Analysis for PhD students (2020); Assignment 1

Problem 1. Prove that for any real $z > 0$ and any natural number N,

$$\sum_{n=0}^{N} \log(z+n) = \Big(z+N+\frac{1}{2}\Big)\log(z+N) - N - \Big(z-\frac{1}{2}\Big)\log z$$
$$+\frac{1}{12}\big((z+N)^{-1}-z^{-1}\big) + \int_0^N \frac{(x-\lfloor x\rfloor)(x-\lfloor x\rfloor-1)+\frac{1}{6}}{2(z+x)^2}\,dx.$$

(10p)

Problem 2. Let $1 < \omega_1 \le \omega_2 \le \cdots$ be an increasing sequence of real numbers satisfying

$$\#\{n \in \mathbb{N} \,:\, \omega_n \le T\} = cT + O(T^{\frac{1}{2}}) \qquad \forall T > 0,$$

where $c > 0$ is some constant. Determine an asymptotic formula for $\prod_{\omega_n < T}(1-\omega_n^{-1})$ as $T \to \infty$.

[Hint: Recall that one can use the logarithm function to transform a product into a sum.]

(10p)

Problem 3. Compute the following limits and justify the calculations:

(a) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{n\log(1+\frac{x}{n})}{x(1+x^2)}\,dx$ (b) $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{1+(nx)^2}{(1+x)^n}\,dx$

(c) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{\cos(\frac{x}{n})}{(1+\frac{x}{n})^n}\,dx$ (d) $\displaystyle\lim_{n\to\infty}\int_0^\infty (n+x)e^{-nx}\,dx$

(15p)

Problem 4. Let $E_1, E_2, \ldots$ be measurable subsets of a measure space (X,μ), with $\mu(E_n) < \infty$ for each n. Let $f \in L^1(\mu)$, and assume that $\lim_{n\to\infty}\int_X |f-\chi_{E_n}|\,d\mu = 0$. Prove that $f(x) \in \{0,1\}$ for μ-almost every $x \in X$.

(10p)

Problem 5. Let $1 \le p < \infty$, let $(X,\mathcal{M},\mu)$ be a fixed measure space, and let (f_n) be a sequence in $L^p = L^p(X,\mathcal{M},\mu)$. Prove that (f_n) is Cauchy in the L^p norm iff the following three conditions all hold:
(i) For every $\varepsilon > 0$, $\mu\big(\{x \,:\, |f_n(x)-f_m(x)| \ge \varepsilon\}\big) \to 0$ as $m,n\to\infty$;
(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |f_n|^p\,dx < \varepsilon$ for every n and every measurable set $E \subset X$ with $\mu(E) < \delta$; and
(iii) for every $\varepsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_{X\setminus E} |f_n|^p\,d\mu < \varepsilon$ for all n.

[Hint: I think that the most difficult part may be the proof of the necessity of (ii). For this, one may combine Folland's Cor. 3.6 with basic facts about the space L^p.]

(15p)

Problem 6. (a) Find an example of a sequence (μ_n) in $M(\mathbb{R})$ such that $\mu_n \to 0$ vaguely, but $\|\mu_n\| \not\to 0$.
(b) Find an example of a sequence (μ_n) in $M(\mathbb{R})$ such that $\mu_n \geq 0$ for every n and $\mu_n \to 0$ vaguely, but there exists some $x \in \mathbb{R}$ such that $\mu_n((-\infty, x]) \not\to 0$.
(c) Let $\mu_n \in M(\mathbb{R})$ be given by $\int_{\mathbb{R}} f \, d\mu_n = \sum_{k=1}^{n} \frac{n-k}{n^2} f(\frac{k}{n})$ for all $f \in C_0(\mathbb{R})$. Prove that the sequence (μ_n) converges vaguely in $M(\mathbb{R})$, and describe the limit measure explicitly.

(15p)

Problem 7. **[Multi-indices]** (a) Prove that for any multi-indices α, β, there is a constant $c_{\alpha,\beta}$ such that

$$\partial^\alpha \left(\frac{1}{x^\beta} \right) = \frac{c_{\alpha,\beta}}{x^{\beta+\alpha}}.$$

Give an explicit formula for $c_{\alpha,\beta}$.
(b) For any multi-index α we write $|\alpha|_\infty := \max(\alpha_1, \ldots, \alpha_n)$. Prove that for any multi-index α, there exist constants $c_{\alpha,m} > 0$ such that

$$\partial^\alpha \exp\Big(\prod_{j=1}^{n} x_j\Big) = \sum_{m=|\alpha|_\infty}^{|\alpha|} c_{\alpha,m} \frac{\prod_{j=1}^{n} x_j^m}{x^\alpha} \exp\Big(\prod_{j=1}^{n} x_j\Big).$$

$\big($Example: $\partial_1^5 \partial_2 \partial_3 \exp(x_1 x_2 x_3) = \big(25 x_2^4 x_3^4 + 11 x_1 x_2^5 x_3^5 + x_1^2 x_2^6 x_3^6\big) \exp(x_1 x_2 x_3).\big)$

(10p)

Problem 8. For any $a > 0$ let $g_a : \mathbb{R} \to \mathbb{R}$ be the function $g_a = a^{-1} \cdot \chi_{(0,a)}$. Let (a_n) be a sequence of positive real numbers and set

$$f_n = g_{a_1} * \cdots * g_{a_n}.$$

(a). Compute $\int_{\mathbb{R}} f_n \, dx$ and $\int_{\mathbb{R}} |f_n| \, dx$.
(b). What is the support of f_n?
(c). Prove that for each $n \geq 2$, $f_n \in C^{n-2}(\mathbb{R})$ but $f_n \notin C^{n-1}(\mathbb{R})$.
(d). Prove that if $\sum_{n=1}^{\infty} a_n = \infty$, then as $n \to \infty$, f_n converges pointwise to 0.

(15p)

[Comment: As (even?) more challenging tasks[1], you may try to prove that if $\sum_{n=1}^{\infty} a_n < \infty$, then as $n \to \infty$, f_n converges uniformly to a function $f \in C_c^\infty(\mathbb{R})$, $f \not\equiv 0$. Also, when $\sum_{n=1}^{\infty} a_n = \infty$, is the convergence $f_n \to 0$ *uniform* or not?]

To be returned: Tuesday, October 6, before midnight. Please send your solutions by email, or put them in my mailbox.

Note: Delayed exercises will in general be ignored. Exceptions are possible, but this requires that you have given me an explanation in advance, which I have approved.

[1] October 21: I have corrected the formulation here in the case $\sum_{n=1}^{\infty} a_n = \infty$.