1}^{n} z_j (T_*\mu)(E_j) = \int_Y \phi\, d(T_*\mu). \tag{3.4}$$

Now let f be an arbitrary function in $L^+(Y,\mathcal{N})$. Then $f\circ T$ is the composition of two measurable functions, hence $f\circ T$ is an $\mathcal{M}$-measurable function $X\to[0,+\infty]$, thus $f\circ T\in L^+(X,\mathcal{M})$. Let $\phi_1,\phi_2,\ldots$ be an increasing sequence of simple functions in $L^+(Y,\mathcal{N})$ such that $\phi_j\to f$ pointwise. Such a sequence exists by Folland's Theorem 2.10. Then $\phi_1\circ T,\phi_2\circ T,\ldots$ is an increasing sequence of simple functions in $L^+(X,\mathcal{M})$, and $\phi_j\circ T\to f\circ T$ pointwise. Hence

$$\int_X (f\circ T)\,d\mu=\lim_{j\to\infty}\int_X(\phi_j\circ T)\,d\mu=\lim_{j\to\infty}\int_X\phi_j\,d(T_*\mu)=\int_X f\,d(T_*\mu).$$

[The first equality holds by the Monotone Convergence Theorem; the second by (3.4), and the third by the Monotone Convergence Theorem.]

Finally let f be an arbitrary function in $L^1(Y,T_*\mu)$. Then $f\circ T$ is an $\mathcal{M}$-measurable function $X\to\mathbb{C}$ and $|f\circ T|=|f|\circ T$ (where $|f|\in L^+(X,\mathcal{M})$) so that $\int_X|f\circ T|\,d\mu=\int_Y|f|\,d(T_*\mu)<\infty$ by what we have already proved; thus $f\circ T\in L^1(X,\mu)$. Finally $\int_X(f\circ T)\,d\mu=\int_Y f\,d(T_*\mu)$ follows by splitting f into its real and imaginary part, and the positive and negative parts of these (viz., using the definition of integrals of complex functions, Folland p. 53), and using the result which we have already proved for L^+-functions. $\square$

Let us now discuss Folland's Theorem 2.44 in this language (a completely similar discussion applies for Theorem 2.42). First of all, Thm 2.44(a) obviously implies Thm 2.44(b): Indeed, given $E\in\mathcal{L}^n$, the function χ_E is a Lebesgue measurable function on $\mathbb{R}^n$; hence Thm 2.44(a) (applied for T^{-1}!) says that $\chi_E\circ T^{-1}\chi_{T(E)}$ is Lebesgue measurable, and $\int\chi_E\,dm=|\det T^{-1}|\int\chi_{T(E)}\,dm$. In other words $T(E)\in\mathcal{L}^n$ and $m(E)=|\det T|^{-1}m(T(E))$, i.e. we have proved Thm 2.44(b). *Conversely*, let us now show that Thm 2.44(b) implies Thm 2.44(a): Note that Thm 2.44(b) (applied for T^{-1}!) says that $T:\mathbb{R}^n\to\mathbb{R}^n$ is $(\mathcal{L}^n,\mathcal{L}^n)$-measurable, and that the push-forward measure T_*m equals $|\det T^{-1}|m=|\det T|^{-1}m$ (equality of measures on $(\mathbb{R}^n,\mathcal{L}^n)$). Hence if f is any Lebesgue measurable function f on $\mathbb{R}^n$, so is $f\circ T$, and furthermore by Proposition 3.3, if $f\geq 0$ or $f\in L^1(m)$ then

$$\int_{\mathbb{R}^n}(f\circ T)\,dm=\int_{\mathbb{R}^n}f\,d(T_*m)=|\det T|^{-1}\int_{\mathbb{R}^n}f\,dm.$$

In other words, we have proved Thm 2.44(a)!

3.6. **Remarks about the Lebesgue measure on $\mathbb{R}^n$.** Recall that Folland (p. 70) defines the Lebesgue measure m^n as the completion of the n-fold product of Lebesgue measure m on $\mathbb{R}$ with itself; its domain is called $\mathcal{L}^n$, the family of *Lebesgue measurable* sets in $\mathbb{R}^n$. Thus "$\mathbb{R}^n$ with Lebesgue measure" is the measure space $(\mathbb{R}^n,\mathcal{L}^n,m^n)$. However, as Folland remarks, we sometimes consider m^n as a *Borel measure*, i.e.

we consider m^n as a measure on the smaller domain $\mathcal{B}_{\mathbb{R}^n}$; the measure space is then $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m^n)$.

Just as Folland, we will most often write simply “m” for m^n, when there is no danger of confusion.

We wish to point out two uniqueness properties of m:

Proposition 3.4. *m is the unique measure on $\mathcal{B}_{\mathbb{R}^n}$ which is invariant under translations and satisfies $m([0,1]^n) = 1$.*

Recall that “invariance under translations” means that $m(E + a) = m(E)$ for every $E \in \mathcal{B}_{\mathbb{R}^n}$ and $a \in \mathbb{R}^n$. The fact that m satisfies this is Folland’s Theorem 2.42. (The invariance relateion even holds for every $E \in \mathcal{L}^n$.)

Proof. (We will use facts about regularity from Folland’s chapter 7.) Assume that μ is a measure on $\mathcal{B}_{\mathbb{R}^n}$ which is invariant under translations and satisfies $\mu([0,1]^n) = 1$; we wish to prove $\mu = m$. Note that every compact set in $\mathbb{R}^n$ is contained in a finite union of translates of the unit cube (i.e. sets $a + [0,1]^n$ with $a \in \mathbb{R}^n$); hence μ is finite on every compact set. Now Folland’s Theorem 7.8 implies that *μ is regular.*

Let us set

$$c = \mu([0,1)^n) \in [0,\infty).$$

Note that for any nonnegative integer k the box $[0,1)^n$ can be expressed as a disjoint union of exactly 2^{nk} translates of the box $\prod_{j=1}^n [0,2^{-k})$; hence $\mu(\prod_{j=1}^n [0,2^{-k})) = c2^{-nk} = cm(\prod_{j=1}^n [0,2^{-k}))$. Now by a simple modification of the proof of Folland’s Lemma 2.43 one proves that every open set $U \subset \mathbb{R}^n$ can be expressed as a countable union of disjoint translates of such cubes $[0,2^{-k})$ $(k \in \mathbb{Z}_{\geq 0})$. Hence, using the fact that both μ and m are countably additive, we have

$$\mu(U) = cm(U) \quad \text{for every open set } U \subset \mathbb{R}^n.$$

Hence since both m and μ are (outer) regular it follows that $\mu(E) = cm(E)$ for each Borel subset $E \subset \mathbb{R}^n$. Finally using $\mu([0,1]^n) = 1$ it follows that $c = 1$ and hence we have proved that $\mu(E) = m(E)$ for each Borel subset $E \subset \mathbb{R}^n$. $\square$

Proposition 3.5. *m is the unique measure on $\mathcal{B}_{\mathbb{R}^n}$ which satisfies $m(\prod_{j=1}^n [a_j,b_j]) = \prod_{j=1}^n (b_j - a_j)$ for all choices of $a_j < b_j$, $j = 1,\ldots,n$.*

(For $n = 1$ this follows from Folland’s Theorem 1.16.)

Proof. Assume that μ is a measure on $\mathcal{B}_{\mathbb{R}^n}$ which satisfies $\mu(\prod_{j=1}^n [a_j,b_j]) = \prod_{j=1}^n (b_j - a_j)$ for all choices of $a_j < b_j$, $j = 1,\ldots,n$. Using the monotonicity of μ we then also have $\mu(\prod_{j=1}^n [a_j,b_j)) = \prod_{j=1}^n (b_j - a_j)$ for all choices of $a_j < b_j$, $j = 1, \ldots, n$, and now the proof of Prop. 3.4 applies to give $\mu = m$. $\square$

3.7. **More about the Riemann integral on** $[A, B]$.

Proposition 3.6. *Let $A < B$ and let f be an arbitrary bounded function $[A, B] \to \mathbb{R}$. Then the Riemann integral $\int_A^B f(x)\,dx$ exists according to Definition 1.4 above if and only if it exists according to Folland's definition (p. 56(bottom)-57(top)); and in this case the two definitions give the same value for $\int_A^B f(x)\,dx$.*

(Note the closely related Prop. 1.14 above, which is valid for arbitrary *complex* functions on $[A, B]$.)

(Note also that what we call "Folland's definition" might be more appropriately called "Darboux's definition/characterization".)

Proof. First assume that the Riemann integral $I = \int_A^B f(x)\,dx$ exists according to Definition 1.4. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $|S(\{x_j\}, \{\xi_j\}) - I| < \varepsilon$ holds for any tagged partition $\langle \{x_j\}, \{\xi_j\} \rangle$ of $[A, B]$ of mesh $\leq \delta$. Now if $P = \{x_j\}_{j=0}^n$ is any partition of $[A, B]$ then by definition we have

$$s_P f = \inf_{\{\xi_j\}} S(\{x_j\}, \{\xi_j\}) \quad \text{and} \quad S_P f = \sup_{\{\xi_j\}} S(\{x_j\}, \{\xi_j\}),$$

where the infimum and the supremum is taken over all choices of $\xi_1, \ldots, \xi_n$ which make $\langle \{x_j\}, \{\xi_j\} \rangle$ into a tagged partition. It follows that

$$s_P f \geq I - \varepsilon \quad \text{and} \quad S_P f \leq I + \varepsilon$$

for any P of mesh $\leq \delta$. On the other hand, note that for any tagged partition $\langle \{x_j\}, \{\xi_j\} \rangle$ of $[A, B]$ we have

$$s_{\{x_j\}} f \leq S(\{x_j\}, \{\xi_j\}) \leq S_{\{x_j\}} f,$$

since $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \leq f(\xi_j) \leq \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$ for each $j \in \{1, \ldots, n\}$. It follows that if P is any partition of $[A, B]$ with mesh $\leq \delta$ then $s_P f < I + \varepsilon$ and $S_P f > I - \varepsilon$. But *every* partition P' of $[A, B]$ has a refinement P with mesh $\leq \delta$, and clearly $s_P f$ is increasing and $S_P f$ is decreasing under refinement of P; hence for every partition P' of $[A, B]$ we have

$$s_{P'} f \leq s_P f < I + \varepsilon \quad \text{and} \quad S_{P'} f \geq S_P f > I - \varepsilon. \tag{3.5}$$

We have thus proved that there *exist* partitions P of $[A, B]$ such that $s_P f \geq I - \varepsilon$ and $S_P f \leq I + \varepsilon$, and that for *every* partition P of $[A, B]$ we have $s_P < I + \varepsilon$ and $S_P > I - \varepsilon$. This holds for every $\varepsilon > 0$; hence by Folland's definition of $\overline{I}_A^B f$ and $\underline{I}_A^B f$ we have $\overline{I}_A^B f = \underline{I}_A^B f =$

I. Hence the Riemann integral $\int_A^B f(x)\,dx$ exists also according to Folland's definition, and equals I.

Conversely, now assume that the Riemann integral $\int_A^B f(x)\,dx$ exists according to Folland's definition, i.e. assume that $\overline{I}_A^B f = \underline{I}_A^B f$. Set $I = \overline{I}_A^B f = \underline{I}_A^B f$. Let $\varepsilon > 0$ be given. Then there exists some partition P of $[A, B]$ with $s_P f \geq I - \varepsilon$ and there exists a partition P with $S_P f \leq I + \varepsilon$; by considering a common refinement of these two we see that there exists a partition $P = \{x_j\}_{j=0}^n$ of $[A, B]$ satisfying both $s_P f \geq I - \varepsilon$ and $S_P f \leq I + \varepsilon$. Set $\mathfrak{M} = \sup\{|f(x)| \,:\, x \in [A, B]\}$ and take

$$\delta := \frac{\varepsilon}{n\mathfrak{M}}.$$

Now let $\langle\{y_k\}_{k=0}^m, \{\xi_k\}_{k=1}^m\rangle$ be an arbitrary tagged partition of $[A, B]$ with mesh $\leq \delta$. Let F be the set of those $k \in \{0, \ldots, m-1\}$ for which the *open* interval (y_{k-1}, y_k) contains at least one of the points $x_1, \ldots, x_{n-1}$. Then $\#F \leq n - 1$, since (y_{k-1}, y_k) and $(y_{k'-1}, y_{k'})$ are disjoint for $k \neq k'$. Furthermore for each $k \in \{0, \ldots, m-1\} \setminus F$ there exists a unique $j_k \in \{1, \ldots, n\}$ such that $[y_{k-1}, y_k] \subset [x_{j_k - 1}, x_{j_k}]$. Now

$$\begin{aligned}
S(\{y_k\}, \{\xi_k\}) &= \sum_{k=1}^m f(\xi_k)(y_k - y_{k-1}) \\
&\leq \sum_{k \in \{0,\ldots,m-1\} \setminus F} M_{j_k}(y_k - y_{k-1}) + \sum_{k \in F} \mathfrak{M}(y_k - y_{k-1}) \\
&\leq S_P f + \#F \cdot \mathfrak{M} \cdot \delta < S_P f + \varepsilon \leq I + 2\varepsilon,
\end{aligned}$$

and similarly

$$S(\{y_k\}, \{\xi_k\}) > s_P f - \varepsilon \geq I - 2\varepsilon.$$

This holds for any tagged partition $\langle\{y_k\}_{k=0}^m, \{\xi_k\}_{k=1}^m\rangle$ of $[A, B]$ with mesh $\leq \delta$. Using the fact that $\varepsilon > 0$ was arbitrary we conclude that $\int_A^B f(x)\,dx$ exists according to Definition 1.4 and $\int_A^B f(x)\,dx = I$. $\square$

Next, let us write out the *proof of Folland's* [4, p. 57, Thm 2.28(a)], with somewhat more details than Folland gives:

Suppose that f is Riemann integrable. For each partition $P = \{t_j\}_{j=0}^n$ of $[a, b]$ we set $M_j = \sup\{f(t) \,:\, t \in [t_{j-1}, t_j]\}$, $m_j = \inf\{f(t) \,:\, t \in [t_{j-1}, t_j]\}$ and define the functions $G_P : [a, b] \to \mathbb{R}$ and $g_P : [a, b] \to$

$\mathbb{R}$ through[11]

(3.6)

$$G_P = M_1 \chi_{[t_0,t_1]} + \sum_{j=2}^{n} M_j \chi_{(t_{j-1},t_j]}, \qquad g_P = m_1 \chi_{[t_0,t_1]} + \sum_{j=2}^{n} m_j \chi_{(t_{j-1},t_j]}.$$

Then

$$\int_{[a,b]} G_P \, dm = S_P f \quad \text{and} \quad \int_{[a,b]} g_P \, dm = s_P f. \tag{3.7}$$

(Proof: $\int_{[a,b]} G_P \, dm$ is the Lebesgue integral and it equals $\sum_{j=1}^{n} M_j (t_j - t_{j-1})$ by the definition given at the beginning of Folland's Sec. 2.2; and this equals $S_P f$ by Folland's definition p.56 bottom.)

Since f is Riemann integrable, it follows from the definition of the Riemann integral $\int_a^b f(t)\, dt$ that there is a sequence $P_1, P_2, P_3, \ldots$ of partitions of $[a, b]$ such that

$$\lim_{k\to\infty} S_{P_k} f = \lim_{k\to\infty} s_{P_k} f = \int_a^b f(t)\, dt \tag{3.8}$$

Note that if for a fixed k we insert some more points in the partition P_k, this *decreases* $S_{P_k} f$ and *increases* $s_{P_k} f$ (equality allowed). It follows that (3.8) remains valid if we modify the sequence $P_1, P_2, P_3, \ldots$ by inserting more points in some or all of the partitions P_k. Now by inserting points appropriately in P_2, then in P_3, then in P_4 etc, we may arrange things so that P_{k+1} is a refinement of P_k for each $k \in \mathbb{N}$, and furthermore $\lim_{k\to\infty} \operatorname{mesh}(P_k) = 0$. Let us define the functions $G : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ by

$$G(t) = \lim_{k\to\infty} G_{P_k}(t), \qquad g(t) = \lim_{k\to\infty} g_{P_k}(t), \qquad (t \in [a, b]).$$

The limits exist in $\mathbb{R}$ since, for each $t \in [a, b]$, $\{G_{P_k}(t)\}_{k=1}^{\infty}$ is a decreasing sequence of real numbers $\geq f(t)$ and $\{g_{P_k}(t)\}_{k=1}^{\infty}$ is an increasing sequence of real numbers $\leq f(t)$; this is because P_{k+1} is a refinement of P_k for each $k \in \mathbb{N}$. It follows that

$$g(t) \leq f(t) \leq G(t), \qquad \forall t \in [a, b]. \tag{3.9}$$

Note also that g and G are Borel measurable, by Folland's Prop. 2.7. By the dominated convergence theorem, taking the majorant function to be e.g. the constant function M, where $M = \sup\{|f(t)| : t \in [a, b]\}$, we have $\lim_{k\to\infty} \int_{[a,b]} G_{P_k} \, dm = \int_{[a,b]} G \, dm$, and here the left hand side equals $\int_a^b f(t)\, dt$ by (3.7) and (3.8). Hence, arguing also in the same

[11]Note that we modify Folland's definition very slightly; this only affects the values $G_P(a)$ and $g_P(a)$; the modification is necessary to make certain statements below valid also at the point $t = a$.

way for $g(t)$, we have

$$\int_{[a,b]} G\,dm = \int_{[a,b]} g\,dm = \int_a^b f(t)\,dt. \tag{3.10}$$

Thus $\int_{[a,b]}(G-g)\,dm = 0$, and using now Folland's Prop. 2.16 and (3.9) it follows that $G = g$ a.e., and therefore $G = f$ a.e. Since G is measurable w.r.t m and m is complete, f is measurable w.r.t. m (by Folland's Prop 2.11(a)), and $\int_{[a,b]} f\,dm = \int_{[a,b]} G\,dm$ (by Folland's Prop 2.23(b)); hence using (3.10) we conclude

$$\int_a^b f(t)\,dt = \int_{[a,b]} f\,dm,$$

and we are done. □

3.8. **Some facts about Jordan content.**

Theorem 3.7. *Given any bounded set $E \subset \mathbb{R}^n$, the following four conditions are equivalent:*
(a). $m(\partial E) = 0$.
(b). ∂E has Jordan content 0.
(c). E has Jordan content (i.e. $\underline{\kappa}(E) = \overline{\kappa}(E)$).
(d). $m(\partial_\varepsilon E) \to 0$ as $\varepsilon \to 0$.
Here in (d), $\partial_\varepsilon E$ is the set of all points in $\mathbb{R}^n$ which have distance $< \varepsilon$ to some point in ∂E.

(Regarding (a), note that for any $E \subset \mathbb{R}^n$ the boundary ∂E is a closed set, hence Lebesgue measurable. Regarding (d), note that for any E and $\varepsilon > 0$ the set $\partial_\varepsilon E$ is open, hence Lebesgue measurable.)

Proof. (a)⇒(b): Assume that $m(\partial E) = 0$. Note that ∂E is closed and bounded, hence compact. Thus by Folland p. 73, lines 8-9, the outer content of ∂E equals 0, i.e. $\overline{\kappa}(\partial E) = 0$. Hence also $\underline{\kappa}(\partial E) = 0$, and ∂E has Jordan content 0.

(b)⇒(c): Recall Folland's definitions on p. 71. Note that for any $k \in \mathbb{Z}$, any cube $Q \in \mathcal{Q}_k$ which satisfies $Q \cap E \neq \emptyset$ and $Q \not\subset E$ must satisfy $Q \cap \partial E \neq \emptyset$. [Proof: Since $Q \cap E \neq \emptyset$ there is a point $x \in Q \cap E$, and since $Q \not\subset E$ there is a point $y \in E \setminus Q$. Now from $x \in E$ and $y \notin E$ it follows that there is some point z on the line segment between x and y satisfying $z \in \partial E$. But Q is convex, hence $z \in Q$, i.e. we have $q \in Q \cap \partial E$.] It follows that $\overline{A}(E,k) \setminus \underline{A}(E,k) \subset \overline{A}(\partial E, k)$. Hence for every $k \in \mathbb{Z}$ we have

$$0 \leq m(\overline{A}(E,k)) - m(\underline{A}(E,k)) \leq m(\overline{A}(\partial E, k)).$$

Now assume that (b) holds; then $\lim_{k\to\infty} m(\overline{A}(\partial E,k)) = 0$ and hence $\lim_{k\to\infty} m(\overline{A}(E,k)) = \lim_{k\to\infty} m(\underline{A}(E,k))$, i.e. $\overline{\kappa}(E) = \underline{\kappa}(E)$, i.e. (c) holds.

(b)⇒(d): Let us write $\mathcal{N}_\varepsilon(F)$ for the ε-neighbourhood of an arbitrary set $F \subset \mathbb{R}^n$, i.e. $\mathcal{N}_\varepsilon(F)$ is the set of all points in $\mathbb{R}^n$ which have distance $< \varepsilon$ to some point in F. Then $\partial_\varepsilon(E) = \mathcal{N}_\varepsilon(\partial E)$, and for any $k \in \mathbb{Z}$, since $\partial E \subset \overline{A}(\partial E, k)$ we have $\partial_\varepsilon E \subset \mathcal{N}_\varepsilon(\overline{A}(\partial E, k))$. Now assume that (b) holds. Let $\eta > 0$ be given. Then there is some $k \in \mathbb{Z}$ such that $m(\overline{A}(\partial E, k)) < \eta$. One verifies easily that for any cube Q in $\mathbb{R}^n$ we have $m(\mathcal{N}_\varepsilon(Q)) \to m(Q)$ as $\varepsilon \to 0$. Hence since $\overline{A}(\partial E, k))$ is a finite union of cubes with pairwise disjoint interior, we also have $m(\mathcal{N}_\varepsilon(\overline{A}(\partial E, k))) \to m(\overline{A}(\partial E, k))$ as $\varepsilon \to 0$. Hence for all sufficiently small ε we have $m(\mathcal{N}_\varepsilon(\overline{A}(\partial E, k))) < \eta$, and thus also $m(\partial_\varepsilon E) < \eta$. But here $\eta > 0$ was arbitrary; hence (d) holds.

(d)⇒(a): Assume that (d) holds. Set $E_j = \partial_{1/j} E$ for $j = 1, 2, \ldots$. Then $E_1 \supset E_2 \supset E_3 \supset \cdots$, all sets E_j are Lebesgue measurable since they are open, and $\partial E \subset \cap_{j=1}^{\infty} E_j$. Also $m(E_1) < \infty$ since E_1 is a bounded set. Now using Folland's Thm. 1.8(a) and (d) we have $m(\partial E) \leq m(\cap_{j=1}^{\infty} E_j) = \lim_{j\to\infty} m(E_j) = 0$, i.e. (a) holds. □

Here's an application:

Example 3.1. If $E \subset \mathbb{R}^n$ is bounded and $m(\partial E) = 0$ then

$$\frac{\#(\mathbb{Z}^n \cap TE)}{T^n} \to m(E) \qquad \text{as } T \to \infty.$$

Proof. Let $\varepsilon > 0$ be given. Since E has Jordan content (by Theorem 3.7), and this Jordan content necessarily equals $m(E)$ (cf. Folland p. 72), there is some $k \in \mathbb{Z}$ such that

$$m(\underline{A}(E, k)) \geq m(E) - \varepsilon \quad \text{and} \quad m(\overline{A}(E, k)) \leq m(E) + \varepsilon. \tag{3.11}$$

Note that for each cube $Q \in \mathcal{Q}_k$ we have, by easy counting, $T^{-n}\#(\mathbb{Z}^n \cap TQ) \to m(Q) = 2^{-kn}$ as $T \to \infty$; hence since $\overline{A}(E, k)$ is a finite union of $N = m(\overline{A}(E, k)) \cdot 2^{kn}$ such cubes (not necessarily disjoint, although they have disjoint interiors) we have

$$\limsup_{T\to\infty} T^{-n}\#(\mathbb{Z}^n \cap T \cdot \overline{A}(E, k)) \leq N \cdot 2^{-kn} = m(\overline{A}(E, k)).$$

Similarly, using the fact that also the *interior* Q° of each cube $Q \in \mathcal{Q}_k$ satisfies $T^{-n}\#(\mathbb{Z}^n \cap TQ^\circ) \to 2^{-kn}$ as $T \to \infty$, and the fact that $\underline{A}(E, k)$ contains a disjoint union of $N' = m(\underline{A}(E, k)) \cdot 2^{kn}$ such Q°, we have

$$\liminf_{T\to\infty} T^{-n}\#(\mathbb{Z}^n \cap T \cdot \underline{A}(E, k)) \geq N' \cdot 2^{-kn} = m(\underline{A}(E, k)).$$

Using also

$$\underline{A}(E, k) \subset E \subset \overline{A}(E, k)$$

and (3.11), it follows that

$$\limsup_{T\to\infty} \frac{\#(\mathbb{Z}^n \cap TE)}{T^n} \leq m(E) + \varepsilon$$

and

$$\liminf_{T\to\infty} \frac{\#(\mathbb{Z}^n \cap TE)}{T^n} \geq m(E) - \varepsilon.$$

This is true for every $\varepsilon > 0$; hence we get the conclusion in Example 3.1. $\square$

Example 3.2. In measure theory we often think “open $\Rightarrow$ nice”. However note that there certainly exist many open and bounded sets $E \subset \mathbb{R}^n$ which do *not* have Jordan content, and for which the counting property in Example 3.1 *fails*.

For example let $q_1, q_2, \ldots$ be an enumeration of all points in the cube $[0,1)^n$ with rational coordinates, let $r_1, r_2, \ldots$ be a sequence of positive real numbers, and set

$$E = \bigcup_{j=1}^{\infty} B_{q_j}(r_j),$$

where $B_q(r)$ is the open ball in $\mathbb{R}^n$ having radius r and center q. The set E is open since it is a union of open sets, and by choosing $r_1, r_2, \ldots$ appropriately we can make $m(E)$ have any value that we want. In particular let us fix a choice of E such that $m(E) < 1$. Now note that for any $T \in \mathbb{N}$ we have $([0,T) \cap \mathbb{Z})^n \subset TE$, since for any $(x_1, \ldots, x_n) \in ([0,T) \cap \mathbb{Z})^n$ we have $T^{-1}(x_1, \ldots, x_n) \in [0,1)^n \cap \mathbb{Q}^n = \{q_1, q_2, \ldots\}$. Hence for $T \in \mathbb{N}$ we have

$$\#(\mathbb{Z}^n \cap TE) \geq \#([0,T) \cap \mathbb{Z})^n = T^n.$$

Hence

$$\lim_{\mathbb{N} \ni T \to \infty} \frac{\#(\mathbb{Z}^n \cap TE)}{T^n} = 1 > m(E),$$

i.e. the property in Example 3.1 *fails*. Hence we also have $m(\partial E) > 0$ and the set E does not have Jordan content. $\square$

Remark 3.8. However, it follows from the proof of Example 3.1 and Folland's Lemma 2.43 that for every open set $E \subset \mathbb{R}^n$,

$$\liminf_{T\to\infty} \frac{\#(\mathbb{Z}^n \cap TE)}{T^n} \geq m(E).$$

4. Lecture 4: Measure and integration theory

This will be a quite abstract (and comparatively difficult?) lecture! I plan to discuss:

* Complex measures (also signed measures), with the Radon-Nikodym Theorem as the central result; Ch. 3.1-3.3.
* As an example of an application of the Radon-Nikodym Theorem I will talk about the fact that $(L^p)^* = L^q$, after first defining the space $L^p(X, \mu)$; Ch. 6.1-6.2.
* Regularity and the Riesz Representation Theorem; Ch. 7.1-7.3.

4.1. **Some facts about $|\nu|$, the total variation measure.** Let $(X, \mathcal{M})$ be a measurable space. As in the lecture we write $M(\mathcal{M})$ for the set of all complex measures on $(X, \mathcal{M})$. In my lecture, for given $\nu \in M(\mathcal{M})$, I define $|\nu|$ as the function $\mathcal{M} \to [0, \infty]$ given by

(4.1)
$$|\nu|(E) = \sup\left\{\sum_{j=1}^{\infty} |\nu(E_j)| \;:\; E_1, E_2, \ldots \in \mathcal{M},\ \text{disjoint}, E = \bigcup_{j=1}^{\infty} E_j\right\}.$$

(This is the definition used e.g. in Rudin, [18].) I also claim that $|\nu|$ is in a strong sense the *smallest* positive measure which dominates ν in the sense that to any $E \in \mathcal{M}$ it gives a measure $\geq |\nu(E)|$.

We will here prove that the definition agrees with Folland's definition (note that this gives a partial solution to Folland's Exercise 21), and prove a precise form of the claim about minimality.

As in Folland, p. 93 (bottom), one sees that there exists a positive measure μ on $(X, \mathcal{M})$ and an $f \in L^1(\mu)$ such that $d\nu = f\, d\mu$, and our task (to prove agreement with Folland's definition of $|\nu|$) is to prove that for every $E \in \mathcal{M}$ we have

(4.2)
$$\int_E |f|\, d\mu = \sup\left\{\sum_{j=1}^{\infty} \left|\int_{E_j} f\, d\mu\right| \;:\; E_1, E_2, \ldots \in \mathcal{M},\ \text{disjoint}, E = \bigcup_{j=1}^{\infty} E_j\right\}.$$

First of all note that if $E_1, E_2, \ldots$ is any sequence of disjoint sets in $\mathcal{M}$ satisfying $E = \cup_{j=1}^{\infty} E_j$ then by Folland's Prop. 2.22 and Thm. 2.15,

$$\sum_{j=1}^{\infty} \left|\int_{E_j} f\, d\mu\right| \leq \sum_{j=1}^{\infty} \int_{E_j} |f|\, d\mu = \int_E |f|\, d\mu.$$

Therefore,

$$\int_E |f|\, d\mu \geq \sup\left\{\sum_{j=1}^{\infty} \left|\int_{E_j} f\, d\mu\right| \;:\; E_1, E_2, \ldots \in \mathcal{M},\ \text{disjoint}, E = \bigcup_{j=1}^{\infty} E_j\right\}.$$

In order to prove the opposite inequality, let us set, for given $E \in \mathcal{M}$ and $N \in \mathbb{N}$ with $N \geq 5$:

$$E_j = f^{-1}(C_j) \qquad \text{for } j = 1, \ldots, N,$$

where

$$C_j = \left\{ z \in \mathbb{C} \setminus \{0\} \, : \, \tfrac{j-1}{N} 2\pi \leq \arg(z) < \tfrac{j}{N} 2\pi \right\} \bigcup \begin{Bmatrix} \{0\} & \text{if } j = 1 \\ \emptyset & \text{if } j > 1 \end{Bmatrix}.$$

Then clearly $E_1, \ldots, E_N \in \mathcal{M}$ and $E = \cup_{j=1}^N E_j$. Also for each j we have

$$\begin{aligned} \Big| \int_{E_j} f \, d\mu \Big| &= \Big| \int_{E_j} e^{-2\pi j/N} f \, d\mu \Big| \geq \Re \int_{E_j} e^{-2\pi j/N} f \, d\mu \\ &= \int_{E_j} \Re\big(e^{-2\pi j/N} f\big) \, d\mu \geq \cos(\tfrac{2\pi}{N}) \int_{E_j} |f| \, d\mu, \end{aligned}$$

since $e^{-2\pi j/N} f(x)$ is (zero or) a complex number with argument in $[-\frac{2\pi}{N}, 0)$ for each $x \in E_j$, and for every such complex number z we have $\Re(z) \geq \cos(\frac{2\pi}{N})|z|$. Hence

$$\sum_{j=1}^N \Big| \int_{E_j} f \, d\mu \Big| \geq \cos(\tfrac{2\pi}{N}) \sum_{j=1}^N \int_{E_j} |f| \, d\mu = \cos(\tfrac{2\pi}{N}) \int_E |f| \, d\mu.$$

Letting $N \to \infty$ we conclude

$$\int_E |f| \, d\mu \leq \sup \Big\{ \sum_{j=1}^\infty \Big| \int_{E_j} f \, d\mu \Big| \, : \, E_1, E_2, \ldots \in \mathcal{M}, \text{ disjoint}, E = \bigcup_{j=1}^\infty E_j \Big\},$$

and this completes the proof of (4.2), and therefore also the proof that our definition of $|\nu|$ is equivalent with Folland’s. □

Next, here’s a precise statement about minimality (cf. Rudin, [18, Sec. 6.1]):

Proposition 4.1. *Let $\nu \in (X, \mathcal{M})$ and let μ be any positive measure on $\mathcal{M}$ such that $|\nu(E)| \leq \mu(E)$ for all $E \in \mathcal{M}$. Then $|\nu|(E) \leq \mu(E)$ for all $E \in \mathcal{M}$.*

Proof. Let $E \in \mathcal{M}$ be given. Then for any sequence $E_1, E_2, \ldots$ of disjoint sets in $\mathcal{M}$ satisfying $E = \cup_{j=1}^\infty E_j$, we have

$$\sum_{j=1}^\infty |\nu(E_j)| \leq \sum_{j=1}^\infty \mu(E_j) = \mu(E).$$

Since this holds for any such sequence $\{E_j\}$, we conclude via (4.1) that $|\nu|(E) \leq \mu(E)$. □

4.2. **Conditional expectation and conditional probability.** We here discuss how the concepts of conditional expectation and conditional probability arise as special cases of the Radon-Nikodym Theorem, using the set-up of Folland's book (cf. also Folland's Exercise 17, p. 93). For a more thorough presentation and development you should consult any standard book on probability theory; cf., e.g., Billingsley, [2, Sections 33–34].

We start by giving a solution to Folland's Exercise 17: Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$. Then $(X, \mathcal{N}, \mu_{|\mathcal{N}})$ is also a finite measure space. Now let $f \in L^1(\mu)$ and let $\lambda \in M(\mathcal{M})$ [12] be given by $d\lambda = f\,d\mu$; then $\lambda \ll \mu$ and hence $\lambda_{|\mathcal{N}}$, which is clearly a complex measure on $\mathcal{N}$, satisfies $\lambda_{|\mathcal{N}} \ll \mu_{|\mathcal{N}}$. Hence by the Radon-Nikodym Theorem there exists a unique function $g \in L^1(\mu_{|\mathcal{N}})$ such that

$$\forall E \in \mathcal{N}: \qquad \int_E f\,d\mu = \int_E g\,d\mu_{|\mathcal{N}}. \tag{4.3}$$

Recall that the uniqueness is understood in the usual sense of L^1 that we identify any two functions that agree a.e. (for us: $\mu_{|\mathcal{N}}$-a.e.). Thus, to be precise, the uniqueness says that if $g' \in L^1(\mu_{|\mathcal{N}})$ is another function satisfying (4.3) then $g = g'$ $\mu_{|\mathcal{N}}$-a.e.

This completes the solution of Folland's Exercise 17.

Let us note that the property (4.3) may equivalently be expressed as:

$$\forall E \in \mathcal{N}: \qquad \int_E f\,d\mu = \int_E g\,d\mu. \tag{4.4}$$

This follows from the fact (definition) that $\int_E g = \int \chi_E g$ (where $\chi_E g$ is $\mathcal{N}$-measurable) and the following lemma:

Lemma 4.2. *If $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, then $\int_X h\,d\mu_{|\mathcal{N}} = \int_X h\,d\mu$ for all $h \in L^1(\mu_{|\mathcal{N}})$.*

Proof. (Cf. Billingsley, [2, Ex. 16.4].) One easily checks (via the definitions in Folland's Sec. 2.3) that it suffices to prove the claim for all $h \in L^+(\mu_{|\mathcal{N}})$. Given such an h, by Folland's Theorem 2.10 there is a sequence $\{h_n\}$ of simple $\mathcal{N}$-measurable functions such that $0 \leq h_1 \leq h_2 \leq \cdots$ and $h_n \to h$ pointwise; and then by the Monotone Convergence Theorem we have $\int_X h\,d\mu_{|\mathcal{N}} = \lim_{n\to\infty} \int_X h_n\,d\mu_{|\mathcal{N}}$. Note that h and each h_n is also $\mathcal{M}$-measurable (since $\mathcal{N} \subset \mathcal{M}$), and by another application of the Monotone Convergence Theorem we have $\int_X h\,d\mu = \lim_{n\to\infty} \int_X h_n\,d\mu$. Hence it now suffices to prove $\int_X h_n\,d\mu_{|\mathcal{N}} = \int_X h_n\,d\mu$ for each n, and thus it suffices to prove that $\int_X h\,d\mu_{|\mathcal{N}} = \int_X h\,d\mu$ whenever h is a simple $\mathcal{N}$-measurable (nonnegative) function. By linearity

[12] As in the lecture we write $M(\mathcal{M})$ for the set of all complex measures on $\mathcal{M}$.

we then reduce to the case when h is a characteristic function: $h = \chi_A$ for some $A \in \mathcal{N}$. But then $\int_X h \, d\mu_{|\mathcal{N}} = \mu_{|\mathcal{N}}(A) = \mu(A) = \int_X h \, d\mu$, and we are done. $\square$

To connect with probability theory, let us now assume that μ *is a probability measure*, i.e. $\mu(X) = 1$. In other words, $(X, \mathcal{M}, \mu)$ is a *probability space!* (A more common notation in probability theory would be to write Ω for X and P for μ; however we will stick to the notation which we are using.) A μ-measurable function on X is now called a *random variable;* in particular our $f \in L^1(\mu)$ is a random variable. Now the function $g \in L^1(\mu_{|\mathcal{N}})$ whose existence we proved above and which satisfies (4.3) and (4.4) is called the *conditional expectation* of f given $\mathcal{N}$, and denoted by $\mathbb{E}[f \| \mathcal{N}]$. Thus, to recapitulate: $\mathbb{E}[f \| \mathcal{N}]$ is the unique function in $L^1(\mu_{|\mathcal{N}})$ satisfying

$$\int_E f \, d\mu = \int_E \mathbb{E}[f \| \mathcal{N}] \, d\mu, \qquad \forall E \in \mathcal{N}.$$

In the special case when f is the characteristic function of a set $A \in \mathcal{M}$; $f = \chi_A$, then $\mathbb{E}[f \| \mathcal{N}]$ is called the *conditional probability of A given $\mathcal{N}$*, and denoted by $\mu[A \| \mathcal{N}]$. Thus:

$$\mu[A \| \mathcal{N}] = \mathbb{E}[\chi_A \| \mathcal{N}],$$

and the defining property (4.4) reads:

$$\mu(A \cap E) = \int_E \mu[A \| \mathcal{N}] \, d\mu, \qquad \forall E \in \mathcal{N}. \tag{4.5}$$

Note that $\mu[A \| \mathcal{N}]$ is a *function*, and not just a number in $[0, 1]$. Informally, $\mu[A \| \mathcal{N}](x)$ may be interpreted as (at least for $\mu_{|\mathcal{N}}$-a.e. x): "The conditional probability that a μ-random element $\omega \in X$ happens to lie in A, *given* that for each $E \in \mathcal{N}$ we have $\omega \in E \Leftrightarrow x \in E$."

We note:

Lemma 4.3. *For any given $A \in \mathcal{M}$, we have $\mu[A \| \mathcal{N}](x) \in [0, 1]$ for $\mu_{|\mathcal{N}}$-a.e. x.*

Proof. Let $A \in \mathcal{M}$ be given. Note that the function $\Re \mu[A \| \mathcal{N}]$ satisfies the same defining property as $\mu[A \| \mathcal{N}]$; hence $\mu[A \| \mathcal{N}](x) \in \mathbb{R}$ must hold for $\mu_{|\mathcal{N}}$-a.e. x, and replacing $\mu[A \| \mathcal{N}]$ by $\Re \mu[A \| \mathcal{N}]$ we may assume $\mu[A \| \mathcal{N}](x) \in \mathbb{R}$ for all $x \in X$.

Now for any given $\varepsilon > 0$, let us set $E = \{x \in X \, : \, \mu[A \| \mathcal{N}](x) \geq 1 + \varepsilon\}$. Then $E \in \mathcal{N}$, and (4.5) gives $\mu(A \cap E) \geq (1 + \varepsilon)\mu(E)$. But $\mu(A \cap E) \leq \mu(E)$; hence $\mu(E) \geq (1 + \varepsilon)\mu(E)$, which forces $\mu(E) = 0$. Letting $\varepsilon = 1/n$ and $n \to \infty$ this implies (using continuity from below for μ; cf. Folland's Thm. 1.8(c)) $\mu(\{x \in X \, : \, \mu[A \| \mathcal{N}](x) > 1\}) = 0$. By an entirely similar argument we also have $\mu(\{x \in X \, : \, \mu[A \| \mathcal{N}](x) < 0\}) = 0$, and this completes the proof. $\square$

We now give two examples to show how the above concept connects with the elementary or intuitive notion of "conditional probability". We will now write P in place of μ and Ω in place of X.

Example 4.1. Let X and Y be two integer valued random variables whose joint distribution is given by probabilities

$$p_{ij} = P(X = i, Y = j), \qquad \forall i, j \in \mathbb{Z}.$$

Thus P is the probability measure on $(\mathbb{Z}^2, \mathcal{P}(\mathbb{Z}^2))$ determined by

$$P(A) = \sum_{\langle i,j \rangle} p_{ij},$$

and we of course have $p_{ij} \geq 0$ for all $i, j \in \mathbb{Z}$, and $\sum_{i,j\in\mathbb{Z}} p_{ij} = P(\mathbb{Z}^2) = 1$. Now let $\mathcal{N}$ be the sub-σ-algebra of $\mathcal{P}(\mathbb{Z}^2)$ given by

$$\mathcal{N} = \{B \times \mathbb{Z} \,:\, B \subset \mathbb{Z}\}.$$

For an arbitrary $A \subset \mathbb{Z}^2$, we wish to determine $P[A\|\mathcal{N}]$, the conditional probability of A given $\mathcal{N}$. The fact that $P[A\|\mathcal{N}]$ is a $\mathcal{N}$-measurable function means that there is a function $g : \mathbb{Z} \to \mathbb{C}$ such that

$$P[A\|\mathcal{N}](i, j) = g(i), \qquad \forall \langle i, j \rangle \in \mathbb{Z}^2.$$

Also the defining property (4.5) says that for every $E \in \mathcal{N}$,

$$P(E \cap A) = \sum_{\langle i,j \rangle \in E} g(i) p_{ij}.$$

In particular taking $E = \{i_0\} \times \mathbb{Z}$ for any $i_0 \in \mathbb{Z}$ (note that this E satisfies $E \in \mathcal{N}$) we conclude:

$$P(A \cap (\{i_0\} \times \mathbb{Z})) = g(i_0) \cdot P(\{i_0\} \times \mathbb{Z}).$$

Using our random variables X and Y the same relation may be expressed as:

$$P(\langle X, Y \rangle \in A \text{ and } X = i_0) = g(i_0) P(X = i_0).$$

Hence for any $i_0 \in \mathbb{Z}$ with $P(X = i_0) > 0$, and any $j \in \mathbb{Z}$, we have

$$P[A\|\mathcal{N}](i_0, j) = g(i_0) = \frac{P(\langle X, Y \rangle \in A \text{ and } X = i_0)}{P(X = i_0)}.$$

Example 4.2. Let X and Y be real-valued random variables taking values in $[0, 1]$, whose joint distribution is given by a probability density function $f \in C([0, 1]^2)$ which is everywhere positive. Thus our probability space is $(\Omega, \mathcal{B}_\Omega, P)$, where $\Omega = [0, 1]^2$, $\mathcal{B}_\Omega$ is the Borel σ-algebra on Ω, and P is the probability measure given by

$$P(A) = \int_{[0,1]^2} f(x)\, dx = \int_0^1 \int_0^1 f(x_1, x_2)\, dx_1\, dx_2, \qquad \forall A \in \mathcal{B}_\Omega,$$

where $dx = dx_1\, dx_2$ is Lebesgue measure. Now let $\mathcal{N}$ be the sub-σ-algebra of $\mathcal{B}_\Omega$ given by

$$\mathcal{N} = \{B \times [0,1] \,:\, B \in \mathcal{B}_{[0,1]}\}.$$

Given any $A \in \mathcal{B}_\Omega$, we wish to determine $P[A\|\mathcal{N}] \in L^1(P_{|\mathcal{N}})$. The fact that this function is $\mathcal{N}$-measurable means that there is a Borel measurable function $g : [0,1] \to \mathbb{R}$ such that

$$P[A\|\mathcal{N}](x_1,x_2) = g(x_1), \qquad \forall (x_1,x_2) \in \Omega.$$

The defininining property (4.5) says that for every $E \in \mathcal{N}$,

$$P(E \cap A) = \int_E g(x_1)\, dP(x),$$

i.e.,

$$\int_{E\cap A} f(x_1,x_2)\, dx_1\, dx_2 = \int_E g(x_1) f(x_1,x_2)\, dx_1\, dx_2,$$

But $E \in \mathcal{N}$ means that $E = B \times [0,1]$ for some $B \in \mathcal{B}_{[0,1]}$; hence the requirement is that the following should hold for every $B \in \mathcal{B}_{[0,1]}$:

$$\int_B \int_0^1 I((x_1,x_2) \in A) f(x_1,x_2)\, dx_2\, dx_1 = \int_B g(x_1) \int_0^1 f(x_1,x_2)\, dx_2\, dx_1.$$

This implies that the following must hold for (Lebesgue-)almost every x_1:

$$\int_0^1 I((x_1,x_2) \in A) f(x_1,x_2)\, dx_2 = g(x_1) \int_0^1 f(x_1,x_2)\, dx_2.$$

Hence since f is continuous and everywhere positive:

$$g(x_1) = \frac{\int_0^1 I((x_1,x_2) \in A) f(x_1,x_2)\, dx_2}{\int_0^1 f(x_1,x_2)\, dx_2}$$

for almost every $x_1 \in [0,1]$.

4.3. **Some remarks about regularity of measures.** Let X be a topological space and $\nu \in M(\mathcal{B}_X)$ (i.e. ν is a complex measure on $(X, \mathcal{B}_X)$). In the lecture I define ν to be *regular* iff $|\nu|$ is regular, and similarly ν to be *outer (inner) regular on* $E \in \mathcal{B}_X$ iff $|\nu|$ is outer (inner) regular on $E \in \mathcal{B}_X$. In order to appreciate these definitions, let us note the following:

Lemma 4.4. *If ν is outer regular on $E \in \mathcal{B}_X$ then for every $\varepsilon > 0$ there is an open set $U_0 \supset E$ such that for every open set U with $E \subset U \subset U_0$ we have $|\nu(U) - \nu(E)| < \varepsilon$. Similarly, if ν is inner regular on $E \in \mathcal{B}_X$ then for every $\varepsilon > 0$ there is a compact set $K_0 \subset E$ such that for every compact set K with $K_0 \subset K \subset E$ we have $|\nu(E) - \nu(K)| < \varepsilon$.*

Proof. Assume that ν is outer regular on $E \in \mathcal{B}_X$, and let $\varepsilon > 0$ be given. Then $|\nu|$ is outer regular on E and hence there is an open set $U_0 \supset E$ such that $|\nu|(U_0) < |\nu|(E) + \varepsilon$. Then for every open set U with $E \subset U \subset U_0$ we have

$$|\nu(U) - \nu(E)| = |\nu(U \setminus E)| \leq |\nu|(U \setminus E) = |\nu|(U) - |\nu|(E) < \varepsilon.$$

The proof of the inner regularity property is entirely similar. □

We also point out:

Lemma 4.5. *A measure $\nu \in M(\mathcal{B}_X)$ is outer regular on a set $E \in \mathcal{B}_X$ if and only if both ν_r and ν_i are outer regular on E, and this holds if and only if all the four positive measures ν_r^+, ν_r^-, ν_i^+, ν_i^- are outer regular on E. The corresponding facts hold for inner regularity.*

Proof. The first claim is an immediate consequence of the fact that $|\nu|(F) \leq |\nu_r|(F) + |\nu_i|(F) \leq 2|\nu|(F)$ for all $F \in \mathcal{B}_X$. It now remains to prove that if $\nu \in M(\mathcal{B}_X)$ is a real (i.e. signed) measure then $|\nu|$ is regular if and only if ν^+ and ν^- is regular. This is immediate using $|\nu| = \nu^+ + \nu^-$. □

Next let us discuss Folland's Theorem 7.8, which says: *Let X be an LCH space in which every open set is σ-compact. Then every (positive) Borel measure on X that is finite on compact sets is regular.* Note that Folland's proof of this theorem makes strong use of the Riesz Representation Theorem, the proof of which is rather complicated and involves constructions with outer measures, Carathéodory's theorem, etc. It is interesting (or at least an amusing exercise!) to ask whether one can give a more direct proof. We will do so in the following; we start by proving a few auxiliary lemmas (perhaps somewhat interesting in themselves).

Lemma 4.6. *Let X be a topological space, let μ be a Borel measure on X, and let $E_1, E_2, \ldots$ be Borel sets in X. If μ is outer regular on each E_j, then μ is outer regular on $\cup_1^\infty E_j$. Similarly, if μ is inner regular on each E_j, then μ is inner regular on $\cup_1^\infty E_j$.*

Proof. Assume that μ is outer regular on each E_j, and set $E = \cup_1^\infty E_j$. If $\mu(E) = \infty$ then there is nothing to prove; hence from now on we assume $\mu(E) < \infty$; then also $\mu(E_j) < \infty$ for each j. Let $\varepsilon > 0$ be given. Then for each j there is an open set U_j in X such that $E_j \subset U_j$ and $\mu(U_j) < \mu(E_j) + \varepsilon 2^{-j}$, i.e. $\mu(U_j \setminus E_j) < \varepsilon 2^{-j}$. Set $U = \cup_1^\infty U_j$. This is an open set and $E \subset U$. Furthermore $U \setminus E \subset \cup_1^\infty (U_j \setminus E_j)$ and thus

$$\mu(U \setminus E) \leq \sum_{j=1}^\infty \mu(U_j \setminus E_j) < \sum_{j=1}^\infty \varepsilon 2^{-j} = \varepsilon.$$

The fact that such an open set exists for each $\varepsilon > 0$ implies that E is outer regular.

Next assume instead that μ is inner regular on each E_j. Let us first prove that μ is then also inner regular on $E_1 \cup E_2$. This is trivial if $\mu(E_1) = \infty$ or $\mu(E_2) = \infty$; hence from now on we assume $\mu(E_1) < \infty$ and $\mu(E_2) < \infty$; thus also $\mu(E_1 \cup E_2) < \infty$. Let $\varepsilon > 0$ be given. Then there exist compact sets K_1, K_2 such that $K_j \subset E_j$ and $\mu(K_j) > \mu(E_j) - \varepsilon$ for $j = 1, 2$, and now $K = K_1 \cup K_2$ is a compact set satisfying $K \subset E_1 \cup E_2$ and $(E_1 \cup E_2) \setminus K \subset (E_1 \setminus K_1) \cup (E_2 \setminus K_2)$, so that $\mu((E_1 \cup E_2) \setminus K) < 2\varepsilon$. The fact that such a compact set K exists for every $\varepsilon > 0$ implies that $E_1 \cup E_2$ is inner regular.

Repeated use of the preceding shows that any finite union $\cup_{j=1}^N E_j$ is inner regular. Let us again set $E = \cup_1^\infty E_j$; we wish to prove that E is inner regular. Let us first assume $\mu(E) < \infty$. Let $\varepsilon > 0$ be given. By Folland's Theorem 1.8(c) we have $\mu(E) = \lim_{N \to \infty} \mu(\cup_1^\infty E_j)$ and hence there exists some N such that $\mu(\cup_1^\infty E_j) > \mu(E) - \varepsilon$. But also $\cup_1^\infty E_j$ is inner regular; hence there exists a compact set K such that $K \subset \cup_1^\infty E_j$ and $\mu(K) > \mu(\cup_1^\infty E_j) - \varepsilon$. Now $K \subset E$ and $\mu(K) > \mu(E) - 2\varepsilon$. The fact that such a compact set K exists for every $\varepsilon > 0$ implies that E is inner regular. The same argument works, mutatis mutandis, when $\mu(E) = \infty$. $\square$

Lemma 4.7. *Let X be an LCH space in which every open set is σ-compact, and let K be a compact subset of X. Then there is a sequence of open sets $U_1 \supset U_2 \supset \ldots$ in X such that $\overline{U}_1$ is compact and $K = \cap_1^\infty U_j$. In particular, if μ is a Borel measure on X which is finite on compact sets, then μ is outer regular on K.*

Proof. Since $X \setminus K$ is open, there exist compact sets $C_1, C_2, \ldots$ such that $X \setminus K = \cup_1^\infty C_j$. For each j, let $V_j = X \setminus C_j$; then V_j is an open set and $K \subset V_j$; hence by Folland's Prop. 4.31 there exists an open set W_j such that $K \subset W_j \subset \overline{W}_j \subset V_j$ and $\overline{W}_j$ is compact. Note that $W_j \subset V_j$ implies $W_j \cap C_j = \emptyset$. Now set $U_k = \cup_{j=1}^k W_k$ for $k \in \mathbb{N}$. Then $U_1, U_2, \ldots$ are open sets, $U_1 \supset U_2 \supset \ldots$, $\overline{U}_1$ is compact, and $K \subset U_k$ for each k. We claim that $K = \cap_1^\infty U_k$. To prove this, let x be an arbitrary point in $X \setminus K$. Then since $X \setminus K = \cup_1^\infty C_j$, there is some j such that $x \in C_j$. Therefore $x \notin W_j$ and hence $x \notin U_k$ for each $k \geq j$, and $x \notin \cap_1^\infty U_k$. This proves the first part of the lemma.

To prove the second part, let μ be a Borel measure on X which is finite on compact sets. Then $\mu(\overline{U}_1) < \infty$ and now by Folland's Theorem 1.8(d), $\mu(K) = \mu(\cap_1^\infty U_k) = \lim_{k \to \infty} \mu(U_k)$, which implies that μ is outer regular on K. $\square$

Lemma 4.8. *Let X be a compact Hausdorff space (thus in particular X is an LCH) in which every open set is σ-compact, and let μ be a finite Borel measure on X. Then μ is regular.*

Proof. Let $\mathcal{F}$ be the family of all Borel subsets $E \subset X$ which have the property that μ is both outer and inner regular on E. Then $\mathcal{F}$ contains every open set in X. Indeed, if $U \subset X$ is open then μ is trivially outer regular on U, and μ is inner regular on U because U is σ-compact (since this implies that there is a sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets of U with $U = \cup_1^\infty K_j$).

We claim that $\mathcal{F}$ is a σ-algebra. Indeed, Lemma 4.6 implies that $\mathcal{F}$ is closed under countable unions; hence it suffices to prove that $\mathcal{F}$ is closed under complements. Thus assume $E \in \mathcal{F}$, and let $\varepsilon > 0$ be given. Since E is inner regular, there is a compact set $K \subset E$ satisfying $\mu(K) > \mu(E) - \varepsilon$; then K^c is an open set, $E^c \subset K^c$, and $\mu(K^c) = \mu(X) - \mu(K) < \mu(X) - \mu(E) + \varepsilon = \mu(E^c) + \varepsilon$. The fact that there exists such an open set for each $\varepsilon > 0$ implies that E^c is outer regular. Similarly, since E is outer regular, there is an open set U such that $E \subset U$ and $\mu(U) < \mu(E) + \varepsilon$. Then U^c is a closed subset of X, hence compact, $U^c \subset E^c$, and $\mu(U^c) = \mu(X) - \mu(U) > \mu(X) - \mu(E) - \varepsilon = \mu(E^c) - \varepsilon$. This proves that E^c is inner regular, and thus $E^c \in \mathcal{F}$. This completes the proof that $\mathcal{F}$ is a σ-algebra.

Now since $\mathcal{F}$ is a σ-algebra containing all the open sets in X, it follows that $\mathcal{F}$ contains the σ-algebra *generated* by the family of open sets in X. But this latter σ-algebra is precisely the Borel σ-algebra, $\mathcal{B}_X$, i.e. we have proved $\mathcal{F} = \mathcal{B}_X$. Hence μ is regular. □

Direct proof of Folland's Theorem 7.8. By assumption X is σ-compact and thus there is a sequence $K_1, K_2, \ldots$ of compact subsets of X with $X = \cup_1^\infty K_j$. Let μ be a Borel measure on X that is finite on compact sets. Let E be an arbitrary Borel set in X. We wish to prove that μ is inner and outer regular on E. Set $E_j = K_j \cap E$; then $E = \cup_1^\infty E_j$ and hence by Lemma 4.6 it suffices to prove that μ is inner and outer regular on each E_j.

Fix an arbitrary index j. Consider K_j with its relative topology. Then $\mathcal{B}_{K_j} = \{F \in \mathcal{B}_X \,:\, F \subset K_j\}$. [Proof: Set $\mathcal{M} = \{F \in \mathcal{B}_X \,:\, F \subset K_j\}$; this is a σ-algebra on K_j which contains every open subset of K_j (viz., every set of the form $K_j \cap U$ with U open in X); hence $\mathcal{B}_{K_j} \subset \mathcal{M}$. On the other hand, let $\mathcal{N} = \{F \in \mathcal{B}_X \,:\, F \cap K_j \in \mathcal{B}_{K_j}\}$. Then $\mathcal{N}$ is a σ-algebra (since $\mathcal{B}_X$ and $\mathcal{B}_{K_j}$ are σ-algebras) and $\mathcal{N}$ contains every open subset of X; hence $\mathcal{N} = \mathcal{B}_X$. This proves that $\mathcal{M} \subset \mathcal{B}_{K_j}$.] Hence μ_j, the restriction of μ to $\{F \in \mathcal{B}_X \,:\, F \subset K_j\}$, is a finite Borel measure on K_j. Also K_j is a compact Hausdorff space in which every open set is σ-compact. Hence by Lemma 4.8, μ_j is regular.

Now let $\varepsilon > 0$ be given. Since μ_j is inner regular on E_j and $E_j \in \mathcal{B}_{K_j}$, there exists a compact subset $C \subset E_j$ such that $\mu_j(C) > \mu_j(E_j) - \varepsilon$. But $\mu_j(C) = \mu(C)$ and $\mu_j(E_j) = \mu(E_j)$; hence $\mu(C) > \mu(E_j) - \varepsilon$ and the fact that such a compact subset exists for any $\varepsilon > 0$ proves that μ is inner regular on E_j. Next since μ_j is outer regular on E_j, there exists an open subset $W \subset K_j$ (wrt the topology of K_j) such that $E_j \subset W$ and $\mu_j(W) < \mu_j(E_j) + \varepsilon$, i.e. $\mu(W) < \mu(E_j) + \varepsilon$. The fact that W is relatively open means that there is an open subset U of X such that $W = K_j \cap U$. Furthermore μ is outer regular on K_j by Lemma 4.7 and hence there exists an open set U' of X such that $K_j \subset U'$ and $\mu(U') < \mu(K_j) + \varepsilon$. Now $U'' = U \cap U'$ is an open subset of X, $E_j \subset U''$, and using $U'' \subset W \cup (U' \setminus K_j)$ we see that $\mu(U'') \leq \mu(W) + \mu(U' \setminus K_j) < \mu(E_j) + \varepsilon + \varepsilon$. The fact that such an open set U'' exists for any $\varepsilon > 0$ implies that μ is outer regular on E_j. This completes the proof. $\square$

4.4. **A fact about σ-compactness.** The following fact is Folland's Exercise 55 on p. 135; it is also mentioned in Folland's Theorem 7.8:

Lemma 4.9. *If X is a second countable LCH space then every open set in X is σ-compact.*

Proof. Let U be an arbitrary open set in X. Let $\mathcal{B}$ be a countable base for the topology of X, and set

$$\mathcal{B}[U] = \{V \in \mathcal{B} \,:\, \overline{V} \text{ is compact and } \overline{V} \subset U\}.$$

Set

$$A = \bigcup_{V \in \mathcal{B}[U]} \overline{V}.$$

Then by construction A is a countable (or finite) union of compact sets and $A \subset U$. Hence to complete the proof it suffices to prove that $U \subset A$. Let x be an arbitrary point in U. By Folland's Prop. 4.30 there is a compact neighborhood N of x such that $N \subset U$, and since $\mathcal{B}$ is a base for the topology of X there is some $V \in \mathcal{B}$ such that $x \in V \subset N$. But then $\overline{V} \subset N$ (since N is closed, by Folland's Prop. 4.24); thus $\overline{V}$ is compact (by Folland's Prop. 4.22) and $\overline{V} \subset U$. This implies that $V \in \mathcal{B}[U]$, and thus by the definition of A we have $\overline{V} \subset A$ and in particular $x \in A$. This completes the proof. $\square$

5. Lecture 5: Fourier analysis

I will start by finishing from Lecture 4:

* Regularity and the Riesz Representation Theorem; Ch. 7.1-7.3.

Next I will go through *parts* of Ch. 8.2, 8.3, 8.6 in Folland, introducing the concepts of convolution and Fourier transforms for functions and measures on $\mathbb{R}^n$, and discussing some of their basic properties.

6. Lecture 6: Fourier analysis

In this lecture I will continue discussing the material in Folland's Ch. 8.2, 8.3, 8.6. Central topics will be approximate units (cf. Theorem 8.14) and the Fourier Inversion Theorem (Theorem 8.26). Towards the end I hope to get time to discuss the Fourier transform on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and some concrete computations, e.g. for the Gauss kernel and the Poisson kernel.

6.1. A fact about uniqueness of limits.

Towards the end of the proof of the Fourier Inversion Theorem (Folland's Theorem 8.26), the following fact is used: *If a sequence of functions $f_1, f_2, \ldots \in L^1(\mathbb{R}^n)$ tends to a function $f \in L^1(\mathbb{R}^n)$ in the L^1-norm, and also for every $x \in \mathbb{R}^n$ the limit $g(x) = \lim_{k\to\infty} f_k(x)$ exists, then $f = g$ a.e. (viz., $f(x) = g(x)$ holds for almost every x).*

This fact is contained in Folland's Section 2.4 (which I haven't discussed in class); for example the fact is an immediate consequence of Folland's Corollary 2.32.

In any case, let us here give a direct proof of the above fact: Note that g is measurable, by Folland's Corollary 2.9. Assume that $f(x) = g(x)$ does *not* hold for almost every x, i.e. $m(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) > 0$. Set $E_j = \{x \in \mathbb{R}^n : |f(x) - g(x)| > j^{-1}\}$. Then $E_1 \subset E_2 \subset \cdots$ and $\{x \in \mathbb{R}^n : f(x) \neq g(x)\} = \cup_1^\infty E_j$; hence by Folland's Theorem 1.8(c) (continuity from below for the measure m) the limit $\lim_{j\to\infty} m(E_j)$ exists and is positive. Thus we may fix some $j \in \mathbb{N}$ for which $m(E_j) > 0$. Next set

$$A_\ell = \{x \in E_j : |f_k(x) - g(x)| < (2j)^{-1}, \forall k \geq \ell\}.$$

Then $\cup_{\ell=1}^\infty A_\ell = E_j$ since $g(x) = \lim_{k\to\infty} f_k(x)$ for every $x \in \mathbb{R}^n$; also $A_1 \subset A_2 \subset \cdots$. Hence using continuity from below for m as before, we conclude that there is some ℓ such that $m(A_\ell) > 0$. Now for every $x \in A_\ell$ and every $k \geq \ell$ we have $|f_k(x) - f(x)| > (2j)^{-1}$, since $|f_k(x) - g(x)| < (2j)^{-1}$ and $|f(x) - g(x)| > j^{-1}$. Hence for every $k \geq \ell$,

$$\|f_k - f\|_1 = \int_{\mathbb{R}^n} |f_k(x) - f(x)|\,dx \geq \int_{A_\ell} (2j)^{-1}\,dx > (2j)^{-1} m(A_\ell),$$

i.e. $\|f_k - f\|_1$ is larger than a fixed positive constant for all $k \geq \ell$. This contradicts the assumption that $f_k \to f$ in $L^1(\mathbb{R}^n)$, and the proof is complete. □

In connection with the above fact one should note that neither of the two types of convergences (convergence in L^1, and pointwise convergence, respectively) implies the other type; see the examples (i)–(iv) in Folland, p. 61.

6.2. **Computing the Poisson kernel.** We wish to calculate the inverse Fourier transform $\phi(x)$ of $\Phi(\xi) = e^{-2\pi|\xi|}$. This function $\phi(x)$ is called the *Poisson kernel;* see Folland p. 260. Thus:

$$\phi(x) = \check{\Phi}(x) = \int_{\mathbb{R}^n} e^{-2\pi|\xi|} e^{2\pi i x\cdot\xi}\, d\xi.$$

Folland outlines a proof of the explicit formula

$$\phi(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} (1 + |x|^2)^{-\frac{n+1}{2}}, \qquad \forall x \in \mathbb{R}^n. \tag{6.1}$$

in his Exercise 26, p. 262. This proof goes via expressing $\Phi(\xi)$ as a superposition of dilated Gauss kernels, and then using the fact that we already know the inverse Fourier transform of these (Prop 8.24). It is a very elegant and fairly short computation! However here we wish to give an alternative proof of (6.1), by pushing through the method which to me seems like the most natural/naive method possible. It turns out that this computation is not at all as nice as the one which Folland outlines in his Exercise 26 (at least not in the way which I carry it out below); however it provides an opportunity to illustrate several important points which are often useful in computations (namely: the fact that polar coordinates can certainly be useful for integration even if the integrand is not radial, and some tips on how to deal with complicated looking integrals and special functions).

Let S_1^{n-1} be the $n-1$ dimensional sphere, which we will always take to be concretely realized as $\mathrm{S}_1^{n-1} = \{x \in \mathbb{R}^n \ : \ |x| = 1\}$, just as in Folland, p. 78. Let σ be the unique Borel measure on S_1^{n-1} described in Folland's Theorem 2.49; this is the natural "$n-1$ dimensional volume measure" on S_1^{n-1}. Then by Theorem 2.49,

$$\begin{aligned}\phi(x) &= \int_0^\infty \int_{\mathrm{S}_1^{n-1}} e^{-2\pi r} e^{2\pi i x\cdot r\omega} r^{n-1}\, d\sigma(\omega)\, dr \\ &= \int_{\mathrm{S}_1^{n-1}} \int_0^\infty e^{2\pi(-1+ix\cdot\omega)r} r^{n-1}\, dr\, d\sigma(\omega).\end{aligned}$$

The inner integral can be evaluated (for any fixed $x \in \mathbb{R}^n$ and $\omega \in \mathrm{S}_1^{n-1}$) by substituting $r = \frac{u}{2\pi(1-ix\cdot\omega)}$. This gives

$$\int_0^\infty e^{2\pi(-1+ix\cdot\omega)r} r^{n-1}\,dr = \frac{1}{(2\pi)^n(1-ix\cdot\omega)^n}\int_C e^{-u}u^{n-1}\,du,$$

where C is the infinite ray in the complex plane which starts at 0 and goes through the point $1 - ix\cdot\omega$. For $R > 0$, let C_R be the part of the ray C which starts at 0 and ends at $z \in C$ with $|z| = R$. Also let D_R be the contour which goes in the circle $\{|z| = R\}$ from the end-point of C_R to $z = R \in \mathbb{R}_{>0}$. Then by the Cauchy integral theorem,

$$\int_{C_R} e^{-u}u^{n-1}\,du + \int_{D_R} e^{-u}u^{n-1}\,du = \int_0^R e^{-u}u^{n-1}\,du.$$

Furthermore using $|e^{-u}u^{n-1}| = e^{-\Re u}|u|^{n-1} = e^{-R\cos(\arg(u))}R^{n-1}$ for all $u \in D_R$ and the fact that $\arg(1-ix\cdot\omega) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we see that (letting $c = \cos(\arg(1-ix\cdot\omega)) = (1+(x\cdot\omega)^2)^{-\frac{1}{2}}$)

$$\left|\int_{D_R} e^{-u}u^{n-1}\,du\right| \leq \int_{D_R} |e^{-u}u^{n-1}|\,|du| \leq \frac{\pi}{2}R^n e^{-cR} \to 0, \quad \text{as } R \to \infty.$$

Hence

$$\int_C e^{-u}u^{n-1}\,du = \lim_{R\to\infty}\int_{C_R} e^{-u}u^{n-1}\,du = \lim_{R\to\infty}\int_0^R e^{-u}u^{n-1} = \int_0^\infty e^{-u}u^{n-1}.$$

However, we recognize this last integral as the *Gamma function;* $= \Gamma(n) = (n-1)!$. (We will say more about the Gamma function in a later lecture, probably in lecture 9.) Thus we conclude:

$$\int_0^\infty e^{2\pi(-1+ix\cdot\omega)r} r^{n-1}\,dr = \frac{\Gamma(n)}{(2\pi)^n(1-ix\cdot\omega)^n}.$$

Hence:

$$\phi(x) = \frac{\Gamma(n)}{(2\pi)^n}\int_{\mathrm{S}_1^{n-1}} (1-ix\cdot\omega)^{-n}\,d\sigma(\omega).$$

We can use the fact that the measure σ is invariant under rotations (this is Folland's Exercise 62 on p. 80; it can be solved using his Theorem 2.49 and his Theorem 2.44 with T being a rotation) to see that $\phi(x)$ is invariant under rotations: If $R : \mathbb{R}^n \to \mathbb{R}^n$ is any rotation about the origin then

$$\begin{aligned}
\phi(Rx) &= \frac{\Gamma(n)}{(2\pi)^n}\int_{\mathrm{S}_1^{n-1}} (1-i(Rx)\cdot\omega)^{-n}\,d\sigma(\omega) \\
&= \frac{\Gamma(n)}{(2\pi)^n}\int_{\mathrm{S}_1^{n-1}} (1-ix\cdot(R^{-1}\omega))^{-n}\,d\sigma(\omega) \\
&= \frac{\Gamma(n)}{(2\pi)^n}\int_{\mathrm{S}_1^{n-1}} (1-ix\cdot\varpi)^{-n}\,d\sigma(\varpi) = \phi(x),
\end{aligned}$$

where in the third equality we substituted $\omega = R(\varpi)$ and used the fact that R is a bijection of S_1^{n-1} onto itself preserving the measure σ. (To be more precise, we used the integration formula for push-forwards of measures, Proposition 3.3 above, together with the fact that $R_*\sigma = \sigma$.) Of course, the fact that $\phi(x)$ is invariant under rotations can alternatively be seen from the very start, using $\phi(x) = \check{\Phi}(x) = \widehat{\widetilde{\Phi}}(x)$ and Folland's Theorem 8.22(b) together with the fact that $\widetilde{\Phi} = \Phi$ is invariant under rotations.

Since $\phi(x)$ is invariant under rotations, it suffices to evaluate $\phi(x)$ when $x = (x_1, 0, \ldots, 0)$, $x_1 \geq 0$. In this case we have, writing $\omega = (\omega_1, \ldots, \omega_n)$:

$$\phi((x_1, 0, \ldots, 0)) = \frac{\Gamma(n)}{(2\pi)^n} \int_{\mathrm{S}_1^{n-1}} (1 - ix_1\omega_1)^{-n} \, d\sigma(\omega).$$

Note that as ω varies over S_1^{n-1}, ω_1 varies over the interval $[-1, 1]$, and it seems clear that the above integral over S_1^{n-1} should be expressible as an integral simply over $\omega_1 \in [-1, 1]$. Indeed, by Proposition 3.3 applied with the map T being $T : \mathrm{S}_1^{n-1} \to [-1, 1]$; $T(\omega) := \omega_1$, we have

$$\phi((x_1, 0, \ldots, 0)) = \frac{\Gamma(n)}{(2\pi)^n} \int_{[-1,1]} (1 - ix_1\omega_1)^{-n} \, d(T_*\sigma)(\omega_1).$$

The question is thus: *What is the push-forward $T_*\sigma$ of the measure σ under the projection $T : \mathrm{S}_1^{n-1} \to [-1, 1]$?* The answer is easily found e.g. using spherical coordinates; cf. Folland's exercise 65 on p. 80.[13]

$$d(T_*\sigma)(\omega_1) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} (1 - \omega_1^2)^{\frac{n-3}{2}} \, d\omega_1. \tag{6.2}$$

(Here $d\omega_1$ is Lebesgue measure, as usual.) Using this we have

$$\phi((x_1, 0, \ldots, 0)) = \frac{\Gamma(n)}{2^{n-1}\pi^{\frac{n+1}{2}}\Gamma(\frac{n-1}{2})} \int_{[-1,1]} (1 - ix_1\omega_1)^{-n} \, (1 - \omega_1^2)^{\frac{n-3}{2}} \, d\omega_1.$$

This explicit integral which perhaps is perhaps not entirely simple to compute. I present one (dirty!) way to compute it below: The two main points I want to make are (1) it is often useful to use a computer algebra package, e.g. Maple, both to get the answer and to learn about e.g. special functions involved, and (2) it is often convenient

[13]Some details: By Folland's Exercise 65 we have for any Borel set $E \subset [-1, 1]$: $(T_*\sigma)(E) = \int_X I(\cos\phi_1 \in E) \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdots \sin\phi_{n-2} \, d\phi_1 \cdots d\phi_{n-2} \, d\theta$, where $X = (0, \pi)^{n-2} \times (0, 2\pi)$. Substituting $\omega_1 = \cos\phi_1$ we have $\int_0^\pi I(\cos\phi_1 \in E) \sin^{n-2}\phi_1 \, d\phi_1 = \int_E (1 - \omega_1^2)^{\frac{n-3}{2}} \, d\omega_1$; also the integral over the remaining variables is recognized as $\sigma_{n-2}(S^{n-2})$, by applying the same exercise 65 with $n-1$ in place of n. But $\sigma_{n-2}(S^{n-2}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ by Folland's Prop. 2.54. Hence the formula (6.2) follows.

to use handbooks of mathematical formulas, such as [5]; also google, Wikipedia, [14, http://dlmf.nist.gov/], etc, can be useful.

This integral can be computed using Maple: Typing

```
> simplify(int((1-omega1^2)^((n-3)/2)*(1-I*x1*omega1)^(-n),omega1=-1..1));
```

gives the answer

```
  1/2     2      (- 1/2 - n/2)
Pi     (x1  + 1)               GAMMA(n/2 - 1/2)
-----------------------------------------------
                  GAMMA(n/2)
```

It is of course always good to try to check where Maple's answers come from. In this case, typing the above without the "`simplify`" we see that the integral is related to the hypergeometric function (a fact which perhaps the more experienced readers could see from start without help). In fact, substituting $\omega_1 = 2u - 1$ we have

$$\begin{aligned}
&\int_{-1}^{1} (1 - ix_1\omega_1)^{-n} \, (1 - \omega_1^2)^{\frac{n-3}{2}} \, d\omega_1 \\
&= 2^{n-2} \int_0^1 u^{\frac{n-3}{2}} (1-u)^{\frac{n-3}{2}} (1 + ix_1 - 2ix_1 u)^{-n} \, du \\
&= 2^{n-2} (1 + ix_1)^{-n} \int_0^1 u^{\frac{n-3}{2}} (1-u)^{\frac{n-3}{2}} \Big(1 - \frac{2ix_1}{1 + ix_1} u\Big)^{-n} \, du
\end{aligned}$$

By [5, 9.111] and [5, 8.384] (cf. also wikipedia) we get

$$2^{n-2}(1+ix_1)^{-n} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} F\Big(n, \frac{n-1}{2}; n-1; \frac{2ix_1}{1+ix_1}\Big),$$

where F is the (Gauss') hypergeometric function (often also denoted by ${}_2F_1$). Next using [5, 9.134.1] we get

$$= 2^{n-2} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} F\Big(\frac{n}{2}, \frac{n+1}{2}; \frac{n}{2}; -x_1^2\Big),$$

and by [5, 9.100–9.102] this is, assuming $|x_1| < 1$:

$$\begin{aligned}
&= 2^{n-2} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} \sum_{j=0}^{\infty} \binom{-(n+1)/2}{j} (-1)^j (-x_1^2)^j \\
&= 2^{n-2} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} (1 + x_1^2)^{-\frac{n+1}{2}}.
\end{aligned}$$

Using also the doubling formula for the Gamma function (see [5, 8.335.1]), $\Gamma(n-1)=\pi^{-\frac{1}{2}}2^{n-2}\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2})$; we conclude:

$$\int_{-1}^{1}(1-ix_1\omega_1)^{-n}\,(1-\omega_1^2)^{\frac{n-3}{2}}\,d\omega_1=\frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}(1+x_1^2)^{-\frac{n+1}{2}}.$$

We have proved this for all $x_1\in(-1,1)$, but since both the left and the right hand sides in this last identity are clearly holomorphic functions in the open connected region

$$x_1\in\mathbb{C}\setminus i\Big((-\infty,-1]\cup[1,\infty)\Big),$$

the identity must hold for all these x_1 by analytic continuation, and in particular the identity holds for all $x_1\in\mathbb{R}$. This validates Maple’s answer! Using this we conclude

$$\begin{aligned}\phi((x_1,0,\ldots,0))&=\frac{\Gamma(n)}{2^{n-1}\pi^{\frac{n+1}{2}}\Gamma(\frac{n-1}{2})}\cdot\frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}(1+x_1^2)^{-\frac{n+1}{2}}\\&=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+x_1^2)^{-\frac{n+1}{2}},\end{aligned}$$

where in the last step we again used the doubling formula for Γ; $\Gamma(n)=\pi^{-\frac{1}{2}}2^{n-1}\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})$. (Note that our two applications of the doubling formula cancel each other; we only used it to check agreement with the Maple output.) Hence, using the fact that ϕ is invariant under rotations, we have

$$\phi(x)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+|x|^2)^{-\frac{n+1}{2}},\qquad\forall x\in\mathbb{R}^n.$$

$\square$

6.3. **On counting integer points in large convex sets.** Given a set $E\subset\mathbb{R}^n$ we are interested in the number of integer points in E, i.e. $\#(\mathbb{Z}^n\cap E)$. If E is “large” and “nice” it seems clear that $\#(\mathbb{Z}^n\cap E)$ should be approximately equal to the *volume* of E. The following result gives a precise error bound for this approximation, when E is replaced by the rescaled set RE and we let $R\to\infty$.

Theorem 6.1. *Assume that E is a bounded open convex set in $\mathbb{R}^n$ ($n\geq 2$) and that there is a constant $C>0$ such that*

$$|\widehat{\chi_E}(\xi)|\leq C(1+|\xi|)^{-\frac{n+1}{2}},\qquad\forall\xi\in\mathbb{R}^n.\tag{6.3}$$

Then there is a constant $C'>0$ such that

$$\big|\#(\mathbb{Z}^n\cap RE)-\mathrm{vol}(RE)\big|\leq C'R^{\frac{(n-1)n}{n+1}},\qquad\forall R\geq 1.\tag{6.4}$$

(Here “vol” denotes volume, i.e. Lebesgue measure: Thus $\mathrm{vol}(RE)=R^n\,\mathrm{vol}(E)=m^n(RE)=R^nm^n(E)$.)

Here are some remarks to put the result in context:

Remark 6.2. We proved in assignment 1, problem 8, that the bound (6.3) holds when C is a *ball.* But in fact (6.3) holds whenever the convex set C has a boundary ∂C which is sufficiently smooth and has everywhere positive gaussian curvature. Cf. Hlawka [9], [8], and Herz, [6].

Remark 6.3. The bound (6.4) with $R^{\frac{(n-1)n}{n+1}}$ replaced by R^{n-1} is "trivial" (note that R^{n-1} is the order of magnitude of the $(n-1)$-volume of the boundary $\partial(RE)$). We proved this in a special case in (1.55), and the proof in the general case is similar.

Remark 6.4. For $n = 2$ the bound in (6.4) is $C'R^{2/3}$; this gives the Voronoi (1903) bound on the Gauss' circle problem. For $n = 3$ the bound in (6.4) is $C'R^{3/2}$.

Remark 6.5. Herz 1962, [7] proves a result similar to Theorem 6.1 but with a more precise discussion on the implied constant, C'. Herz' method of proof is similar to the one below except that Herz uses convolution with characteristic functions only, i.e. no smooth bump functions. However the method below is quite standard and useful also in many other problems.

Remark 6.6. With virtually the same proof one can strengthen (6.4) to a result *uniform over all translations of* RE, namely:

$$\left|\#(\mathbb{Z}^n \cap (x + RE)) - \mathrm{vol}(RE)\right| \leq C'R^{\frac{(n-1)n}{n+1}}, \qquad \forall R \geq 1,\ x \in \mathbb{R}^n.$$

Proof of Theorem 6.1 (with a motivating discussion). The starting idea is to try to apply the Poisson summation formula to χ_{RE}, i.e. "$\sum_{k\in\mathbb{Z}^n} \chi_{RE}(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\chi_{RE}}(k)$", the point being that the left hand side in this relation equals $\#(\mathbb{Z}^n \cap RE)$, the number of lattice points in RE. However some modification of is necessary since the sum $\sum_{k\in\mathbb{Z}^n} \widehat{\chi_{RE}}(k)$ typically does not converge!

In order to make the Fourier transform decay more rapidly we convolve χ_{RE} with a smooth bump function acting like an approximate identity. Thus let us fix a function $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\phi \geq 0$, $\int \phi = 1$ and $\mathrm{supp}(\phi) \subset B_1^n$, the unit ball centered at the origin, and let us convolve χ_{RE} with ϕ_δ with $\delta > 0$ small, where

$$\phi_\delta(x) = \delta^{-n}\phi(\delta^{-1}x)$$

as usual. (Thus $\mathrm{supp}\,\phi_\delta \subset B_\delta^n$.) Note that $\chi_{RE} * \phi_\delta$ equals χ_{RE} except in the δ-neighbourhood of $\partial\chi_{RE}$. However, the naive way of using this fact to bound $\left|\sum_{k\in\mathbb{Z}^n} \chi_{RE}(k) - \sum_{k\in\mathbb{Z}^n} (\chi_{RE} * \phi_\delta)(k)\right|$ gives only $\ll R^{n-1}$ and we are stuck with the trivial bound (cf. Remark 6.3).

Thus instead let us try to *bound* χ_{RE} *from above and below*, i.e. let us replace $\chi_{RE} * \phi_\delta$ by a function which is $\geq \chi_{RE}$, and another function which is $\leq \chi_{RE}$ (both being as 'near' χ_{RE} as possible). Clearly, if

$(RE)^{-\delta}$ is the set RE minus the δ-neighbourhood[14] of its boundary, and $(RE)^{+\delta}$ is the set RE together with the δ-neighbourhood of its boundary, then

$$\chi_{(RE)^{-\delta}} * \phi_\delta \le \chi_{RE} \le \chi_{(RE)^{+\delta}} * \phi_\delta.$$

However in place of $(RE)^{\pm}$ we would like to use some set whose Fourier transform we have direct information about. Here the convexity of E comes in handy: Since E is convex, we have[15]

$$aE + bE = (a+b)E, \qquad \forall a, b \ge 0. \tag{6.5}$$

To use this, let us fix $r > 0$ so that E contains a ball of radius r (this is possible since E is open); then there is some $x_0 \in \mathbb{R}^n$ such that $x_0 + B_r^n \subset E$, i.e. $B_r^n \subset E - x_0$. This implies that for any $R > 0$ and $\delta > 0$, the set $RE + \frac{\delta}{r}(E - x_0)$ contains $RE + B_\delta^n$, i.e. the δ-neighbourhood of RE. By (6.5) we have $RE + \frac{\delta}{r}(E - x_0) = (R + \frac{\delta}{r})E - \frac{\delta}{r}x_0$, and we have thus proved that *for any* $R > 0$ *and* $\delta > 0$, *the set*

$$E_{R,\delta}^+ := \Big(R + \frac{\delta}{r}\Big)E - \frac{\delta}{r}x_0$$

contains the δ*-neighbourhood of* RE. By a completely similar argument one also proves that *for any* $\delta > 0$ *and* $R > \frac{\delta}{r}$, *the* δ*-neighbourhood of the set*

$$E_{R,\delta}^- := \Big(R - \frac{\delta}{r}\Big)E + \frac{\delta}{r}x_0$$

is contained in RE. It follows from these two facts that, if $R > \frac{\delta}{r}$,

$$\chi_{E_{R,\delta}^-} * \phi_\delta \le \chi_{RE} \le \chi_{E_{R,\delta}^+} * \phi_\delta.$$

Here the functions $\chi_{E_{R,\delta}^-} * \phi_\delta$ and $\chi_{E_{R,\delta}^+} * \phi_\delta$ are in C_c^∞ and hence we can apply the Poisson summation formula to them. This gives (in the case of $E_{R,\delta}^+$):

$$\sum_{k \in \mathbb{Z}^n} (\chi_{E_{R,\delta}^+} * \phi_\delta)(k) = \sum_{k \in \mathbb{Z}^n} \widehat{\chi_{E_{R,\delta}^+} * \phi_\delta}(k) = \sum_{k \in \mathbb{Z}^n} \widehat{\chi_{E_{R,\delta}^+}}(k)\widehat{\phi}(\delta k).$$

Here $\widehat{\phi}(0) = \int \phi = 1$ and

$$\widehat{\chi_{E_{R,\delta}^+}}(0) = \mathrm{vol}(E_{R,\delta}^+) = \Big(R + \frac{\delta}{r}\Big)^n \mathrm{vol}(E).$$

[14] By the δ-neigbourhood of a set $F \subset \mathbb{R}^n$ we mean the set of all points in $\mathbb{R}^n$ which have distance $< \delta$ to some point in F.

[15] Here we use the notation $tE = \{tx : x \in E\}$ and $E + E' = \{x + y : x \in E, y \in E'\}$, for any $E, E' \subset \mathbb{R}^n$.

Hence we conclude

$$\#(\mathbb{Z}^n \cap RE) = \sum_{k\in\mathbb{Z}^n} \chi_{RE}(k) \leq \sum_{k\in\mathbb{Z}^n} (\chi_{E^+_{R,\delta}} * \phi_\delta)(k)$$
$$= \Big(R+\frac{\delta}{r}\Big)^n \operatorname{vol}(E) + \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \widehat{\chi_{E^+_{R,\delta}}}(k)\widehat{\phi}(\delta k).$$

Similarly using $\chi_{RE} \geq \chi_{E^-_{R,\delta}} * \phi_\delta$ we get, if $R > \frac{\delta}{r}$,

$$\#(\mathbb{Z}^n \cap RE) \geq \Big(R-\frac{\delta}{r}\Big)^n \operatorname{vol}(E) + \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \widehat{\chi_{E^-_{R,\delta}}}(k)\widehat{\phi}(\delta k).$$

From now on we keep $R \geq 1$ and $\delta \leq 1$, and we note that $(R \pm \frac{\delta}{r})^n = R^n + O(\delta R^{n-1})$. Here and in any later big-O bound the implied constant may depend on n, r, E but not on R, δ. Hence we conclude:

$$\big|\#(\mathbb{Z}^n \cap RE) - R^n \operatorname{vol}(E)\big|$$
$$\ll \delta R^{n-1} + \max\Big(\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \Big|\widehat{\chi_{E^-_{R,\delta}}}(k)\widehat{\phi}(\delta k)\Big|, \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \Big|\widehat{\chi_{E^+_{R,\delta}}}(k)\widehat{\phi}(\delta k)\Big|\Big).$$

Now since $\phi \in C_c^\infty(\mathbb{R}^n)$ we have, for any fixed $A > 0$:

$$|\widehat{\phi}(\xi)| \ll (1+|\xi|)^{-A}, \qquad \forall \xi \in \mathbb{R}^n$$

(the implied constant may depend on A). Furthermore, since $E^\pm_{R,\delta}$ is a translate of $(R \pm \frac{\delta}{r})E$ we have, using our assumption (6.1) (and cf. Folland Thm. 8.22(a)-(b)):

$$|\widehat{\chi_{E^\pm_{R,\delta}}}(\xi)| = \Big(R\pm\frac{\delta}{r}\Big)^n \Big|\chi_E\Big(\Big(R\pm\frac{\delta}{r}\Big)\xi\Big)\Big| \ll \Big(R\pm\frac{\delta}{r}\Big)^n \Big(1+\Big(R\pm\frac{\delta}{r}\Big)|\xi|\Big)^{-\frac{n+1}{2}}.$$

Recall that we are assuming $R \geq 1$ and $\delta \leq 1$. Let us now also assume $R \geq \frac{2}{r}$, so that $R \geq 2\frac{\delta}{r}$. Then $R \ll R \pm \frac{\delta}{r} \ll R$, and we conclude:

$$|\widehat{\chi_{E^\pm_{R,\delta}}}(\xi)| \ll R^n(1+R|\xi|)^{-\frac{n+1}{2}} \leq R^{\frac{n-1}{2}}|\xi|^{-\frac{n+1}{2}}, \qquad \forall \xi \in \mathbb{R}^n.$$

Hence:

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \Big|\widehat{\chi_{E^\pm_{R,\delta}}}(k)\widehat{\phi}(\delta k)\Big| \ll R^{\frac{n-1}{2}} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{-\frac{n+1}{2}}(1+|\delta k|)^{-A}. \tag{6.6}$$

The last sum can be treated using dyadic decomposition: We split the set $\mathbb{Z}^n \setminus \{0\}$ into the annuli $B^n_{2^m} \setminus B^n_{2^{m-1}}$, $m = 1, 2, 3, \ldots$, where B^n_r is the open n-dimensional ball of radius r centered at the origin. (Note that since we take B^n_r to be open every point in $\mathbb{Z}^n \setminus \{0\}$ really belongs to one of our annuli.) For $k \in B^n_{2^m} \setminus B^n_{2^{m-1}}$ we have

$$|k|^{-\frac{n+1}{2}}(1+|\delta k|)^{-A} \ll \begin{cases} 2^{-\frac{(n+1)m}{2}} & \text{if } \delta \leq 2^{-m} \\ 2^{-\frac{(n+1+2A)m}{2}}\delta^{-A} & \text{if } \delta \geq 2^{-m}. \end{cases}$$

Also the number of integer points in $B^n_{2^m} \setminus B^n_{2^{m-1}}$ is $\leq$ the number of points in B_{2^m}, which is $\ll 2^{mn}$. Hence, if we now assume that A has been fixed to be a constant $> \frac{n-1}{2}$ (say $A = n$, for definiteness), we get that (6.6) is

$$\ll R^{\frac{n-1}{2}} \Biggl(\sum_{1 \leq m \leq -\log_2 \delta} 2^{mn - \frac{(n+1)m}{2}} + \sum_{m > -\log_2 \delta} 2^{mn - \frac{(n+1+2A)m}{2}} \delta^{-A} \Biggr) \\ \ll R^{\frac{n-1}{2}} \delta^{-\frac{n-1}{2}}.$$

(Another way to obtain this bound, $\sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{-\frac{n+1}{2}} (1 + |\delta k|)^{-A} \ll \delta^{-\frac{n-1}{2}}$, would be to prove that this sum is $\ll$ the corresponding integral, $\int_{\mathbb{R}^n \setminus B^n_{1/2}} |x|^{-\frac{n+1}{2}} (1 + |\delta x|)^{-A} \, dx$; this integral is easily bounded using polar coordinates.)

Collecting our results we have now proved:

$$\bigl| \#(\mathbb{Z}^n \cap RE) - R^n \operatorname{vol}(E) \bigr| \ll \delta R^{n-1} + R^{\frac{n-1}{2}} \delta^{-\frac{n-1}{2}},$$

for all $R \geq \max(\frac{2}{r}, 1)$ and all $\delta \leq 1$. We now choose δ optimally for given R: The best choice is seen to be $\delta = R^{-\frac{n-1}{n+1}}$. (Note that this is ≤ 1 since $R \geq 1$, i.e. this is a valid choice of δ.) With this choice we obtain

$$\bigl| \#(\mathbb{Z}^n \cap RE) - R^n \operatorname{vol}(E) \bigr| \ll R^{\frac{(n-1)n}{n+1}}.$$

Hence we have proved that (6.4) holds for all $R \geq \max(\frac{2}{r}, 1)$. Of course we also have $|\#(\mathbb{Z}^n \cap RE) - R^n \operatorname{vol}(E)| \ll 1$ for all $R \in [1, \max(\frac{2}{r}, 1)]$; hence (6.4) is proved for all $R \geq 1$. $\square$

7. Lecture 9: Special functions and asymptotic expansions

In the first part of this lecture I will discuss some more stuff in Fourier analysis: The Poisson summation formula, and an application to counting lattice points in convex sets (see Section 6.3 above).

I will then start discussing asymptotic bounds and expansions, and I will start by discussing how to bound and estimate *positive* integrals. Regarding *bounding*, I have already given two examples of using *dyadic decomposition;* see below (6.6) and around (1.5). I also recommend the example given on www.tricki.org, and the proof of Theorem 8.15 in Folland's book. Thus in this lecture I will not say more about dyadic decomposition, but will instead discuss other ways of bounding, and then estimating positive integrals.

As an application I will discuss the Γ-function and how to obtain an asymptotic formula for $\Gamma(z)$ as $z \to +\infty$ along the real axis using the formula $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt$.

7.1. **Notation: "big O", "little o", "$\ll$", "$\gg$", "$\asymp$" and "$\sim$".**

Note: The exact conventions regarding these symbols may differ slightly between different books and papers. The following are the conventions which we will use throughout the present course.

"Big O": If a is a *non-negative* number, the symbol "$O(a)$" is used to denote any number b for which $|b| \leq Ca$, where C is a positive "constant", called *the implied constant*. We write "constant" within quotation marks since C may well be allowed to depend on certain parameters: When using the big-O notation it is very important to always be clear about *which* parameters C is allowed to depend on. Furthermore, it must always be clear for which variable ranges the bound holds. For example: "$f(x) = O(x^3)$ as $x \to \infty$" means that there is some constant $C > 0$ such that *for all sufficiently large x* we have $|f(x)| \leq Cx^3$. On the other hand, "$f(x) = O(x^3)$ for $x \geq 1$" means that there is some constant $C > 0$ such that $|f(x)| \leq Cx^3$ *holds for all $x \geq 1$.*

If the implied constant can be taken to be independent of *all* parameters present in the problem, then the implied constant is said to be *absolute.*

Note that whenever we use the notation "$O(a)$" we require that $a \geq 0$.

"little o": We write "$f(x) = o(g(x))$ as $x \to a$" to denote that $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$; we will only use this notation when $g(x) > 0$ for all x sufficiently near a! Thus for example if we write "$\sum_{n=1}^N a_n =$

$\frac{2}{3}N^{\frac{3}{2}} - \frac{6}{7}N^{\frac{7}{6}}(1 + o(1))$ as $N \to \infty$" then "$o(1)$" denotes some function $f(N)$ which satisfies $\lim_{N\to\infty} f(N) = 0$. Note that, unlike the "big O"-notation *the "little o"-notation can only be used when we are taking a limit.*

"$\ll$": "$b \ll a$" means the same as $b = O(a)$.

"$\gg$": "$b \gg a$" means that there is a constant $C > 0$ (again called the *implied constant*) such that $|b| \geq Ca \geq 0$.

Thus note that "$a \ll b$" is in general not equivalent with "$b \gg a$" – but they are equivalent whenever both a and b are nonnegative.

"$\asymp$": "$b \asymp a$" means $[b \ll a$ and $b \gg a]$.

"$\sim$": We write "$f(x) \sim g(x)$ as $x \to a$" to denote that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. Thus this notation can only be used when $g(x) \neq 0$ for all x sufficiently near a. We may note that *if* $g(x) \neq 0$ for all x sufficiently near a, then "$f(x) \sim g(x)$ as $x \to a$" is equivalent with "$f(x) = g(x)(1 + o(1))$ as $x \to a$".

7.2. **The Γ-function; basic facts.** For easy reference we here collect the basic facts about the Gamma function, mostly without proofs. For more details, cf., e.g., Ahlfors [1, Ch. 6.2.4-5], Olver [16, Ch. 2.1], or a number of other sources.

The Γ-function is commonly defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad \text{for } z \in \mathbb{C} \text{ with } \Re z > 0, \tag{7.1}$$

together with the relation

$$\Gamma(z + 1) = z\Gamma(z) \tag{7.2}$$

which can be used to extend $\Gamma(z)$ to a meromorphic function for all $z \in \mathbb{C}$, the only poles being at $z \in \{0, -1, -2, -3, \ldots\}$, and each pole being simple. (Note that (7.1) defines $\Gamma(z)$ as an analytic function in the region $\{\Re z > 0\}$, and using integration by parts one proves that (7.2) holds in this region; it is then easy to prove that if (7.2) is used to define $\Gamma(z)$ also when $\Re z \leq 0$ then we get a meromorphic function as claimed.)

We have

$$\Gamma(n) = (n - 1)!, \qquad \forall n \in \mathbb{N}.$$

The Γ-function is also given by the following infinite product formula (which is sometimes used as a definition):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n}\Big) e^{-z/n}, \tag{7.3}$$

where γ is *Euler's constant,* defined so that $\Gamma(1) = 1$, i.e.

$$\gamma := -\log\Big(\prod_{n=1}^{\infty}\Big(1+\frac{1}{n}\Big)e^{-1/n}\Big) = \lim_{N\to\infty}\Big(-\log N + \sum_{n=1}^{N}\frac{1}{n}\Big) = 0.57722\ldots. \tag{7.4}$$

The product in (7.3) is a so called Weierstrass product (cf. Wikipedia), and since $\sum_{n=1}^{\infty} n^{-2} < \infty$ the product in (7.3) is uniformly absolutely convergent[16] on compact subsets of $\mathbb{C}$, and (thus) $\Gamma(z)^{-1}$ is an entire function which has simple zeros at each point $z = 0, -1, -2, \ldots$, and no other zeros.

The Γ-function satisfies the following relations (identities between meromorphic functions of $z \in \mathbb{C}$):

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}; \tag{7.5}$$

$$\Gamma(2z) = \pi^{-\frac{1}{2}}2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}). \tag{7.6}$$

(The relation (7.6) is called Legendre's duplication formula.)

An important formula involving the Γ-function is the following:

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{7.7}$$

true for all $a, b \in \mathbb{C}$ with $\Re a, \Re b > 0$. (Cf. Folland p. 77, Exercise 60.) The above function (as a function of a and b) is called the *beta function*, $B(a, b)$. Many other integrals can be transformed into a beta function – namely any convergent integral of the form $\int_A^B L_1(x)^{\alpha} L_2(x)^{\beta}\,dx$ (where A or B may be $\pm\infty$) where L_1 and L_2 are two affine linear forms of x such that $L_1(x)$ is 0 or ∞ at $x = A$ and $L_2(x)$ is 0 or ∞ at $x = B$.

We give two final formulas involving the Γ-function: The $(n-1)$-dimensional volume of the unit sphere S_1^{n-1} is $\frac{2\pi^{n/2}}{\Gamma(\frac{1}{2}n)}$, and (hence) the volume of the n-dimensional unit ball is $\frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)}$. Cf. Folland, Prop. 2.54 and Cor. 2.55.

7.3. **Stirling's formula.**

We have the following asymptotic formula for $\Gamma(z)$ for z large:

Theorem 7.1. ***(Stirling's formula.)*** *For any fixed* $\varepsilon > 0$ *we have*

$$\log \Gamma(z) = \big(z - \tfrac{1}{2}\big)\log z - z + \log\sqrt{2\pi} + O\big(|z|^{-1}\big), \tag{7.8}$$

[16]Recall that a product $\prod_{n=1}^{\infty}(1 + u_n(z))$ is said to be absolutely convergent if $\sum_{n=1}^{\infty}|u_n(z)| < \infty$. Hence in the present case we should set $u_n(z) = (1 + \frac{z}{n}\Big)e^{-z/n} - 1$, and the claim is that then $\sum_{n=1}^{\infty}|u_n(z)|$ converges, and converges uniformly with respect to z in any compact subset of $\mathbb{C}$.

for all z with $|z| \geq 1$ and $\left|\arg z\right| \leq \pi - \varepsilon$. (The implied constant depends on ε but of course not on z. Also in the right hand side we use the principal branch of the logarithm function.) In fact we have the following more precise asymptotic formula, for any $m \in \mathbb{Z}_{\geq 0}$:

(7.9)

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \log z - z + \log \sqrt{2\pi} + \sum_{k=0}^{m} \frac{B_{2k+2}}{(2k+2)(2k+1)} z^{-2k-1} + O\big(|z|^{-2m-3}\big)$$

for all z with $|z| \geq 1$ and $\left|\arg z\right| \leq \pi - \varepsilon$. (The implied constant depends on m and ε but of course not on z.) Here B_r is the rth Bernoulli number; cf. Definition 1.18.

Exponentiating, (7.8) gives:

$$\Gamma(z) = \sqrt{2\pi} \cdot \frac{z^{z-\frac{1}{2}}}{e^z} \cdot e^{O(|z|^{-1})} = \sqrt{2\pi} \cdot \frac{z^{z-\frac{1}{2}}}{e^z} \cdot \Big(1 + O(|z|^{-1})\Big)$$

for all z with $|z| \geq 1$ and $\left|\arg z\right| \leq \pi - \varepsilon$. Here if z is general complex one has to remember that $z^{z-\frac{1}{2}}$ is by definition the same as $\exp((z-\frac{1}{2}) \log z)$ where the principal branch of the logarithm is used.

There is a slight modification of Stirling's formula which is often convenient to remember:

Corollary 7.2. *For any fixed $\varepsilon > 0$ and $\alpha \in \mathbb{C}$ we have*

$$\log \Gamma(z + \alpha) = \left(z + \alpha - \tfrac{1}{2}\right) \log z - z + \log \sqrt{2\pi} + O\big(|z|^{-1}\big), \tag{7.10}$$

for all z with $|z| \geq 1$, $|z + \alpha| \geq 1$ and $\left|\arg(z + \alpha)\right| \leq \pi - \varepsilon$. (The implied constant depends on ε and α but of course not on z.)

This corollary follows more or less immediately from (7.8) by using $\log(z+\alpha) = \log z + \frac{\alpha}{z} + O(|z|^2)$ for $|z|$ large (viz., the Taylor expansion of $\log(1 + \alpha z^{-1})$ for z large); we postpone the details to an appendix (see Sec. 7.5). It is important to note that if we are interested in finer asymptotics as in (7.9) then the transformation from z to $z + \alpha$ is *not* as simple as in (7.10) (but it can of course be worked out, using Taylor expansions).

For example, taking $\alpha = 1$ in Corollary 7.10 gives

$$\Gamma(z + 1) = \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z} (1 + O(|z|^{-1})), \tag{7.11}$$

and in particular:

$$n! = \sqrt{2\pi} \cdot \frac{n^{n+\frac{1}{2}}}{e^n} (1 + O(n^{-1})), \qquad \forall n \geq 1.$$

Let us now discuss the *proof* of Stirling's formula: One method of proof is to use the product formula, (7.3), take the logarithm, and then apply the Euler-MacLaurin summation formula, Theorem 1.19. Let us discuss this in some detail. First, since $\Gamma(z)$ is a meromorphic function of $z \in \mathbb{C}$ with no zeros, and simple poles at $z = 0, -1, -2, \ldots$ and no other points, we can define a branch of $\log \Gamma(z)$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ [17] (cf., e.g., [18, Thm. 13.11(h)]). This branch is uniquely determined by requiring that $\log \Gamma(z) > 0$ for all large $z \in \mathbb{R}_{>0}$. Now the product formula, (7.3), implies:

(7.12)

$$\log \Gamma(z) = -\log z - \gamma z + \sum_{n=1}^{\infty} \Big(\frac{z}{n} - \log\Big(1 + \frac{z}{n}\Big)\Big), \qquad \forall z \in \mathbb{C} \setminus (-\infty, 0],$$

where in the right hand side the principal branch of the logarithm is used throughout. (Outline of details: One checks that the sum in (7.12) is uniformly absolutely convergent for z in compact subsets of $\mathbb{C} \setminus (-\infty, 0]$; hence the right hand side defines an analytic function in the region $\mathbb{C} \setminus (-\infty, 0]$. One immediately checks that this function is > 0 for all large $z \in \mathbb{R}_{>0}$, and that the exponential of this function equals $\Gamma(z)$. Hence the right hand side indeed coincides with the branch of $\log \Gamma(z)$ which we defined above.)

By writing $\sum_{n=1}^{\infty}$ as $\lim_{N \to \infty} \sum_{n=1}^{N}$ and then using the fact that $\sum_{n=1}^{N} \frac{z}{n} - z \log N = \gamma z$ (cf. (7.4)), the formula (7.12) can be rewritten as ($\forall z \in \mathbb{C} \setminus (-\infty, 0]$):

$$\log \Gamma(z) = \lim_{N \to \infty} \Big(z \log N - \sum_{n=0}^{N} \log(z + n) + \sum_{n=1}^{N} \log n \Big). \tag{7.13}$$

This is the formula to which we apply the Euler-MacLaurin summation formula, Theorem 1.19. For the details of how this leads to Theorem 7.1, cf., e.g., Olver [16, Ch. 8.4].

7.4. **Γ-asymptotics directly from the integral.** As an example of techniques for bounding and estimating positive integrals, we will here discuss the asymptotics of $\Gamma(s)$ as $s \to \infty$ along the real axis, using the formula $\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, dx$ (cf. (7.1)).

This section should be compared with Olver, [16, Ch. 3.8].

Thus from now on we assume $s > 0$ and we will focus on the case of s large. We consider the integral

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, dx,$$

and we note that the integrand is positive for all x. As a first step we determine the monotonicity properties of the integrand. Set

$$f(x) = e^{-x} x^{s-1}.$$

[17] That is, an analytic function $g : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ satisfying $e^{g(z)} = \Gamma(z)$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$.

We compute:

$$f'(x) = (s-1-x)x^{s-2}e^{-x}.$$

Already here we see that it is convenient to assume $s > 1$, so let's assume this. Now we see that $f(x)$ is increasing for $0 < x < s-1$ and decreasing for $x > s-1$; thus $f(x)$ attains its maximum at $x = s-1$. When studying how quickly $f(x)$ descends when x moves away from $s-1$ it is convenient to set $x = s-1+y$, and consider the logarithm of $f(x)$:

$$\begin{aligned}\log f(s-1+y) &= -(s-1+y) + (s-1)\log(s-1+y) \\ &= (s-1)\Big(-1 - \frac{y}{s-1} + \log(s-1) + \log\Big(1+\frac{y}{s-1}\Big)\Big).\end{aligned}$$

We see that it is convenient to take $u = \frac{y}{s-1}$ as a new variable. Then:

$$\log f\Big((s-1)(1+u)\Big) = (s-1)\log\Big(\frac{s-1}{e}\Big) - (s-1)\Big(u - \log(1+u)\Big),$$

or in other words:

$$f((s-1)(1+u)) = \Big(\frac{s-1}{e}\Big)^{s-1} e^{-(s-1)(u-\log(1+u))}.$$

Substituting $x = (s-1)(1+u)$ in our integral we get:

$$\Gamma(s) = (s-1)^s e^{1-s} \int_{-1}^{\infty} e^{-(s-1)(u-\log(1+u))}\, du. \tag{7.14}$$

Set

$$g(u) = u - \log(1+u).$$

It is clear from the above analysis that $g(u)$ attains its minimum at $u = 0$; this can of course also easily be checked at this point: We have $g'(u) = \frac{u}{1+u}$, thus $g'(u) < 0$ for $-1 < u < 0$ and $g'(u) > 0$ for $u > 0$. Using the Taylor expansion for $\log(1+u)$ we see that for any fixed $0 < c_0 < 1$ we have

$$g(u) = \frac{1}{2}u^2 + O(u^3), \qquad \forall u \in [-c_0, c_0].$$

Hence for any $0 < \alpha < c_0$:

$$\begin{aligned}\int_{-\alpha}^{\alpha} e^{-(s-1)(u-\log(1+u))}\, du &= \int_{-\alpha}^{\alpha} e^{-\frac{1}{2}(s-1)(u^2+O(u^3))}\, du \qquad \Big\{\text{Set } u = \sqrt{\frac{2}{s-1}}t\Big\} \\ &= \sqrt{\frac{2}{s-1}} \int_{-\alpha\sqrt{(s-1)/2}}^{\alpha\sqrt{(s-1)/2}} e^{-t^2 + O((s-1)^{-\frac{1}{2}}t^3)}\, dt.\end{aligned}$$

If we assume that $\alpha \leq 10(s-1)^{-\frac{1}{3}}$ (say) then $(s-1)^{-\frac{1}{2}}t^3$ in the last integral always has bounded absolute value (namely $\leq 2^{-\frac{3}{2}}10^3$) and hence we may continue as follows:

$$= \sqrt{\frac{2}{s-1}} \int_{-\alpha\sqrt{(s-1)/2}}^{\alpha\sqrt{(s-1)/2}} e^{-t^2} \Big(1 + O((s-1)^{-\frac{1}{2}} t^3)\Big) \, dt$$
$$= \sqrt{\frac{2}{s-1}} \int_{-\alpha\sqrt{(s-1)/2}}^{\alpha\sqrt{(s-1)/2}} e^{-t^2} \, dt + O((s-1)^{-1}).$$

Let us note that for any $A > 0$ we have

$$\text{(7.15)} \quad \int_{-A}^{A} e^{-t^2} \, dt = \int_{-\infty}^{\infty} e^{-t^2} \, dt - 2\int_{A}^{\infty} e^{-t^2} \, dt = \sqrt{\pi} - O(A^{-1} e^{-A^2}).$$

[Proof of the last error bound (note that this is in itself a simple but useful example on how to bound positive integrals): $\int_A^\infty e^{-t^2}\,dt = \int_0^\infty e^{-(A+u)^2}\,du \leq \int_0^\infty e^{-A^2-2Au}\,du = (2A)^{-1}e^{-A^2} \ll A^{-1}e^{-A^2}$. We remark that one easily sees that the bound is optimal so long as A is bounded away from zero, i.e. we have also $\int_A^\infty e^{-t^2}\,dt \gg A^{-1}e^{-A^2}$ for all $A \geq c > 0$.] Hence from the above computation we get:

$$\int_{-\alpha}^{\alpha} e^{-(s-1)(u-\log(1+u))} \, du = \sqrt{\frac{2\pi}{s-1}} + O\Big((s-1)^{-1} + \frac{e^{-\frac{1}{2}(s-1)\alpha^2}}{\alpha\sqrt{s-1}}\Big).$$

This has been proved for any $\alpha \in (0, c_0)$ ($c_0 < 1$) satisfying $\alpha \leq 10(s-1)^{-\frac{1}{3}}$. But recall that our real task was to compute $\int_{-1}^\infty e^{-(s-1)(u-\log(1+u))}\,du$; hence it remains to bound the integrals $\int_{-1}^{-\alpha}$ and $\int_\alpha^\infty$. The first of these is easily handled using the fact that $u - \log(1+u) \geq \frac{1}{2}u^2$ for all $u \in (-1, 0]$; this is clear e.g. from the Taylor expansion of $\log(1+u)$. Hence we get:

$$\int_{-1}^{-\alpha} e^{-(s-1)(u-\log(1+u))} \, du \leq \int_{-1}^{-\alpha} e^{-\frac{1}{2}(s-1)u^2} \, du$$
$$\leq \sqrt{\frac{2}{s-1}} \int_{\alpha\sqrt{(s-1)/2}}^{\infty} e^{-t^2} \, dt \ll \frac{e^{-\frac{1}{2}(s-1)\alpha^2}}{\alpha(s-1)}.$$

(Cf. (7.15) regarding the last bound.) Finally to bound the $\int_\alpha^\infty$ we use the fact that there is an absolute constant $c_1 > 0$ such that $u - \log(1+u) \geq c_1 u^2$ for all $u \in [0, 1]$. (Prove this fact as an exercise! In fact the optimal choice of c_1 is $c_1 = 1 - \log 2 = 0.3068\ldots$.) Furthermore there is an absolute constant $c_2 > 0$ such that for all $u \geq 1$ we have $u - \log(1+u) \geq c_2 u$. (Prove this fact as an exercise!) Hence

$$\int_{\alpha}^{\infty} e^{-(s-1)(u-\log(1+u))} \, du \leq \int_{\alpha}^{1} e^{-c_1(s-1)u^2} \, du + \int_{1}^{\infty} e^{-c_2(s-1)u} \, du$$
$$\ll \frac{e^{-c_1(s-1)\alpha^2}}{\alpha(s-1)} + \frac{e^{-c_2(s-1)}}{s-1}.$$

Adding together the integrals we have now proved that (assuming $s \geq 2$, say, so that $\frac{1}{s-1} \ll \frac{1}{\sqrt{s-1}}$ etc):

$$\int_{-1}^{\infty} e^{-(s-1)(u-\log(1+u))}\,du = \sqrt{\frac{2\pi}{s-1}} + O\Big((s-1)^{-1} + \frac{e^{-\frac{1}{2}(s-1)\alpha^2}}{\alpha\sqrt{s-1}} + \frac{e^{-c_1(s-1)\alpha^2}}{\alpha(s-1)}\Big).$$

(The other error terms are trivially subsumed by the error terms written out.) This has been proved for any $\alpha \in (0, c_0)$ ($c_0 < 1$) satisfying $\alpha \leq 10(s-1)^{-\frac{1}{3}}$. Making now a definite choice of α, say $\alpha = \frac{1}{2}(s-1)^{-\frac{1}{3}}$ (this is $\leq \frac{1}{2}$ for all $s \geq 2$, hence ok) we trivially have that the last two error terms decay exponentially and are subsumed by the first error term, $O((s-1)^{-1})$. Hence, recalling (7.14), we have proved:

$$\Gamma(s) = \sqrt{2\pi}(s-1)^{s-\frac{1}{2}}e^{1-s}\Big(1 + (s-1)^{-\frac{1}{2}}\Big), \qquad \forall s \geq 2.$$

Note that this agrees with (7.11) above, but with a worse error term.

In order to get a better error term, and even an asymptotic expansion of $\Gamma(s)$, we modify the treatment of (7.14) as follows: We have seen that $g(u) = u - \log(1+u)$ is strictly decreasing for $u \in (-1, 0)$ and strictly increasing for $u \in (0, \infty)$, since $g'(u) = \frac{u}{1+u}$. Hence the function g has continuous *inverses*, $h_1 : [0, \infty) \to (-1, 0]$ (a decreasing function with $h_1(0) = 0$, satisfying $g(h_1(v)) = v$, $\forall v \in \mathbb{R}_{\geq 0}$) and $h_2 : [0, \infty) \to [0, \infty)$ (an increasing function with $h_2(0) = 0$, satisfying $g(h_2(v)) = v$, $\forall v \in \mathbb{R}_{\geq 0}$). We note that h_1 and h_2 are C^∞ in $(0, \infty)$, and now substituting $u = h_1(v)$ and $u = h_2(v)$ in the integral in (7.14) we obtain:

$$\int_{-1}^{\infty} e^{-(s-1)(u-\log(1+u))}\,du = \int_{-1}^{0} \cdots + \int_{0}^{\infty} \cdots$$

$$= \int_0^\infty e^{-(s-1)v}(-h_1'(v))\,dv + \int_0^\infty e^{-(s-1)v}h_2'(v)\,dv. \tag{7.16}$$

By implicit differentiation using $g(h_j(v)) = v$ we see that

$$h_j'(v) = 1 + h_j(v)^{-1}, \qquad j = 1, 2.$$

In particular if $c_3 > 0$ is any fixed constant then for all $v \geq c_3$ both $-h_1'(v)$ and $h_2'(v)$ and *bounded* and positive. (Namely: $0 < -h_1'(v) \leq -1 - h_1(c_3)^{-1}$ and $0 < h_2'(v) \leq 1 + h_2(c_3)^{-1}$.) Hence the contribution from $v \geq c_3$ to the integrals in (7.16) is:

$$\ll \int_{c_3}^{\infty} e^{-(s-1)v}\,dv = \frac{e^{-c_3(s-1)}}{s-1}, \tag{7.17}$$

i.e. exponentially small.

It remains to treat the integrals for v near 0, and here we will use the power series expansion of $h_j(v)$. In fact, since $g(u)$ is analytic for

$u \in \mathbb{C}$ with $|u| < 1$ with the power series

$$g(u) = u - \log(1+u) = \frac{1}{2}u^2 - \frac{1}{3}u^3 + \frac{1}{4}u^4 - \dots,$$

it follows that there exists some open disc $\Omega \subset \mathbb{C}$ centered at 0, and an analytic function $H : \Omega \to \mathbb{C}$, such that $H(0) = 0$, $H(w) > 0$ for $w \in \Omega \cap \mathbb{R}_{>0}$, and $g(H(w)) = w^2$ for all $w \in \Omega$. The power series for $H(w)$ can be found by substituting in $g(H(w)) = w^2$, and we compute:

$$H(w) = \sqrt{2}w + \frac{2}{3}w^2 + \frac{1}{9\sqrt{2}}w^3 + \dots, \qquad \forall w \in \Omega. \tag{7.18}$$

Let $r > 0$ be the radius of Ω. Now since $v \mapsto H(\sqrt{v})$ for $v \in [0, r^2)$ is a continuous function mapping 0 to 0, increasing at least for small v, and whose composition with g is the identity function, we conclude that $h_2(v) = H(\sqrt{v})$ for all $v \in [0, r^2)$. Similarly $h_1(v) = H(-\sqrt{v})$ for all $v \in [0, r^2)$. Hence by differentiating we obtain

$$h_j'(v) = \frac{(-1)^j}{2\sqrt{v}} H'((-1)^j\sqrt{v}) = \frac{(-1)^j}{2\sqrt{v}}\Big(\sqrt{2} + \frac{4}{3}(-1)^j\sqrt{v} + \frac{1}{3\sqrt{2}}v + \dots\Big),$$

for all $v \in (0, r^2)$. Hence if we choose $c_3 = \frac{1}{2}r^2$, say, then we have

$$\int_0^{c_3} e^{-(s-1)v}(-h_1'(v))\,dv = \int_0^{c_3} e^{-(s-1)v}\Big(\frac{1}{\sqrt{2v}} - \frac{2}{3} + \frac{\sqrt{v}}{6\sqrt{2}} + O(v)\Big)\,dv \tag{7.19}$$

Lemma 7.3. *For any fixed $c, c' > 0$ and $\alpha > -1$, we have*

$$\int_0^c e^{-Av}v^\alpha\,dv = \Gamma(\alpha+1)A^{-1-\alpha} + O(A^{-1}e^{-cA}), \qquad \forall A \geq c'.$$

Proof. Substituting $v = A^{-1}x$ we get

$$\int_0^c e^{-Av}v^\alpha\,dv = A^{-1-\alpha}\int_0^{Ac} e^{-x}x^\alpha\,dx = A^{-1-\alpha}\Big(\Gamma(\alpha+1) - \int_{Ac}^\infty e^{-x}x^\alpha\,dx\Big). \tag{7.20}$$

Here if $\alpha \leq 0$ then x^α is a decreasing function of $x > 0$ and thus $\int_{Ac}^\infty e^{-x}x^\alpha\,dx \leq (Ac)^\alpha \int_{Ac}^\infty e^{-x}\,dx = (Ac)^\alpha e^{-Ac}$. On the other hand if $k - 1 < \alpha \leq k$ with $k \in \mathbb{N}$ then we may integrate by parts k times to get

$$\begin{aligned}\int_{Ac}^\infty e^{-x}x^\alpha\,dx &= e^{-Ac}\sum_{j=0}^{k-1}\Big(\prod_{m=0}^{j-1}(\alpha-m)\Big)(Ac)^{\alpha-j} + \prod_{m=0}^{k-1}(\alpha-m)\int_{Ac}^\infty e^{-x}x^{\alpha-k}\,dx \\ &\leq e^{-Ac}\sum_{j=0}^{k}\Big(\prod_{m=0}^{j-1}(\alpha-m)\Big)(Ac)^{\alpha-j} \ll e^{-Ac}A^\alpha.\end{aligned}$$

Using this in (7.20) completes the proof of the lemma. □

Using the lemma we obtain from (7.19):

$$\int_0^{c_3} e^{-(s-1)v}(-h_1'(v))\,dv = \frac{\Gamma(\frac{1}{2})}{\sqrt{2}}(s-1)^{-\frac{1}{2}} - \frac{2}{3}\Gamma(1)(s-1)^{-1} + \frac{\Gamma(\frac{3}{2})}{6\sqrt{2}}(s-1)^{-\frac{3}{2}} + O\Big((s-1)^{-2}\Big)$$

(since the error term from $O(\frac{e^{-c_3(s-1)}}{s-1})$, is clearly subsumed by $O((s-1)^{-2})$). Similarly

$$\int_0^{c_3} e^{-(s-1)v}h_2'(v)\,dv = \frac{\Gamma(\frac{1}{2})}{\sqrt{2}}(s-1)^{-\frac{1}{2}} + \frac{2}{3}\Gamma(1)(s-1)^{-1} + \frac{\Gamma(\frac{3}{2})}{6\sqrt{2}}(s-1)^{-\frac{3}{2}} + O\Big((s-1)^{-2}\Big).$$

Adding these together, and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ and the fact that the contribution from $v \geq c_3$ is bounded by (7.17), we conclude:

(7.21)
$$\int_{-1}^{\infty} e^{-(s-1)(u-\log(1+u))}\,du = \sqrt{2\pi}\Big((s-1)^{-\frac{1}{2}} + \frac{1}{12}(s-1)^{-\frac{3}{2}} + O\big((s-1)^{-2}\big)\Big).$$

It is clear from the $\pm$ symmetry that if we keep track of one more term in (7.18) then the $(s-1)^{-2}$ cancels, i.e. we may actually improve the error in (7.21) to $O((s-1)^{-\frac{5}{2}})$. Hence we obtain from (7.14):

(7.22)
$$\Gamma(s) = \sqrt{2\pi}(s-1)^{s-\frac{1}{2}}e^{1-s}\Big(1 + \frac{1}{12}(s-1)^{-1} + O((s-1)^{-2})\Big), \qquad \forall s \geq 2.$$

Clearly by keeping track of more terms in (7.18) we can obtain an asymptotic expansion of arbitrary precision.

Finally let us note that (7.22) agrees with Stirling's formula, Theorem 7.1: We have that (7.22) is equivalent with (keeping $s \geq 2$):

$$\begin{aligned}\log\Gamma(s) &= \log\sqrt{2\pi} + (s-\tfrac{1}{2})\log(s-1) + 1 - s + \log\Big(1 + \tfrac{1}{12}(s-1)^{-1} + O((s-1)^{-2})\Big)\\ &= \log\sqrt{2\pi} + (s-\tfrac{1}{2})\log s + (s-\tfrac{1}{2})\log(1-s^{-1}) + 1 - s + \tfrac{1}{12}(s-1)^{-1} + O((s-1)^{-2})\\ &= \log\sqrt{2\pi} + (s-\tfrac{1}{2})\log s + (s-\tfrac{1}{2})\big(-s^{-1} - \tfrac{1}{2}s^{-2} + O(s^{-3})\big) + 1 - s + \tfrac{1}{12}s^{-1} + O(s^{-2})\\ &= \log\sqrt{2\pi} + (s-\tfrac{1}{2})\log s - s + \tfrac{1}{12}s^{-1} + O(s^{-2})\end{aligned}$$

This agrees with (7.9) for $m = 0$ (but with a worse error term), since $B_2 = \frac{1}{6}$.

7.5. **Appendix: Proof of Corollary 7.10.** By Stirling's formula, Theorem 7.1, we have

(7.23)

$$\log\Gamma(z+\alpha)=\big(z+\alpha-\tfrac12\big)\log(z+\alpha)-(z+\alpha)+\log\sqrt{2\pi}+O\big(|z+\alpha|^{-1}\big),$$

for all z with $|z+\alpha|\geq 1$ and $\big|\arg(z+\alpha)\big|\leq\pi-\varepsilon$. Here and below, for definiteness, we consider the argument function to take its values in $(-\pi,\pi]$, i.e. $\arg:\mathbb{C}\setminus\{0\}\to(-\pi,\pi]$.

Let us fix a constant $C>1$ so large that $\big|\arg(1+w)\big|<\frac12\varepsilon$ for all $w\in\mathbb{C}$ with $|w|\leq C^{-1}$. Then note that if $|z|\geq C|\alpha|$ and $|z|\geq 1$ then $\arg(z+\alpha)=\arg(z(1+\alpha/z))\equiv\arg(z)+\arg(1+\alpha/z)\pmod{2\pi}$ together with $\big|\arg(z+\alpha)\big|\leq\pi-\varepsilon$ and $\big|\arg(1+\alpha/z)\big|<\frac12\varepsilon$ imply that $\big|\arg(z)\big|\leq\pi-\frac12\varepsilon$ and $\arg(z+\alpha)=\arg(z)+\arg(1+\alpha/z)$. Hence

$$\log(z+\alpha)=\log z+\log\Big(1+\frac{\alpha}{z}\Big),$$

where in all three places we use the principal branch of the logarithm function. Since $|\alpha/z|\leq C^{-1}<1$ we can continue:

$$\log(z+\alpha)=\log z+\frac{\alpha}{z}+O\Big(\frac{\alpha^2}{z^2}\Big)=\log z+\frac{\alpha}{z}+O\big(|z|^{-2}\big)$$

(since we allow the implied constant to depend on α). Using this in (7.23) we get

$$\begin{aligned}\log\Gamma(z+\alpha)&=\big(z+\alpha-\tfrac12\big)\Big(\log z+\frac{\alpha}{z}+O\big(|z|^{-2}\big)\Big)-(z+\alpha)+\log\sqrt{2\pi}+O\big(|z+\alpha|^{-1}\big)\\&=\big(z+\alpha-\tfrac12\big)\log z-z+\log\sqrt{2\pi}+O\big(|z|^{-1}\big)+O\big(|z+\alpha|^{-1}\big)\\&=\big(z+\alpha-\tfrac12\big)\log z-z+\log\sqrt{2\pi}+O\big(|z|^{-1}\big),\end{aligned}$$

where in the last step we used the fact that $|z+\alpha|\geq|z|-|\alpha|=|z|\big(1-|\alpha/z|\big)\geq(1-C^{-1})|z|\gg|z|$. Hence we have proved the desired formula for all z satisfying $|z|\geq 1$, $|z+\alpha|\geq 1$, $|\arg(z+\alpha)|\leq\pi-\varepsilon$ and $|z|\geq C|\alpha|$.

It remains to treat z satisfying $|z|\geq 1$, $|z+\alpha|\geq 1$, $|\arg(z+\alpha)|\leq\pi-\varepsilon$ and $|z|\leq C|\alpha|$. This is trivial: These set of such z is *compact* and $\log\Gamma(z+\alpha)-\big(z+\alpha-\frac12\big)\log z+z-\log\sqrt{2\pi}$ is continuous on this set, hence bounded. Also $|z|$ is bounded on the set; hence $|z|^{-1}$ is bounded from below. Hence by adjusting the implied constant we have $\log\Gamma(z+\alpha)-\big(z+\alpha-\frac12\big)\log z+z-\log\sqrt{2\pi}=O\big(|z|^{-1}\big)$ for all z in our compact set, as desired. $\square$

8. Lecture 10: Special functions and asymptotic expansions

In this lecture I did the following:

* Finish discussing the proof of an asymptotic formula for $\Gamma(s)$ as $s \to +\infty$ (Sec. 7.4 above).

* Discuss briefly what we mean by an *asymptotic expansion* in general (e.g. the expansion in (7.9)) and why it is useful.

* Say a few words about the most basic and standard asymptotic expansion of all: The *Taylor expansion* of a function, both in one and several variables.

9. Lecture 11: Special functions and asymptotic expansions

In this lecture I introduced the J-Bessel function.

9.1. **The J-Bessel function.**

The J-Bessel function can be defined by the following Taylor series expansion around $z = 0$:

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{m=0}^\infty \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\Big(\frac{z}{2}\Big)^{2m}. \tag{9.1}$$

This definition works for any $\nu \in \mathbb{C}$ and any $z \in \mathbb{C} \setminus (-\infty, 0]$, say, and for fixed ν we see that $J_\nu(z)$ is an analytic function of $z \in \mathbb{C} \setminus (-\infty, 0]$. ($J_\nu(z)$ is also jointly analytic in the variables ν, z.) Note that if ν happens to be a nonnegative integer then $J_\nu(z)$ is in fact an *entire* function, i.e. an analytic function in $z \in \mathbb{C}$.

For given $\nu \in \mathbb{C}$, the function $z \mapsto J_\nu(z)$ is a solution of the so called *Bessel differential equation*,

$$f''(z) + \frac{1}{z} f'(z) + \Big(1 - \frac{\nu^2}{z^2}\Big) f(z) = 0. \tag{9.2}$$

The solution (9.1) of (9.2) is easily obtained using the Frobenius-Fuchs method (cf. e.g., mathworld), that is one makes the Ansatz $f(z) = z^\nu \sum_{n=0}^\infty a_n z^n$ and seeks possible values of ν and $a_0, a_1, \ldots \in \mathbb{C}$ ($a_0 \neq 0$) which make f solve (9.2). Note that also $J_{-\nu}(z)$ is a solution to (9.2) (since the equation (9.2) remains unchanged when replacing ν by $-\nu$) and in fact if $\nu \notin \mathbb{Z}$ then $\{J_\nu(z), J_{-\nu}(z)\}$ form a *fundamental system* of solutions, i.e. any solution to (9.2) can be expressed as a linear combination of these two. In the (important!) special case $\nu = n \in \mathbb{Z}$ however, we have

$$J_{-n}(z) = (-1)^n J_n(z) \qquad (n \in \mathbb{Z})$$

(verify this directly from (9.1)!) and another function is needed to obtain a fundamental system of solutions to (9.2).

Let us record some basic recurrence relations for the Bessel functions, both of which can be proved directly from (9.1):

$$J_{\nu-1}(z) + J_{\nu+1}(z) = (2\nu/z)J_\nu(z); \tag{9.3}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z). \tag{9.4}$$

From these we also deduce:

$$J_{\nu+1}(z) = \frac{\nu}{z}J_\nu(z) - J_\nu'(z); \tag{9.5}$$

$$J_{\nu-1}(z) = \frac{\nu}{z}J_\nu(z) + J_\nu'(z). \tag{9.6}$$

An alternative formula which can be taken as the definition of the J-Bessel function when $\Re\nu > -\frac{1}{2}$ is the following (cf. [5, 8.411.10]):

$$\begin{aligned} J_\nu(z) &= \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt \\ &= \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-\pi/2}^{\pi/2} e^{iz\sin\theta}(\cos\theta)^{2\nu}\,d\theta. \end{aligned} \tag{9.7}$$

Proof of (9.7) *using* (9.1). The second formula follows from the first by substituting $t = \sin\theta$; hence it now suffices to prove the first formula. Using $e^{izt} = \sum_{n=0}^{\infty}\frac{(izt)^n}{n!}$ (which is true for all z, t) we have

$$\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt = \int_{-1}^{1}\sum_{n=0}^{\infty}\frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

Here we may change order of summation and integration, since

$$\begin{aligned} \int_{-1}^{1}\sum_{n=0}^{\infty}\Big|\frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\Big|\,dt &= \int_{-1}^{1}\sum_{n=0}^{\infty}\frac{|izt|^n}{n!}(1-t^2)^{\Re\nu-\frac{1}{2}}\,dt \\ = \int_{-1}^{1} e^{|zt|}(1-t^2)^{\Re\nu-\frac{1}{2}}\,dt &\le e^{|z|}\int_{-1}^{1}(1-t^2)^{\Re\nu-\frac{1}{2}}\,dt < \infty, \end{aligned}$$

where the last step holds since $\Re\nu > -\frac{1}{2}$. Hence

$$\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt = \sum_{n=0}^{\infty}\int_{-1}^{1}\frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

But here for each odd n the integrand is an odd function of t and hence the integral vanishes. Thus we may continue:

$$= \sum_{m=0}^{\infty}\int_{-1}^{1}\frac{(izt)^{2m}}{(2m)!}(1-t^2)^{\nu-\frac{1}{2}}\,dt = \sum_{m=0}^{\infty}\frac{(-1)^m z^{2m}}{(2m)!}\cdot 2\int_0^1 t^{2m}(1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

Substituting $t = \sqrt{u}$ in the integral and then using (7.7) and (7.6) we get:

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\Gamma(2m+1)} \cdot \int_0^1 u^{m-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}\, du \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\pi^{-\frac{1}{2}} 2^{2m} \Gamma(m+\frac{1}{2})\Gamma(m+1)} \frac{\Gamma(m+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(m+\nu+1)} \\
&= \sqrt{\pi}\, \Gamma(\nu+\tfrac{1}{2}) \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m}}{m!\, \Gamma(m+\nu+1)}.
\end{aligned}
$$

Hence, comparing with (9.1) and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have proved (9.7). $\square$

For $\nu = n \in \mathbb{Z}_{\geq 0}$ we have the following alternative integral formula for $J_n(z)$ (cf. [5, 8.411.1]):

$$
J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin\theta - in\theta}\, d\theta. \tag{9.8}
$$

(This looks quite similar to the second formula in (9.7) but I don't know of any easy direct way to derive either from the other!)

Proof of (9.8). Let $n \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{C}$ be given. We have $e^{iz\sin\theta} = \sum_{k=0}^{\infty} \frac{(iz\sin\theta)^k}{k!}$, with absolute convergence uniformly over $\theta \in [-\pi, \pi]$; hence

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin\theta - in\theta}\, d\theta = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sin\theta)^k e^{-in\theta}\, d\theta.
$$

Here use $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ to get $(\sin\theta)^k = (2i)^{-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{(k-2j)i\theta}$. From this we see that

$$
\frac{1}{2\pi} \int_0^{2\pi} (\sin\theta)^k e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{if } k < n \text{ or } k \not\equiv n \bmod 2; \\ (2i)^{-k} (-1)^{\frac{1}{2}(k-n)} \binom{k}{(k-n)/2} & \text{if } k \geq n \text{ and } n \equiv k \bmod 2. \end{cases}
$$

Writing $k = n + 2m$ we thus get

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin\theta - in\theta}\, d\theta &= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{n+2m}}{(n+2m)!} \binom{n+2m}{m} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{n+2m}}{m!(n+m)!} = J_n(z),
\end{aligned}
$$

where the last equality holds because of (9.1). $\square$

In fact we have already encountered the J-Bessel function: In problem 8 of Assignment 1 we considered the Fourier transform of the characteristic function of the unit ball B_1^n in $\mathbb{R}^n$:

$$\begin{aligned}\widehat{\chi}_{B_1^n}(\xi) &= \frac{\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}(n+1))}\int_{-1}^{1}(1-x^2)^{\frac{1}{2}(n-1)}\,e^{-2\pi i|\xi|x}\,dx\\ &= |\xi|^{-n/2}J_{n/2}(2\pi|\xi|).\end{aligned} \tag{9.9}$$

(Here $\frac{\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}(n+1))}$ is the volume of the $(n-1)$-dimensional unit ball.) The last identity of (9.9) is a special case of (9.7)!

Similarly, the Fourier transform of the surface measure σ on the unit sphere S_1^{n-1} (viewed as a Borel measure on $\mathbb{R}^n$ in the usual way, i.e. $\sigma(E)=\sigma(E\cap \mathrm{S}_1^{n-1})$ for any Borel subset $E\subset\mathbb{R}^n$) is also given by the J-Bessel function (cf. Folland's Exercise 22, p. 256):

$$\begin{aligned}\widehat{\sigma}(\xi) &= \int_{\mathrm{S}_1^{n-1}} e^{-2\pi i\xi\cdot\omega}\,d\sigma(\omega) = \int_{\mathrm{S}_1^{n-1}} e^{-2\pi i|\xi|\omega_1}\,d\sigma(\omega)\\ &= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_{-1}^{1} e^{-2\pi i|\xi|\omega_1}(1-\omega_1^2)^{\frac{n-3}{2}}\,d\omega_1 = 2\pi|\xi|^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(2\pi|\xi|),\end{aligned}$$

where we first used the rotational symmetry, then used (6.2), and finally used (9.7). It follows that if $F\in L^1(\mathbb{R}^n)$ is *any* radial function, i.e. a function such that $F(\xi)$ only depends on $|\xi|$; say $F(\xi)=f(|\xi|)$, then

$$\begin{aligned}\widehat{F}(\xi) &= \int_{\mathbb{R}^n} F(x)e^{-2\pi i\xi\cdot x}\,dx = \int_0^\infty f(\rho)\int_{\mathrm{S}_1^{n-1}} e^{-2\pi i\xi\cdot\rho\omega}\,d\sigma(\omega)\,\rho^{n-1}\,d\rho\\ &= \int_0^\infty f(\rho)\widehat{\sigma}(\rho\xi)\rho^{n-1}\,d\rho = 2\pi|\xi|^{1-\frac{n}{2}}\int_0^\infty f(\rho)\rho^{\frac{n}{2}}J_{\frac{n}{2}-1}(2\pi\rho|\xi|)\,d\rho.\end{aligned} \tag{9.10}$$

Remark 9.1. The fact that (9.9) is obtained as a special case of (9.10) when taking $f=\chi_{[0,1]}$, is easily seen to be equivalent with the following identity (writing $Y=2\pi|\xi|\geq 0$)

$$Y^{\frac{n}{2}}J_{\frac{n}{2}}(Y) = \int_0^Y u^{\frac{n}{2}}J_{\frac{n}{2}-1}(u)\,du.$$

This identity follows from the fact that $Y^{\frac{n}{2}}J_{\frac{n}{2}}(Y)$ equals 0 at $Y=0$, and $\frac{d}{dY}(Y^{\frac{n}{2}}J_{\frac{n}{2}}(Y)) = Y^{\frac{n}{2}}(\frac{n/2}{Y}J_{\frac{n}{2}}(Y)+J'_{\frac{n}{2}}(Y)) = Y^{\frac{n}{2}}J_{\frac{n}{2}-1}(Y)$, cf. (9.6).

Remark 9.2. Formula (9.10) gives an explicit expression for the Fourier transform of a radial function (which is again a radial function). Applying now *Fourier inversion*, we conclude that for any "nice" $f:\mathbb{R}_{\geq 0}\to\mathbb{C}$

[18]

$$\widetilde{f}(r) = 2\pi r^{1-\frac{n}{2}} \int_0^\infty f(\rho)\rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi r\rho)\, d\rho,$$

then f can be recovered by applying exactly the same integral transform once more, i.e.:

$$f(\rho) = 2\pi \rho^{1-\frac{n}{2}} \int_0^\infty \widetilde{f}(r) r^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi \rho r)\, dr, \tag{9.11}$$

This can be seen as a special case of the so called *Hankel transform:* For any fixed $\nu \geq -\frac{1}{2}$, the Hankel transform of a function $g : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is defined by

$$\mathcal{H}_\nu g(r) = \int_0^\infty g(\rho)\rho J_\nu(r\rho)\, d\rho,$$

and the Hankel inversion formula says that for any “nice” g we have $\mathcal{H}_\nu \mathcal{H}_\nu g = g$. We note that the inversion formula (9.11) follows from $\mathcal{H}_\nu \mathcal{H}_\nu g = g$ applied with $\nu = \frac{n}{2} - 1$ and $g(\rho) = f(\rho)\rho^{\frac{n}{2}-1}$. Indeed, then $\widetilde{f}(r) = 2\pi r^{1-\frac{n}{2}}[\mathcal{H}_\nu g](2\pi r)$, and therefore

$$\begin{aligned}
&2\pi \rho^{1-\frac{n}{2}} \int_0^\infty \widetilde{f}(r) r^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi \rho r)\, dr \\
&= (2\pi)^2 \rho^{1-\frac{n}{2}} \int_0^\infty [\mathcal{H}_\nu g](2\pi r) r J_{\frac{n}{2}-1}(2\pi\rho r)\, dr \\
&= \rho^{1-\frac{n}{2}} \int_0^\infty [\mathcal{H}_\nu g](r) r J_{\frac{n}{2}-1}(\rho r)\, dr \\
&= \rho^{1-\frac{n}{2}} [\mathcal{H}_\nu \mathcal{H}_\nu g](\rho) = \rho^{1-\frac{n}{2}} g(\rho) = f(\rho).
\end{aligned}$$

[18]More precisely, for any $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ such that both $\int_0^\infty |f(\rho)|\rho^{n-1}\, d\rho < \infty$ and $\int_0^\infty |\widetilde{f}(\rho)|\rho^{n-1}\, d\rho < \infty$.

9.2. **The Dirichlet eigenvalues in a disk.** In this section we will see how the J-Bessel function shows up when seeking the *Dirichlet eigenfunctions and eigenvalues in a disk.* Specifically, let Ω be the disk $\Omega = B_a^2 = \{|x| < a\}$ in $\mathbb{R}^2$, and consider the following PDE (for real-valued $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $\lambda \geq 0$):

$$\begin{cases} \Delta u + \lambda u = 0 \quad \text{in all } \Omega \\ u_{|\partial\Omega} = 0. \end{cases} \tag{9.12}$$

The first equation says that u is an eigenvalue of the Laplace operator $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ with eigenvalue $-\lambda$; the second equation says that u should satisfy the Dirichlet boundary conditions, i.e. u should vanish along the boundary of Ω.

Physically, the eigenvalues λ of the above problem corresponds to the eigenfrequencies of vibration of an idealized circular "drum" of radius a in the plane; and the eigenfunctions u give the corresponding "vibration patterns". Note also that solving problem (9.12) is a first step in solving e.g. the heat or wave equation in a *cylinder* domain, using separation of variables.

Recall that using Green's formula one easily sees that all eigenvalues λ to the above problem (as well as for the corresponding Dirichlet eigenvalue problem in any nice domain in $\mathbb{R}^n$) are *positive.* Hence in the following investigation we will assume $\lambda > 0$ from start.

Let us now try to solve (9.12) (i.e. to find all solution pairs u, λ) by expressing u in polar coordinates and separating variables.[19] Thus we write (by slight abuse of notation) $u(r, \theta)$ for the value of u at the points $(r\cos\theta, r\sin\theta) \in \mathbb{R}^2$. Recalling that the Laplacian in polar coordinates is given by $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$, we see that the task is to solve:

$$\begin{cases} \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u + \lambda u = 0, & 0 < r < a,\ 0 \leq \theta \leq 2\pi, \\ u(a, \theta) = 0, & 0 \leq \theta \leq 2\pi. \end{cases}$$

Separating the variables r and θ means making the Ansatz that u is of the form $u(r, \theta) = R(r)\Phi(\theta)$. Then we get:

$$\begin{cases} \Big(R''(r) + \frac{1}{r}R'(r) + \lambda R(r)\Big)\phi(\theta) = -\frac{1}{r^2}R(r)\phi''(\theta); \\ R(a) = 0; \\ \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi). \end{cases}$$

If the first equation has a non-vanishing solution then there must exist a constant $\mu \in \mathbb{R}$ such that

$$R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = \frac{\mu}{r^2}R(r), \qquad 0 < r < a,$$

[19] We here follow Pinchover and Rubinstein, [17, Sec. 9.5.3], to which we refer for more details. Our exposition will be slightly sloppy regarding certain details.

and

$$\phi''(\theta) = -\mu\phi(\theta), \qquad \forall\theta \in [0, 2\pi].$$

This last equation has the general solution

$$\phi(\theta) = \begin{cases} Ae^{\sqrt{-\mu}\theta} + Be^{-\sqrt{-\mu}\theta} & \text{if } \mu < 0 \\ A + B\theta & \text{if } \mu = 0 \\ A\cos(\sqrt{\mu}\theta) + B\cos(\sqrt{\mu}\theta) & \text{if } \mu > 0. \end{cases}$$

One easily checks that if $\mu < 0$ then the there is no choice of A, B other than $A = B = 0$ which makes the boundary conditions $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$ hold; if $\mu = 0$ then the boundary conditions are satisfied iff $B = 0$ (i.e. ϕ is a constant function), and if $\mu > 0$ then the boundary conditions are satisfied iff $\sqrt{\mu} \in \mathbb{N}$. Thus we may – incorporating also the case $\mu = 0$ – from now on write $\mu = \mu_n = n^2$ ($n \in \mathbb{Z}_{\geq 0}$) and the general ϕ-solution is

$$\phi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \qquad (A_n, B_n \in \mathbb{R}),$$

where B_0 is "redundant" since anyway $\sin(0 \cdot \theta) \equiv 0$. The equation for $R(r)$ now reads:

$$R''(r) + \frac{1}{r}R'(r) + \Big(\lambda - \frac{n^2}{r^2}\Big)R(r) = 0, \qquad 0 < r < a,$$

Applying the change of variables $s = \sqrt{\lambda}r$, i.e. writing

$$\psi(s) = R(s/\sqrt{\lambda}),$$

the equation takes the form

$$\psi''(s) + \frac{1}{s}\psi'(s) + \Big(1 - \frac{n^2}{s^2}\Big)\psi(s) = 0, \qquad 0 < s < \sqrt{\lambda}a,$$

and bdry conditions $\lim_{s\to 0}\psi(s)$ should exist and be finite and $\psi(\sqrt{\lambda}a) = 0$. This is the *Bessel differential equation!* For the present case, i.e. order $= n$, it can be shown that the only solution to this equation for which $\lim_{s\to 0^+}\psi(s)$ exists and is finite, is $\psi(s) = J_n(s)$ (up to multiplication with a constant). Remember that we also have the boundary condition $J_n(\sqrt{\lambda}a) = \psi(\sqrt{\lambda}a) = R(a) = 0$. That is, $\sqrt{\lambda}a$ must be a zero of $J_n(z)$. Let us write $0 < j_{n,1} < j_{n,2} < \ldots$ for the full set of positive zeros of $J_n(z)$. Then we conclude: *The general solution $\langle\lambda, u\rangle$ of our Dirichlet eigenvalue problem* (9.12) *with u being of the form $u(r, \theta) = R(r)\Phi(\theta)$, is:*

$$\lambda = (j_{n,m}/a)^2, \qquad u_{n,m}(r, \theta) = J_n\Big(\frac{j_{n,m}}{a}r\Big)\big(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta\big),$$

where $\langle n, m\rangle$ runs through $\mathbb{Z}_{\geq 0} \times \mathbb{N}$. (Recall that each $B_{0,m}$ is "redundant".)

10. Lecture 12: Special functions and asymptotic expansions

In this lecture I plan to:

* Discuss the method of stationary phase (treating oscillatory integrals).

* Discuss *uniform* asymptotic expansions, by presenting an example, a uniform asymptotic formula for the J-Bessel function.

10.1. The Method of Stationary phase: heuristic discussion.

In this section we discuss Kelvin's *method of stationary phase* (also called the *method of critical points*), following Olver [16, Sec. 3.11-13].

Consider the integrals

$$\int_a^b \cos(xp(t))q(t)\,dt, \qquad \int_a^b \sin(xp(t))q(t)\,dt,$$

in which $a, b, p(t)$ and $q(t)$ are independent of the parameter x. For large x (i.e. $x \to +\infty$) the integrands oscillate rapidly and cancel themselves over most of the range. Cancellation does not occur, however, in the neighborhoods of the following points: (i) the endpoints a and b (when finite), owing to lack of symmetry; (ii) zeros of $p'(t)$, because $p(t)$ changes relatively slowly near these. The zeros of $p'(t)$ are called the "stationary points" for the above integrals.

Both integrals are covered simultaneously by combining them into

$$I(x) = \int_a^b e^{ixp(t)}q(t)\,dt. \tag{10.1}$$

Let us first discuss *heuristically* what the contributions from the points in (i) and (ii) above should be!

First consider the endpoints: In the neighborhood of $t = a$ the integrand in (10.1) is approximately

$$\exp\Big(ix\big(p(a) + (t-a)p'(a)\big)\Big)q(a).$$

Let us assume $p'(a) \neq 0$; then a primitive function to the last function is:

$$\frac{\exp\Big(ix\big(p(a) + (t-a)p'(a)\big)\Big)q(a)}{ixp'(a)}.$$

(Of course we can add any constant to this; however the above is the unique primitive function whose "long-time average" is zero.) Using this suggests that the contribution from the lower endpoint in (10.1) should be

$$\approx -\frac{e^{ixp(a)}q(a)}{ixp'(a)}. \tag{10.2}$$

Similarly the contribution from the upper endpoint in (10.1) should be (if $p'(b) \neq 0$)

$$\approx \frac{e^{ixp(b)}q(b)}{ixp'(b)}. \tag{10.3}$$

Next we consider any stationary point, of the integral in (10.1): Thus assume that $t_0 \in (a, b)$ is a stationary point of $p(t)$, i.e. $p'(t_0) = 0$. Assume that $p''(t_0) \neq 0$ and $q(t_0) \neq 0$. Then for t near t_0 the integrand in (10.1) is approximately

$$e^{ix(p(t_0)+\frac{1}{2}p''(t_0)(t-t_0)^2)}q(t_0).$$

Let us pursue our belief that only the neighborhood of t_0 matters; then we may just as well extend the limits of integration to $-\infty$ and $+\infty$, and so conclude that the contribution from our stationary point should be approximately

$$\int_{-\infty}^{\infty} e^{ix(p(t_0)+\frac{1}{2}p''(t_0)(t-t_0)^2)}q(t_0)\,dt = \frac{q(t_0)e^{ixp(t_0)}\sqrt{2}}{\sqrt{x|p''(t_0)|}} \int_{-\infty}^{\infty} e^{\operatorname{sgn}(p''(t_0))iu^2}\,du$$

(we substituted $t = t_0 + (\frac{2}{x|p''(t_0)|})^{1/2}u$). However the integral $\int_{-\infty}^{\infty} e^{\pm iu^2}\,du$ can be computed explicitly: $\int_{-\infty}^{\infty} e^{\pm iu^2}\,du = e^{\pm\pi i/4}\sqrt{\pi}$ (proof: split into $u < 0$ and $u > 0$; substitute $u = \pm\sqrt{v}$, and use Lemma 10.1 below). Hence we get:

$$= q(t_0)\sqrt{\frac{2\pi}{|xp''(t_0)|}}e^{ixp(t_0)}e^{\operatorname{sgn}(p''(t_0))\pi i/4}. \tag{10.4}$$

It should be noted that (10.4) is $\asymp x^{-1/2}$ but (10.2) and (10.3) are $\asymp x^{-1}$, i.e. the contribution from the stationary point(s) is of larger order of magnitude than the contribution from the endpoints as $x \to \infty$.

In a similar way one can get heuristic expressions for the contribution from stationary points of higher order, i.e. in cases when $p'(t_0) = p''(t_0) = 0$, as well as certain cases when $q(t_0) = 0$. The approximate value of the integral $I(x)$ for large x is obtained by summing expressions of the form (10.4) over the various stationary points in the range of integration and adding the contributions (10.2) and (10.3) from the endpoints. Of course our discussion so far has been heuristic, but we shall see below how to obtain rigorous results using the ideas just outlined.

10.2. **The Method of Stationary Phase: rigorous discussion.** In this section we give some useful lemmas and then some examples of how the method of Stationary Phase can be used to derive asymptotic formulas. We continue to follow Olver [16, Sec. 3.11-13] rather closely. The key step is to substitute $v = p(t)$ in the integral! (This is the same idea that we used in (7.16).)

First, let us note that the case in which stationary points are absent is an exercise in integration by parts. Indeed, to be specific assume that a and b are finite, $p \in C^2([a,b])$, $q \in C^1([a,b])$, and $p'(t) \neq 0$ for all $t \in [a,b]$. Then either $p'(t) > 0$ for all $t \in [a,b]$ or else $p'(t) < 0$ for all $t \in [a,b]$; in particular $p(t)$ is either strictly increasing or strictly decreasing, and hence we may take $v = p(t)$ as new integration variable, giving

$$I(x) = \int_{p(a)}^{p(b)} e^{ixv} f(v)\, dv, \qquad \text{where } f(v) = \frac{q(t)}{p'(t)}.$$

Note that our assumptions imply $f \in C^1([a,b])$; hence we may integrate by parts to get

$$= \Big[\frac{e^{ixv}}{ix} f(v)\Big]_{v=p(a)}^{v=p(b)} - \int_{p(a)}^{p(b)} \frac{e^{ixv}}{ix} f'(v)\, dv$$
$$= \frac{ie^{ixp(a)}q(a)}{xp'(a)} - \frac{ie^{ixp(b)}q(b)}{xp'(b)} + \frac{i}{x}\int_{p(a)}^{p(b)} e^{ixv} f'(v)\, dv.$$

But by the Riemann-Lebesgue lemma the integral $\int_{p(a)}^{p(b)} e^{ixv} f'(v)\, dv$ tends to zero as $x \to \pm\infty$. Hence we conclude

$$I(x) = \frac{ie^{ixp(a)}q(a)}{xp'(a)} - \frac{ie^{ixp(b)}q(b)}{xp'(b)} + o(x^{-1}) \qquad \textit{as } x \to \infty.$$

Note that if $f \in C^2([a,b])$ then we may even integrate by parts once more and get the better error term $O(x^{-2})$; and if we assume even more smoothness we may even get an asymptotic expansion. (Cf. Olver [16, Sec. 3.5].)

Next we give a couple of lemmas which are useful for the treatment of the contributions from stationary points.

Lemma 10.1. *For any* $0 < \alpha < 1$ *and* $x \in \mathbb{R} \setminus \{0\}$,

$$\int_0^\infty e^{ixv} v^{\alpha-1}\, dv = e^{\mathrm{sgn}(x)\alpha\frac{\pi}{2}i}\Gamma(\alpha)|x|^{-\alpha}.$$

Proof. (Sketch.) We use the fact that the integrand is an analytic function of $v \in \mathbb{C} \setminus (-\infty, 0]$. If $x > 0$ then we change contour from the positive real axis to the positive imaginary axis. (To do this rigorously one takes some $0 < r < R$ (r small, R large) and applies Cauchy's integral theorem for the contour $C = C_1 + C_2 + C_3 + C_4$, where C_1 is the line segment going from r to R along the real axis, C_2 is the arc in the circle $\{|z| = R\}$ which goes from R to iR; C_3 is the line segment going from iR to ir, and C_4 is the arc in the circle $\{|z| = r\}$ which goes from ir to r. One proves that the integrals along C_2 and along C_4 tend to 0 as $r \to 0$ and $R \to \infty$, and the conclusion is that $\int_{C_1} e^{ixv} v^{\alpha-1}\, dv$ and $\int_{-C_3} e^{ixv} v^{\alpha-1}\, dv$ have the same limits as $r \to 0$, $R \to \infty$.) Writing $v = it$ and taking t as new variable of integration, this gives:

$$\int_0^\infty e^{ixv} v^{\alpha-1}\, dv = \int_0^\infty e^{-xt} (it)^{\alpha-1}\, i\, dt = e^{\alpha\frac{\pi}{2}i} \int_0^\infty e^{-xt} t^{\alpha-1}\, dt$$
$$= e^{\alpha\frac{\pi}{2}i} x^{-\alpha} \int_0^\infty e^{-s} s^{\alpha-1}\, dt = e^{\alpha\frac{\pi}{2}i} \Gamma(\alpha) x^{-\alpha}.$$

If $x < 0$ then we instead change the contour from the positive real axis to the negative imaginary axis. □

Lemma 10.2. *For any $\alpha < 1$, $\kappa > 0$ and $x > 0$ we have*

$$\left| \int_\kappa^\infty e^{ixv} v^{\alpha-1}\, dv \right| \leq \frac{2\kappa^{\alpha-1}}{x}.$$

(The main point for us will be that the above is $O_{\kappa,\alpha}(x^{-1})$.)

Proof. By integration by parts,

$$\int_\kappa^\infty e^{ixv} v^{\alpha-1}\, dv = \left[\frac{e^{ixv}}{ix} v^{\alpha-1} \right]_{v=\kappa}^{v=\infty} - \int_\kappa^\infty \frac{e^{ixv}}{ix} (\alpha - 1) v^{\alpha-2}\, dv$$
$$= -\frac{e^{ix\kappa}}{ix} \kappa^{\alpha-1} - \frac{\alpha - 1}{ix} \int_\kappa^\infty e^{ixv} v^{\alpha-2}\, dv.$$

Hence

$$\left| \int_\kappa^\infty e^{ixv} v^{\alpha-1}\, dv \right| \leq \frac{\kappa^{\alpha-1}}{x} + \frac{1 - \alpha}{x} \int_\kappa^\infty v^{\alpha-2}\, dv = \frac{2\kappa^{\alpha-1}}{x}.$$

□

Example 10.1. Let us seek an asymptotic formula for $J_\nu(x)$ for fixed $\nu \in \mathbb{C}$ with $\Re\nu > -\frac{1}{2}$, as $x \to +\infty$, using the formula (9.7),

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{izt} (1 - t^2)^{\nu-\frac{1}{2}}\, dt.$$

We see that the only critical points are the end-points of integration; hence the determination of the asymptotics boils down to the integration by parts procedure given in the solution to Assignment 1, problem 8 (we typically need to integrate by parts several times due to the high-order vanishing of $(1 - t^2)^{\nu-\frac{1}{2}}$ at $t = \pm 1$ when $\Re\nu$ is large). In fact the solution given there can be refined (cf., [3, p. 97, Exercise 3.15]) to yield an *asymptotic expansion* of $J_\nu(x)$): For any fixed ν as above and any fixed $m \in \mathbb{Z}_{\geq 0}$, we have for all $x \geq 1$:

$$J_\nu(x) = \Big(\frac{2}{\pi x}\Big)^{1/2}\Big\{\cos\big(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\big)\sum_{j=0}^{m}(-1)^j\frac{A_{2j}(\nu)}{x^{2j}}$$

$$-\sin\big(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\big)\sum_{j=0}^{m}(-1)^j\frac{A_{2j+1}(\nu)}{x^{2j+1}} + O(x^{-2m-2})\Big\}, \tag{10.5}$$

where

$$A_j(\nu) = \frac{(4\nu^2-1)(4\nu^2-3)\cdots(4\nu^2-(2j-1)^2)}{j!8^j}. \tag{10.6}$$

(In particular $A_0(\nu) = 1$ and $A_1(\nu) = \frac{1}{2}\nu^2 - \frac{1}{8}$.)

Example 10.2. Let us seek an asymptotic formula for $J_n(x)$ for fixed $n \in \mathbb{Z}_{\geq 0}$ and $x \to +\infty$, using the formula (9.8),

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin t}e^{-int}\,dt \qquad (n \in \mathbb{Z}_{\geq 0},\ x > 0).$$

(Of course the result which we will obtain will be subsumed by the result in Example 10.1; however it is a good illustration of stationary phase.) Note that the integrand gets conjugated when replacing t by $2\pi - t$; hence we have

$$J_n(x) = \Re\big(I(n,x)\big)$$

where

$$I(w,x) := \frac{1}{\pi}\int_0^{\pi} e^{ix\sin t - iwt}\,dt, \qquad w, x \in \mathbb{R}.$$

We will analyse the integral $I(w,x)$ for a general fixed $w \in \mathbb{R}$, as $x \to +\infty$. [Caution: For general $w \in \mathbb{R}$, $\Re(I(w,x))$ does not give the J-Bessel function $J_w(x)$ but rather the so called Anger function, $\mathbf{J}_w(x)$. Cf. [16, p. 103].]

Thus $p(t) = \sin t$ and $q(t) = e^{-iwt}$ in the previous discussion. The unique stationary point for $t \in [0,\pi]$ is at $t = \frac{\pi}{2}$. We "know" from the heuristic discussion that stationary points contribute more than the endpoints, hence let's in the first place focus our attention to t near $\frac{\pi}{2}$. Thus we split the integration into $[0,\frac{\pi}{2}]$ and $[\frac{\pi}{2},\pi]$ and in each interval we take $v = \sin t$ as a new variable of integration. In fact to make the computations slightly more normalized we take $v = 1 - \sin t$. Thus we write

$$I(w,x) = I_1(w,x) + I_2(w,x)$$

where

$$I_1(w,x) = \frac{1}{\pi}\int_0^{\pi/2} e^{ix\sin t - iwt}\,dt; \qquad I_2(w,x) = \frac{1}{\pi}\int_{\pi/2}^{\pi} e^{ix\sin t - iwt}\,dt.$$

Let's first treat $I_1(w,x)$. Substituting $v = 1 - \sin t$ we have $t = \arcsin(1-v)$, and

$$I_1(w,x) = \frac{1}{\pi}\int_0^1 e^{ix(1-v)-iw\arcsin(1-v)}\,\frac{dv}{\sqrt{2v-v^2}}.$$
$$= \frac{e^{ix-iw\frac{\pi}{2}}}{\pi}\bigg(\int_0^1 e^{-ixv}\,\frac{dv}{\sqrt{2v}} + \int_0^1 e^{-ixv}\,\phi(v)\,dv\bigg),$$

where

$$\phi(v) = \frac{e^{iw(\frac{\pi}{2}-\arcsin(1-v))}}{\sqrt{2v-v^2}} - \frac{1}{\sqrt{2v}}.$$

Here by Lemma 10.1 and Lemma 10.2 we have ($\forall x \geq 1$)

$$\int_0^1 e^{-ixv}\,\frac{dv}{\sqrt{2v}} = \int_0^\infty e^{-ixv}\,\frac{dv}{\sqrt{2v}} - \int_1^\infty e^{-ixv}\,\frac{dv}{\sqrt{2v}} = \sqrt{\frac{\pi}{2}}\frac{e^{-\pi i/4}}{\sqrt{x}} + O(x^{-1}).$$

In order to treat the remaining integral $\int_0^1 e^{-ixv}\,\phi(v)\,dv$ we need to understand the behavior of $\phi(v)$ and $\phi'(v)$, especially in the limit $v \to 0$. We compute that, for all $v \in (0,1]$ (and allowing the implied constant in the big-O-estimates to depend on w):

$$\phi(v) = \frac{e^{iw(\frac{\pi}{2}-\arcsin(1-v))}}{\sqrt{2v-v^2}} - \frac{1}{\sqrt{2v}} = \frac{1}{\sqrt{2v}}\Big(e^{O(\sqrt{v})}\big(1-\tfrac{1}{2}v\big)^{-\frac{1}{2}} - 1\Big)$$
$$= \frac{1}{\sqrt{2v}}\Big((1+O(\sqrt{v}))(1+O(v)) - 1\Big) = \frac{1}{\sqrt{2v}}\cdot O(\sqrt{v}) = O(1),$$

and

$$\phi'(v) = -2^{-\frac{3}{2}}v^{-\frac{3}{2}}\Big(e^{iw(\frac{\pi}{2}-\arcsin(1-v))}\big(1-\tfrac{1}{2}v\big)^{-\frac{1}{2}} - 1\Big)$$
$$+\frac{1}{\sqrt{2v}}\bigg(\frac{iwe^{-iw(\frac{\pi}{2}-\arcsin(1-v))}}{\sqrt{2v}(1-\frac{1}{2}v)} + \frac{e^{iw(\frac{\pi}{2}-\arcsin(1-v))}}{4(1-\frac{1}{2}v)^{3/2}}\bigg)$$
$$= v^{-\frac{3}{2}}\cdot O(\sqrt{v}) + O\Big(v^{-\frac{1}{2}}\big(v^{-\frac{1}{2}}+1\big)\Big) = O(v^{-1}).$$

Hence we see that we cannot integrate by parts all the way down to $v = 0$ in $\int_0^1 e^{-ixv}\,\phi(v)\,dv$, since $\phi'(v)$ grows too quickly. Instead let's break up the integral at some point $a \in (0,1]$ which we'll fix later:

$$\int_0^1 e^{-ixv}\phi(v)\,dv = \int_0^a e^{-ixv}\phi(v)\,dv + \bigg[\frac{e^{-ixv}}{-ix}\phi(v)\bigg]_a^1 - \int_a^1 \frac{e^{ixv}}{ix}\phi'(v)\,dv$$
$$= \int_0^a O(1)\,dv + O(x^{-1}) + O\Big(x^{-1}\int_a^1 v^{-1}\,dv\Big)$$
$$= O\Big(a + x^{-1} + x^{-1}\log(a^{-1})\Big).$$

Now choose a optimally by taking $a = x^{-1}$; this satisfies the requirement $a \in (0,1]$ as long as we assume $x \geq 1$; we then get that the above is $O(x^{-1}(1+\log x))$. Let us from now on assume $x \geq 2$; then the last error can be written in the simpler form $O(x^{-1}\log x)$, and we have now proved:

$$I_1(w,x)=\sqrt{\frac{1}{2\pi x}}e^{i(x-\frac{\pi}{2}w-\frac{\pi}{4})}+O(x^{-1}\log x),\qquad \forall x\geq 2.$$

The treatment of $I_2(w,x)$ is completely similar, and gives the same result:

$$I_2(w,x)=\sqrt{\frac{1}{2\pi x}}e^{i(x-\frac{\pi}{2}w-\frac{\pi}{4})}+O(x^{-1}\log x),\qquad \forall x\geq 2.$$

Adding up we conclude:

$$I(w,x)=\sqrt{\frac{2}{\pi x}}e^{i(x-\frac{\pi}{2}w-\frac{\pi}{4})}+O(x^{-1}\log x),\qquad \forall x\geq 2.$$

(Recall that we allow the implied constant to depend on w.)

In particular we obtain for the J-Bessel function:

(10.7)

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\cos\big(x-\tfrac{\pi}{2}n-\tfrac{\pi}{4}\big)+O_n(x^{-1}\log x),\qquad \forall n\in\mathbb{Z}_{\geq 0},\ x\geq 2.$$

Example 10.3. Let us seek an asymptotic formula in the limit $x\to+\infty$ for the (so called) Airy function of negative argument:

$$\mathrm{Ai}(-x)=\frac{1}{\pi}\int_0^\infty\cos(\tfrac{1}{3}w^3-xw)\,dw\qquad (x>0).$$

Note that this integral should be considered as a generalized integral, i.e. $\mathrm{Ai}(-x)=\lim_{A\to+\infty}\frac{1}{\pi}\int_0^A\cos(\frac{1}{3}w^3-xw)\,dw$, since we have $\int_0^\infty|\cos(\frac{1}{3}w^3-xw)|\,dw=\infty$.

The stationary points of the integrand are $w=\pm\sqrt{x}$, and only $w=\sqrt{x}$ lies in the range of integration. Substituting $w=\sqrt{x}(1+t)$ we get

$$\begin{aligned}\mathrm{Ai}(-x)&=\frac{\sqrt{x}}{\pi}\int_{-1}^\infty\cos\big(x^{3/2}\big(-\tfrac{2}{3}+t^2+\tfrac{1}{3}t^3\big)\big)\,dt\\ &=\frac{\sqrt{x}}{\pi}\Re\bigg(e^{-\frac{2}{3}x^{3/2}i}\int_{-1}^\infty e^{ix^{3/2}(t^2+\frac{1}{3}t^3)}\,dt\bigg).\end{aligned}\tag{10.8}$$

In order to find the asymptotic behavior of the last integral we split the range of integration into $(-1,0)$ and $(0,\infty)$, and for each part we take $v=v(t)=t^2+\frac{1}{3}t^3$ as a new variable of integration. Note that $v'(t)=2t+t^2=t(2+t)$, i.e. $v(t)$ is strictly decreasing for $t\in[-1,0]$ and strictly increasing for $t\in[0,\infty)$. Let us write $u_1:[0,\frac{2}{3}]\to[-1,0]$ and $u_2:[0,\infty)\to[0,\infty)$ for the primitive function of $t\mapsto v(t)$ in these two ranges. Then we get:

$$\int_{-1}^\infty e^{ix^{3/2}(t^2+\frac{1}{3}t^3)}\,dt=-\int_0^{2/3}e^{ix^{3/2}v}u_1'(v)\,dv+\int_0^\infty e^{ix^{3/2}v}u_2'(v)\,dv.$$

Since $v(t)$ is an analytic function of $t \in \mathbb{C}$ and $v(0) = v'(0) = 0$, $v''(0) > 0$, it follows that there exists some open disc $\Omega \subset \mathbb{C}$ centered at 0, and an analytic function $H : \Omega \to \mathbb{C}$, such that $H(0) = 0$, $H(w) > 0$ for $w \in \Omega \cap \mathbb{R}_{>0}$, and $H(w)^2 + \frac{1}{3}H(w)^3 = v(H(w)) = w^2$ for all $w \in \Omega$. The power series for $H(w)$ can be found by substituting in $H(w)^2 + \frac{1}{3}H(w)^3 = v(H(w)) = w^2$, and we compute:

$$H(w) = w - \frac{1}{6}w^2 + \frac{5}{72}w^3 + \ldots. \tag{10.9}$$

Let $r > 0$ be the radius of Ω. Now since $v \mapsto H(\sqrt{v})$ for $v \in [0, r^2)$ is a continuous function mapping 0 to 0, increasing at least for small v, and whose composition with g is the identity function, we conclude that $u_2(v) = H(\sqrt{v})$ for all $v \in [0, r^2)$. Similarly $u_1(v) = H(-\sqrt{v})$ for all $v \in [0, r^2)$. Hence by differentiating we obtain, for all $v \in [0, r^2)$,

$$u_j'(v) = \frac{(-1)^j}{2\sqrt{v}} H'((-1)^j\sqrt{v}) = \frac{(-1)^j}{2\sqrt{v}}\Big(1 - \frac{1}{3}(-1)^j\sqrt{v} + \frac{5}{24}v + \ldots\Big). \tag{10.10}$$

Now

$$\int_0^\infty e^{ix^{3/2}v} u_2'(v)\, dv = \int_0^\infty e^{ix^{3/2}v}\frac{dv}{2\sqrt{v}} + \int_0^\infty e^{ix^{3/2}v}\Big(u_2'(v) - \frac{1}{2\sqrt{v}}\Big)\, dv.$$

Here the first integral is $\frac{1}{2}e^{\frac{\pi}{4}i}\sqrt{\pi}x^{-3/4}$, by Lemma 10.1. We handle the second integral by integration by parts:

$$\int_0^\infty e^{ix^{3/2}v}\Big(u_2'(v) - \frac{1}{2\sqrt{v}}\Big)\, dv$$

$$= \lim_{V\to\infty}\Bigg(\bigg[\frac{e^{ix^{3/2}v}}{ix^{3/2}}\Big(u_2'(v) - \frac{1}{2\sqrt{v}}\Big)\bigg]_{v=0}^{v=V} - \int_0^V \frac{e^{ix^{3/2}v}}{ix^{3/2}} \cdot \frac{d}{dv}\Big(u_2'(v) - \frac{1}{2\sqrt{v}}\Big)\, dv\Bigg). \tag{10.11}$$

Here by (10.10) (and the formula obtained by differentiating once more, using the analyticity) we have $u_2'(v) - \frac{1}{2\sqrt{v}} = O(1)$ and $\frac{d}{dv}(u_2'(v) - \frac{1}{2\sqrt{v}}) = O(v^{-1/2})$ as $v \to 0$; and thus there are problems at "$v \to 0$" in the above expression. In order to determine the behavior of $u_2'(v)$ and $u_2''(v)$ for large v we first note that $v = u_2(v)^2 + \frac{1}{3}u_2(v)^3$ and $u_2(v) > 0$ together force

$$u_2(v) \sim (3v)^{1/3}, \qquad \text{as } v \to +\infty. \tag{10.12}$$

[Proof, with a more precise estimate: For all $v > 0$ we have $\frac{1}{3}u_2(v)^3 < v$; thus $u_2(v) < (3v)^{1/3}$. Hence also $v = u_2(v)^2 + \frac{1}{3}u_2(v)^3 < (3v)^{2/3} + \frac{1}{3}u_2(v)^3$, i.e. $u_2(v)^3 > 3(v - (3v)^{2/3})$, i.e. $u_2(v) > (3(v - (3v)^{2/3}))^{1/3} = ((3v)(1 - O(v^{-1/3})))^{1/3} = (3v)^{1/3}(1 - O(v^{-1/3})) = (3v)^{-1/3} - O(1)$. Hence we have proved that there is an absolute constant $C > 0$ such that for all $v \geq 1$ (say), $(3v)^{-1/3} - C < u_2(v) < (3v)^{-1/3}$. This is clearly a more precise statement than (10.12).] Now by differentiating the relation $v = u_2(v)^2 + \frac{1}{3}u_2(v)^3$ and using (10.12) we get $u_2'(v) = (2u_2(v) + u_2(v)^2)^{-1} \sim (3v)^{-\frac{2}{3}}$ as $v \to \infty$. Similarly, differentiating the relation $v = u_2(v)^2 + \frac{1}{3}u_2(v)^3$ twice we have $u_2''(v) = -\frac{2+2u_2(v)}{2u_2(v)+u_2(v)^2}u_2'(v)^2 \sim -2(3v)^{-\frac{5}{3}}$ as $v \to \infty$. Using these asymptotics (or just upper bounds) in (10.11) we conclude:

$$\int_0^\infty e^{ix^{3/2}v}\left(u_2'(v) - \frac{1}{2\sqrt{v}}\right)dv = O(x^{-\frac{3}{2}}),$$

and hence

$$\int_0^\infty e^{ix^{3/2}v}u_2'(v)\,dv = \frac{1}{2}e^{\frac{\pi}{4}i}\sqrt{\pi}x^{-3/4} + O(x^{-\frac{3}{2}}), \qquad \forall x \geq 1.$$

The treatment of the $u_1'(v)$-integral is similar but easier, using both Lemma 10.1 and Lemma 10.2, and we get:

$$-\int_0^{2/3} e^{ix^{3/2}v}u_1'(v)\,dv = \frac{1}{2}e^{\frac{\pi}{4}i}\sqrt{\pi}x^{-3/4} + O(x^{-\frac{3}{2}}), \qquad \forall x \geq 1.$$

Adding these two and inserting in (10.8) we conclude:

$$\begin{aligned}\mathrm{Ai}(-x) &= \frac{\sqrt{x}}{\pi}\Re\left(e^{-\frac{2}{3}x^{3/2}i}e^{\frac{\pi}{4}i}\sqrt{\pi}x^{-3/4} + O(x^{-\frac{3}{2}})\right)\\ &= \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\cos\left(\tfrac{2}{3}x^{3/2} - \tfrac{\pi}{4}\right) + O(x^{-1}), \qquad \forall x \geq 1.\end{aligned}$$

10.3. **Uniform asymptotics for $J_\nu(x)$ for ν large.** Using methods for asymptotic expansions of solutions to (ordinary) second order differential equations, Olver [15] (cf. also [16, Ch. 11, (10.18)]) has proved the following formula. For all $\nu \geq 1$ and all $t > 0$:

$$J_\nu(\nu t) = \nu^{-\frac{1}{3}} \Big(\frac{4\zeta}{1-t^2}\Big)^{1/4} \Big\{ \mathrm{Ai}(\xi) + O\Big(\nu^{-1} \frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}}\Big) \Big\} \tag{10.13}$$

where $\zeta = \zeta(t) = (\frac{3}{2}u(t))^{2/3} \operatorname{sgn}(1-t)$ and $\xi = \nu^{2/3}\zeta$ with

$$u(t) = \begin{cases} \operatorname{arctanh}\big(\sqrt{1-t^2}\big) - \sqrt{1-t^2} & \text{if } 0 < t \leq 1 \\ \sqrt{t^2-1} - \arctan\big(\sqrt{t^2-1}\big) & \text{if } t \geq 1. \end{cases}$$

We also use the notation $\xi^+ = \max(\xi, 0)$.

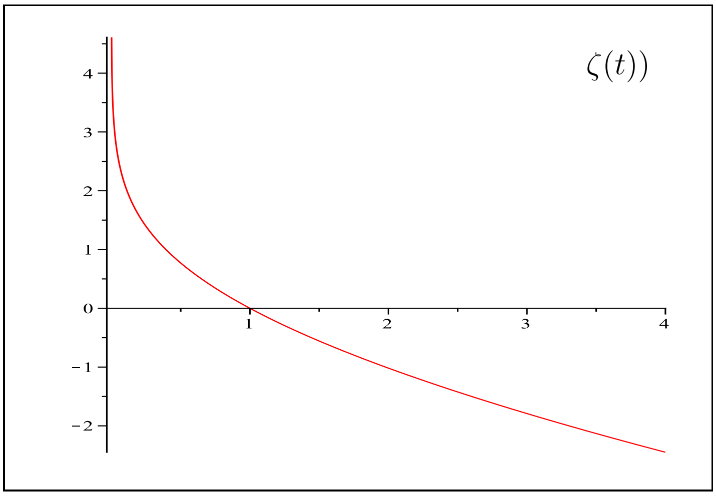

Figure 1 – The auxiliary function $\zeta(t)$.

We stress: In (10.13), the implied constant in the big-O is *absolute!* By contrast, the implied constant in (10.5) depends on ν (as well as on m)! See Example 10.4 for a more detailed comparison.

In (10.13) appears the *Airy function:*

$$\mathrm{Ai}(\xi) = \frac{1}{\pi} \int_0^\infty \cos(\tfrac{1}{3}w^3 + \xi w)\, dw, \qquad \xi \in \mathbb{R}$$

(cf. Example 10.3 above). This is a smooth function satisfying the differential equation $\mathrm{Ai}''(\xi) = \xi\, \mathrm{Ai}(\xi)$ with $\mathrm{Ai}(0) = (3^{2/3}\Gamma(2/3))^{-1}$, $\mathrm{Ai}'(0) = -(3^{1/3}\Gamma(1/3))^{-1}$. The following asymptotic relations hold:

$$\mathrm{Ai}(\xi) = \frac{1}{2\sqrt{\pi}} \xi^{-1/4} e^{-\frac{2}{3}\xi^{3/2}} \left(1 + O\left(\xi^{-3/2}\right)\right) \qquad \text{as } \xi \to \infty, \tag{10.14}$$

(10.15)

$$\mathrm{Ai}(\xi) = \frac{1}{\sqrt{\pi}} |\xi|^{-1/4} \left(\cos\left(\tfrac{2}{3}|\xi|^{3/2} - \tfrac{\pi}{4}\right) + O\left(|\xi|^{-3/2}\right)\right) \qquad \text{as } \xi \to -\infty.$$

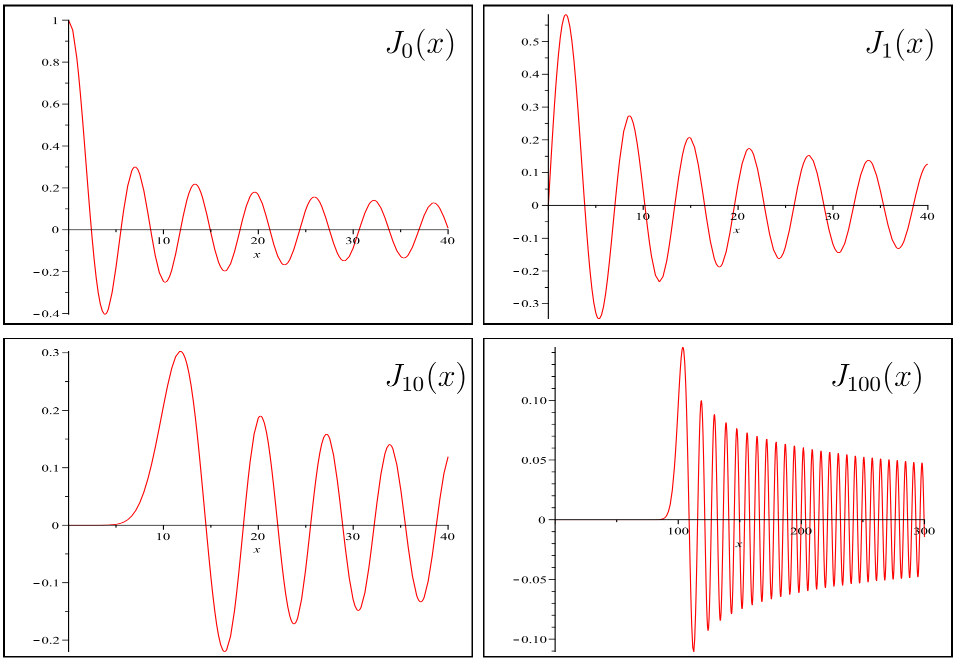

Figure 2 – The J-Bessel function $J_\nu(x)$ for $\nu = 0, 1, 10, 100$. We remark that for every $\nu \geq 1$ the graph of the right hand side in (10.13) (without the error term) is practically indistinguishable from the $J_\nu(x)$-graph; already for $\nu = 1$ the relative error is typically below 0.01!

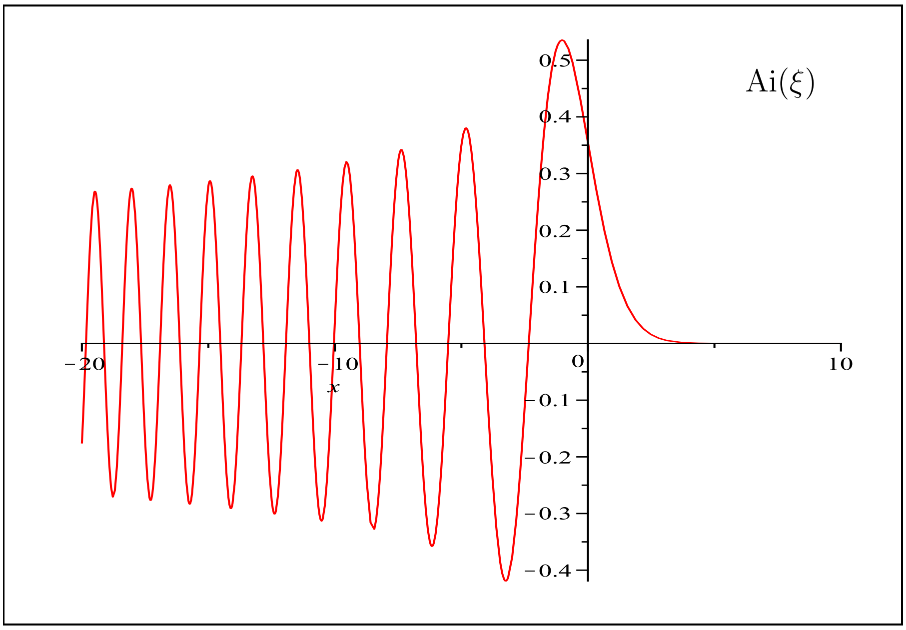

Figure 3 – The Airy function $\mathrm{Ai}(\xi)$.

Cf. [16, Ch. 11.1]. Cf. also Example 10.3 above, where we prove a weaker version of the second formula.

In order to better understand what (10.13) really means, let us derive from it an asymptotic relation involving only elementary functions:

Proposition 10.3. *Fix an arbitrary $C > 0$. Then for all $\nu \geq 1$ and $x > 0$ we have*

(10.16)

$$J_\nu(x) = \begin{cases} \dfrac{e^{\sqrt{\nu^2-x^2}}}{\sqrt{2\pi}\sqrt[4]{\nu^2-x^2}\left(\frac{\nu}{x}+\sqrt{(\frac{\nu}{x})^2-1}\right)^\nu}\left(1+O\Big(\dfrac{\sqrt{\nu}}{(\nu-x)^{3/2}}\Big)\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \leq \nu - C\nu^{\frac13} & \\ O(\nu^{-\frac13}) \qquad\qquad\qquad\qquad\qquad \text{if } |x-\nu| \leq C\nu^{\frac13} & \\ \dfrac{\sqrt{2}}{\sqrt{\pi}\sqrt[4]{x^2-\nu^2}}\left\{\cos\Big(\sqrt{x^2-\nu^2}-\nu\arccos\big(\dfrac{\nu}{x}\big)-\dfrac{\pi}{4}\Big)+O\Big(\dfrac{\sqrt{\nu}}{(x-\nu)^{3/2}}+\dfrac{1}{\nu}\Big)\right\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \geq \nu + C\nu^{\frac13}. & \end{cases}$$

Here the implied constant in each "big-O" depends only on C, i.e. it is independent of ν and x. The bound in the case $|x-\nu| \leq C\nu^{\frac13}$ may be complemented by the following fact: There exist absolute constants $C_1, C_2 > 0$ such that

$$J_\nu(x) \gg \nu^{-\frac13}, \qquad \forall\, \nu \geq C_1,\ x \in [\nu - C_2\nu^{\frac13}, \nu + C_2\nu^{\frac13}]. \tag{10.17}$$

Remark 10.4. As will be seen in the proof, for $x < \nu$ we have

$$\frac{e^{\sqrt{\nu^2-x^2}}}{\left(\frac{\nu}{x}+\sqrt{(\frac{\nu}{x})^2-1}\right)^\nu} = \exp\left\{-\nu\left(\operatorname{arctanh}\Big(\sqrt{1-\tfrac{x^2}{\nu^2}}\Big)-\sqrt{1-\tfrac{x^2}{\nu^2}}\right)\right\},$$

and using this format in the first case in (10.16) we see that the proposition implies that $J_\nu(x) \ll |x+\nu|^{-\frac14}|x-\nu|^{-\frac14}$ whenever $|x-\nu| \geq C\nu^{\frac13}$; thus in particular $J_\nu(x) \ll \nu^{-\frac12}$ whenever $x \geq 1.01\nu$ or $x \leq 0.99\nu$. By contrast, in the comparatively small interval $|x-\nu| \ll \nu^{\frac13}$ the function $J_\nu(x)$ is of order of magnitude $\nu^{-\frac13}$, i.e. much larger than elsewhere!

Proof. We apply (10.13) with $t = x/\nu$. Let us first assume $x \leq \nu - C\nu^{\frac13}$. Then $t < 1$, and we note that for all $0 < t < 1$ we have $\zeta(t) \gg 1-t$ (cf. Figure 1 and Section 10.6); hence our assumption $\nu - x \geq C\nu^{\frac13}$ implies that $\zeta(t) \gg \frac{\nu-x}{\nu} \gg \nu^{-\frac23}$ (here and in the rest of the proof, the implied constant in any $\ll$, $\gg$ or big-O depends only on C) and therefore $\xi = \nu^{2/3}\zeta \gg \nu^{-\frac13}(\nu-x) \gg 1$, i.e. ξ is bounded from below by a positive constant which only depends on C. Hence by (10.13) and

(10.14) we have

$$J_\nu(x) = \nu^{-\frac{1}{3}}\Big(\frac{4\zeta}{1-(x/\nu)^2}\Big)^{1/4} \cdot \frac{1}{2\sqrt{\pi}}\xi^{-1/4}e^{-\frac{2}{3}\xi^{3/2}}\Big(1+O(\xi^{-\frac{3}{2}}+\nu^{-1})\Big)$$
$$= \frac{e^{-\frac{2}{3}\xi^{3/2}}}{\sqrt{2\pi}\sqrt[4]{\nu^2-x^2}}\Big(1+O\Big(\frac{\sqrt{\nu}}{(\nu-x)^{3/2}}+\nu^{-1}\Big)\Big).$$

The error term may be simplified by using the fact that $\frac{\sqrt{\nu}}{(\nu-x)^{3/2}} > \nu^{-1}$. (It may appear that our treatment of the error term was wasteful in the case of t near 0, since in this case $\zeta(t)$ is of higher order of magnitude than $1-t$. However note that in this case we anyway have $\frac{\sqrt{\nu}}{(\nu-x)^{3/2}} \asymp \nu^{-1}$, i.e. the error term in (10.16) matches the error term in (10.13), i.e. we have not been wasteful.) Finally to express $e^{-\frac{2}{3}\xi^{3/2}}$ in terms of x,ν, note that

$$-\frac{2}{3}\xi^{3/2} = -\frac{2}{3}\nu\zeta^{\frac{3}{2}} = -\nu\Big(\operatorname{arctanh}\Big(\sqrt{1-(x/\nu)^2}\Big) - \sqrt{1-(x/\nu)^2}\Big)$$
$$= -\nu\log\Big(\tfrac{\nu}{x}+\sqrt{(\tfrac{\nu}{x})^2-1}\Big)+\sqrt{\nu^2-x^2}.$$

Hence we obtain the formula in (10.16), in the case $\nu - x \geq C\nu^{\frac{1}{3}}$.

Let us next assume $x \geq \nu + C\nu^{\frac{1}{3}}$. Then $t > 1$. Note that if $1 < t \leq 2$ then $-\zeta(t) \gg t-1$ (cf. Figure 1 and Section 10.6) and thus our assumption $x-\nu \geq C\nu^{\frac{1}{3}}$ implies that $-\zeta(t) \gg \frac{x-\nu}{\nu} \gg \nu^{-\frac{2}{3}}$ and $-\xi = -\nu^{\frac{2}{3}}\zeta \gg \nu^{-\frac{1}{3}}(x-\nu) \gg 1$. In the remaining case $t > 2$ we have $-\zeta(t) \gg 1$ so that certainly $-\xi \gg 1$ again. Thus our assumption $x \geq \nu + C\nu^{\frac{1}{3}}$ implies that $-\xi$ is bounded from below by a positive constant which only depends on C. Hence by (10.13) and (10.15) we have

$$J_\nu(x) = \frac{1}{\sqrt{\pi}}\nu^{-\frac{1}{3}}\Big(\frac{4\zeta}{1-(x/\nu)^2}\Big)^{\frac{1}{4}}|\xi|^{-\frac{1}{4}}\Big(\cos\big(\tfrac{2}{3}|\xi|^{3/2}-\tfrac{\pi}{4}\big)+O\left(|\xi|^{-\frac{3}{2}}+\nu^{-1}\right)\Big)$$
$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{x^2-\nu^2}}\Big(\cos(\tfrac{2}{3}|\xi|^{3/2}-\tfrac{\pi}{4})+O\Big(\frac{\sqrt{\nu}}{(x-\nu)^{3/2}}+\frac{1}{\nu}\Big)\Big),$$

where the form of the error term is clear from the previous discussion in the case $1 < t \leq 2$, while in the case $t > 2$ it holds since $|\xi| = \nu^{\frac{2}{3}}|\zeta| \gg \nu^{\frac{2}{3}}$, thus $|\xi|^{-\frac{3}{2}} \ll \nu^{-1}$. Finally to express $\cos(\frac{2}{3}|\xi|^{3/2}-\frac{\pi}{4})$ in terms of x,ν we note that

$$\tfrac{2}{3}|\xi|^{\frac{3}{2}} = \tfrac{2}{3}\nu|\zeta|^{\frac{3}{2}} = \nu\Big(\sqrt{(x/\nu)^2-1}-\arctan\Big(\sqrt{(x/\nu)^2-1}\Big)\Big)$$
$$= \sqrt{x^2-\nu^2}-\nu\arccos(\nu/x),$$

and we again obtain the formula in (10.16).

Finally assume $|x-\nu| \leq C\nu^{\frac{1}{3}}$. Then $|t-1| = \frac{|x-\nu|}{\nu}\nu^{-\frac{2}{3}}$, and hence if $\nu \geq (2C)^{3/2}$ we have $|t-1| \leq \frac{1}{2}$ and thus $|\zeta(t)| \ll |t-1|$ and

$|\xi| = \nu^{\frac{2}{3}}|\zeta| \ll 1$ and using this (10.13) is seen to imply $J_\nu(x) = O(\nu^{-\frac{1}{3}})$. Finally this bound is extended to $\nu \in [1, (2C)^{3/2}]$ by using the continuity of $J_\nu(x)$ (in both variables) and the fact that the set

$$\{(\nu, x) \,:\, \nu \in [1, (2C)^{3/2}],\ |x - \nu| \leq C\nu^{\frac{1}{3}}\}$$

is compact. The lower bound (10.17) can be proved by a similar discussion, using the fact that $\mathrm{Ai}(\xi) \gg 1$ for all ξ sufficiently near 0. (We leave out the details.) $\square$

Remark 10.5. The result(s) of Proposition 10.3 can also be naturally derived using other methods, such as steepest descent. Cf., e.g., [3, Exercises 7.17, 7.18] and [16, Ch. 4.9].

Example 10.4. Comparison (10.5) **vs** [(10.13)≈**Prop. 10.3**]. Note that (10.5) with $m = 0$ implies that for all $\nu > -\frac{1}{2}$ and $x \geq 1$,

$$J_\nu(x) = \Big(\frac{2}{\pi x}\Big)^{1/2}\Big\{\cos\big(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\big) + O_\nu(x^{-1})\Big\}, \tag{10.18}$$

where we write "O_ν" to stress that the implied constant depends on ν. Let us from now on keep ν *large* (say $\nu \geq 100$). By comparing with Prop. 10.3 (the case $x \geq \nu + C\nu^{\frac{1}{3}}$) we can now determine how large x has to be (depending on ν) for the asymptotic formula (10.18) to be at all relevant. Clearly a necessary condition for this is that the two amplitudes are near each other, i.e. that $\frac{\sqrt[4]{x^2-\nu^2}}{\sqrt{x}}$ is near one, say $\frac{\sqrt[4]{x^2-\nu^2}}{\sqrt{x}} > 0.99$. This is seen to imply $x > 5\nu$. Next, note that the difference between the two cos-arguments, i.e. between $x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi$ and $\sqrt{x^2 - \nu^2} - \nu\arccos(\frac{\nu}{x}) - \frac{\pi}{4}$, tends to 0 as $x \to +\infty$ for any fixed ν. Through differentiation w.r.t. x we see that this absolute difference equals $\int_x^\infty (1 - \sqrt{1 - (\nu/x')^2})\, dx'$; in particular it is a strictly decreasing function of x for fixed ν. Clearly another necessary condition for (10.18) to be relevant for all $x \geq x_0 = x_0(\nu)$ is that the difference under consideration is *small* (i.e. less than some fixed small positive constant) for $x = x_0$. Recall that we have already noted that we must have $x_0 > 5\nu$. Using the fact that $1 - \sqrt{1 - t^2} = \frac{1}{2}t^2 + O(t^4)$ for $0 < t \leq \frac{1}{5}$ we obtain

$$\int_{x_0}^\infty (1 - \sqrt{1 - (\nu/x')^2})\, dx' = \int_{x_0}^\infty \Big(\frac{\nu^2}{2x'^2} + O\Big(\frac{\nu^4}{x'^4}\Big)\Big)\, dx' = \frac{\nu^2}{2x_0} + O\Big(\frac{\nu^4}{x_0^3}\Big).$$

Hence if $x_0 = \nu^2$ and ν is sufficiently large (i.e. larger than a certain absolute constant) then the above expression is $\in [0.49, 0.51]$; and thus for such large ν the difference under consideration is ≥ 0.49 for all $x_0 \leq \nu^2$, i.e. we must have $x_0 > \nu^2$ for the formula (10.18) to be relevant!

On the other hand for $x_0 > C\nu^2$ with C a not too small constant $C > 1$, the same type of analysis as above shows that the asymptotic formula (10.18) really *is* starting to be relevant...

Of course, for x *sufficiently* large (as depends on ν), the error term in (10.18) is *better* than the error term in Prop. 10.3! In order to say something more precise, note that it is certainly possible to keep track on the dependence on ν in the computations leading to (10.5), and one result from such an analysis is the following (cf. [11, (B.35)][20]): For all $\nu \geq 0$ and all $x \geq 1 + \nu^2$ we have

$$(10.19) \qquad J_\nu(x) = \Big(\frac{2}{\pi x}\Big)^{1/2} \Big\{ \cos\big(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\big) + O\Big(\frac{1+\nu^2}{x}\Big) \Big\},$$

where the implied constant is absolute. For large ν the error term here is better than the error term in Prop. 10.3 iff x/ν^3 is sufficiently large!

10.4. **Asymptotics for the zeros of $J_n(x)$.** We have seen in Section 9.2 that the (positive) zeros of $J_n(x)$ are important. We here state, without proofs, some asymptotics for these zeros. 0(The main tools for the proofs are the asymptotic formulas for $J_n(x)$ itself, together with the Bessel differential equation, (9.2).)

As in Section 9.2 we write $0 < j_{n,1} < j_{n,2} < \ldots$ for the full set of positive zeros of $J_n(z)$ ($n \in \mathbb{Z}_{\geq 0}$). For any *fixed* $n \in \mathbb{Z}_{\geq 0}$, we have $j_{n,s} = (s + \frac{1}{2}n - \frac{1}{4})\pi + O_n(s^{-1})$, $\forall s \in \mathbb{N}$ (cf. Olver, [16, p. 247 (6.03)]). By using the result in (10.19) we may in fact prove that the implied constant in the last big-O bound is $\ll 1 + n^2$ when $s \geq 1 + n^2$. Thus: For all $n \geq 0$ and all $s \geq 1 + n^2$ we have

$$(10.20) \qquad j_{n,s} = (s + \tfrac{1}{2}n - \tfrac{1}{4})\pi + O\Big(\frac{1+n^2}{s}\Big),$$

where the implied constant is absolute.

In order to give a uniform asymptotic formula, let us write $0 > a_1 > a_2 > \ldots$ for the zeros of $\mathrm{Ai}(\xi)$ (cf. Figure 3). Let us also write $t : \mathbb{R} \to \mathbb{R}_{>0}$ for the inverse of the function $\zeta(t)$ (cf. Figure 1); thus $t(\zeta)$ is a smooth, strictly decreasing function satisfying $t(0) = 1$, $\lim_{\zeta\to\infty} t(\zeta) = 0$ and $\lim_{\zeta\to-\infty} t(\zeta) = +\infty$. By Olver [15, Sec. 7] we have

$$(10.21) \qquad j_{n,s} = n \cdot t(n^{-\frac{2}{3}} a_s) + O(n^{-1}), \qquad \forall n, s \geq 1,$$

the implied constant again being absolute. Furthermore it is clear from [15] that $j_{n,1} > n$ for all $n \geq 1$. To complement the formula (10.21), let

[20]This reference just gives a statement without a proof or precise reference; I have not yet checked the statement carefully, or located a reference containing a proof. Note however that the statement seems quite reasonable in view of the format of (10.5), (10.6) (considering higher terms in the expansion).

us also note the following asymptotic formula for the zeros of $\mathrm{Ai}(\xi)$:

$$a_s = -\left(\tfrac{3}{8}\pi(4s-1)\right)^{\frac{2}{3}}\left(1+O(s^{-2})\right), \qquad \forall s \geq 1. \tag{10.22}$$

(cf. [15, p. 367 (A19), (A20)]; note that this statement is consistent with (10.14), but it is not a consequence of (10.14) by itself).

Let us compare (10.20) and (10.21) in the case $s \geq 1+n^2$ and n large. First of all note that the definition of $\zeta(t)$ implies

$$\zeta(t) = -\left(\tfrac{3}{2}\left(t - \tfrac{\pi}{2} + \tfrac{1}{2}t^{-1} + O(t^{-3})\right)\right)^{2/3}, \qquad \text{as } t \to \infty,$$

from which we easily deduce

$$t(\zeta) = \tfrac{2}{3}|\zeta|^{3/2} + \tfrac{\pi}{2} - \tfrac{3}{4}|\zeta|^{-3/2} + O\left(|\zeta|^{-3}\right), \qquad \text{as } \zeta \to -\infty.$$

Using this together with (10.21) and (10.22) we conclude that, when $s \geq 1+n^2$ (thus $|a_s| \gg n^{\frac{4}{3}}$) and n is sufficiently large:

$$\begin{aligned} j_{n,s} &= n\Big(\pi n^{-1}(s-\tfrac{1}{4})(1+O(s^{-2})) - \frac{\pi}{2} - \frac{n}{2\pi(s-\frac{1}{4})}(1+O(s^{-2})) + O(n^2 s^{-2})\Big) + O(n^{-1}) \\ &= (s - \tfrac{1}{2}n - \tfrac{1}{4})\pi + \frac{n^2}{2\pi s} + O(n^{-1}). \end{aligned}$$

This is consistent with (10.20), and it is more precise than (10.20) as long as s/n^3 is small, while on the other hand (10.20) is more precise than (10.21) when s/n^3 is large!

10.5. **Some applications of the uniform Bessel asymptotics.** Recall from Section 9.2 that the Dirichlet eigenvalues for a unit disk in $\mathbb{R}^2$ are given by $j_{n,m}^2$ where $\langle n, m\rangle$ runs through $\mathbb{Z}_{\geq 0} \times \mathbb{N}$, and for $n \geq 1$ each such eigenvalue comes with multiplicity 2.

Let us now seek an asymptotic formula for the number of eigenvalues $\leq T$, as $T \to \infty$! Let us count the contribution from each n. For $n = 0$ the number is $\#\{m \in \mathbb{N} \,:\, j_{0,m} \leq \sqrt{T}\} = \pi^{-1}\sqrt{T} + O(1)$, by (10.20), and for each $n \geq \sqrt{T}$ the number is zero, since $j_{n,m} \geq j_{n,1} > n$. Now assume $1 \leq n < \sqrt{T}$: For such n, the number of eigenvalues $\leq T$ is equals to 2 times the largest $s \in \mathbb{N}$ for which $j_{n,s} \leq \sqrt{T}$, i.e. (by (10.21)) the largest s such that

$$n \cdot t(n^{-\frac{2}{3}}a_s) + O(n^{-1}) \leq \sqrt{T},$$

i.e.

$$a_s \geq n^{\frac{2}{3}}\zeta\Big(n^{-1}\sqrt{T} + O(n^{-2})\Big).$$

Using (10.22) and writing $t_{n,T} := n^{-1}\sqrt{T} + O(n^{-2})$ we find that this is the largest s for which:

$$-\Big(\tfrac{3}{8}\pi(4s-1))(1+O(s^{-2}))\Big)^{\frac{2}{3}} \geq n^{\frac{2}{3}}\zeta(t_{n,T}),$$

i.e.

$$\pi(s-\tfrac{1}{4})(1+O(s^{-2})) \leq n\Big(\sqrt{t_{n,T}^2-1}-\arctan\Big(\sqrt{t_{n,T}^2-1}\Big)\Big).$$

Using $\frac{d}{dt}(\sqrt{t^2-1}-\arctan(\sqrt{t^2-1})) = \sqrt{1-t^{-2}} \leq 1$ we see that this implies:

(10.23)
$$s \leq \pi^{-1}n\Big(\sqrt{n^{-2}T-1}-\arctan(\sqrt{n^{-2}T-1})\Big)+\tfrac{1}{4}+O(n^{-1}+s^{-1}).$$

The largest s satisfying this clearly satisfies

$$s_{max} = \pi^{-1}n\Big(\sqrt{n^{-2}T-1}-\arctan(\sqrt{n^{-2}T-1})\Big)+O(1). \tag{10.24}$$

Hence we obtain that the *total* number of eigenvalues is

$$\begin{aligned}\#\{\lambda \leq T\} &= O(\sqrt{T})+\frac{2}{\pi}\sum_{1\leq n<\sqrt{T}} n\Big(\sqrt{n^{-2}T-1}-\arctan(\sqrt{n^{-2}T-1})\Big)\\ &= O(\sqrt{T})+\frac{2}{\pi}\sqrt{T}\sum_{1\leq n<\sqrt{T}}\Big(\sqrt{1-(n/\sqrt{T})^2}-\frac{n}{\sqrt{T}}\arccos(n/\sqrt{T})\Big).\end{aligned}$$

By comparing with a Riemann integral we see that the last expression is $\sim \pi^{-1}T\int_0^1\big(\sqrt{1-x^2}-x\arccos x\big)\,dx = T/8$ as $T\to\infty$. In fact the error is easily estimated to be $O(\sqrt{T})$, and hence we have proved:

$$\#\{\lambda \leq T\} = \frac{T}{4}+O(\sqrt{T}).$$

Note that this is the standard Weyl law[21] (with the standard error term, optimal for certain compact manifolds such as the sphere)! However we wish to stress that the asymptotics here (i.e. before we did the wasteful step going from (10.23) to (10.24)) would really allow a much more precise understanding of the error term in terms of a lattice counting problem — and it seems likely that the $O(\sqrt{T})$ can be improved quite a bit, without too much difficulty!

Note also that the above analysis leads to an asymptotic formula for the number of eigenvalues $\leq \sqrt{T}$ of each "*type* $n \leq \sqrt{T}$!"

10.6. **Appendix: Some notes on how to extract** (10.13) **from Olver** [16, Ch. 11.10]**.** Olver's statement is much more complicated than (10.13) for three reasons: (1) he considers general complex argument in the J-Bessel function; (2) he is interested in allowing to extract *numerical* bounds on the error term; (3) he actually gives an *asymptotic expansion* of $J_\nu(\nu t)$, of which (10.13) is just the main term!

[21] which says that for any nice d-dim domain D the number of (Dirichlet, say) eigenvalues $\leq T$ is $\sim (2\pi)^{-d}\,\mathrm{vol}(\mathcal{B}_1^d)\,\mathrm{vol}(D)T^{d/2}$ as $T\to\infty$.

Let us first derive our formula for $\zeta(t)$. Olver's definition is (for $t \in \mathbb{C}$) [16, (10.05)]:

$$\frac{2}{3}\zeta^{3/2} = \log\Big(\frac{1+\sqrt{1-t^2}}{t}\Big) - \sqrt{1-t^2}, \tag{10.25}$$

where the branches take their principal values when $t \in (0,1)$ and $\zeta \in (0,\infty)$, and are continuous elsewhere. Thus for $t \in (0,1)$ (and also for $t=1$) we have

$$\frac{2}{3}\zeta^{3/2} = \operatorname{arctanh}\big(\sqrt{1-t^2}\big) - \sqrt{1-t^2} = \operatorname{arccosh}(t^{-1}) - \sqrt{1-t^2}.$$

To study the behavior of this function as $t \to 1^-$ we use $\operatorname{arctanh} z - z = \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7 + \ldots$ (true when $|z|<1$) and (writing $t = 1-w$, with $w > 0$ near 0): $\sqrt{1-(1-w)^2} = \sqrt{2-w}\sqrt{w} = \sqrt{2w}(1 - \frac{1}{4}w - \frac{1}{32}w^2 - \ldots)$ to conclude:

$$\frac{2}{3}\zeta(1-w)^{3/2} = \frac{2\sqrt{2}}{3}w^{\frac{3}{2}} + \frac{3\sqrt{2}}{10}w^{\frac{5}{2}} + O(|w|^{\frac{7}{2}}).$$

Hence since $\zeta(t)$ is analytic at $t=1$ and positive for $t<1$ near 1, we must have $\zeta(1-w) = 2^{\frac{1}{3}}w + \frac{3\cdot 2^{\frac{1}{3}}}{10}w^2 + \ldots$, i.e.

$$\zeta(1+w) = -2^{\frac{1}{3}}w + \tfrac{3}{10}\cdot 2^{\frac{1}{3}}\cdot w^2 + \ldots \tag{10.26}$$

for all $w \in \mathbb{C}$ near 0. Now consider (10.25) for $t > 1$; we wish to determine which branches to use for the various functions appearing; to start let's assume $\sqrt{1-t^2} = \varepsilon_j i\sqrt{t^2-1}$ where $\varepsilon_1, \varepsilon_2 \in \{1,-1\}$ are for the first and the second appearance of "$\sqrt{1-t^2}$", respectively. Note that $t^{-1}(1+\varepsilon_1 i\sqrt{t^2-1})$ has absolute value 1 and real part t^{-1}; from this we conclude (since we are using the branch of log which tends to 0 as its argument tends to 1):

$$\log\Big(\frac{1+\varepsilon_1\sqrt{t^2-1}}{t}\Big) = i\varepsilon_1 \arctan\big(\sqrt{t^2-1}\big) = i\varepsilon_1 \arccos(t^{-1}).$$

Hence:

$$\frac{2}{3}\zeta^{3/2} = i\varepsilon_1 \arctan\big(\sqrt{t^2-1}\big) - i\varepsilon_2\sqrt{t^2-1}.$$

Now $\arctan z + z = 2z + O(|z|^3)$ but $\arctan z - z = -\frac{1}{3}z^3 + O(|z|^5)$ as $z \to 0$; hence since we know that the right hand side above must behave like $w^{\frac{3}{2}}$ when $t = 1+w$, $w \to 0^+$, we conclude that $\varepsilon_1 = \varepsilon_2$; thus

$$\frac{2}{3}\zeta(1+w)^{3/2} = i\varepsilon_1\Big(-\frac{2\sqrt{2}}{3}w^{\frac{3}{2}} + \frac{3\sqrt{2}}{10}w^{\frac{5}{2}} + O(|w|^{\frac{7}{2}})\Big).$$

Comparing with (10.26) we see that we can take either $\varepsilon_1 = 1$ or -1, so long as we take the corresponding correct branch when raising to $\frac{2}{3}$; either way we obtain

$$\zeta(t) = -\Big(\tfrac{3}{2}\big(\sqrt{t^2-1} - \arctan\big(\sqrt{t^2-1}\big)\big)\Big)^{2/3} \qquad \text{for } t > 0.$$

This completes the proof of the formula for $\zeta(t)$.

Now to get (10.13) we apply [16, (10.18)] with $n = 0$, noticing that Olver's "$A_0(\zeta)$" equals 1:

$$J_\nu(\nu t) = \frac{1}{1+\delta_1}\nu^{-1/3}\Big(\frac{4\zeta}{1-t^2}\Big)^{1/4}\Big\{\mathrm{Ai}(\xi) + \varepsilon_{1,0}(\nu,\zeta)\Big\}$$

(where we use our notation $\xi = \nu^{2/3}\zeta$). Here $|\delta_1| \ll \nu^{-1}$ by [16, (10.20) and p. 422], and from [16, (10.20) and p. 395] we get

$$|\varepsilon_{1,0}(\nu,\zeta)| \ll \nu^{-1}\frac{M(\xi)}{E(\xi)} \ll \nu^{-1}\frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}}$$

(this may be improved for t near 0). Hence for ν larger than a certain absolute constant we have

$$J_\nu(\nu t) = \nu^{-1/3}\Big(\frac{4\zeta}{1-t^2}\Big)^{1/4}\bigg\{\mathrm{Ai}(\xi) + O\bigg(\nu^{-1}\frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}}\bigg)\bigg\}\Big(1+O(\nu^{-1})\Big).$$

Finally the last factor $(1+O(\nu^{-1}))$ can be multiplied into the expression using (10.14) and (10.15), and we obtain (10.13). [22] □

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[22] Note that the above was obtained for ν being lager than a sufficiently large absolute constant $C > 0$. In order to treat the remaining case of $\nu \in [1, C]$ (if $C \geq 1$) one may refer to the easier asymptotic formulas for $J_\nu(x)$ when $x \to 0$ and $x \to \infty$ for ν in a *compact* set. (Alternatively, it seems to me to be clear from the discussion on [16, p. 423] that δ_{2n+1} stays bounded away from -1 for all $\nu \geq 1$ but I haven't completely convinced myself about this.)

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