Analysis for PhD students (2020); Assignment 2

Problem 1. Prove that if $f \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} \big|\xi \widehat{f}(\xi)\big|\, d\xi < \infty$, then f coincides almost everywhere with a differentiable function f_0 whose derivative is

$$f_0'(x) = 2\pi i \int_{-\infty}^{\infty} \xi \, \widehat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi \qquad (\forall x \in \mathbb{R}).$$

(15p)

Problem 2. [**Formulation corrected, Nov. 6.**] Prove that if μ is a positive Borel measure on $\mathbb{T}$ with $\mu(\mathbb{T}) = 1$, then $|\widehat{\mu}(k)| < 1$ for all $k \neq 0$ unless μ is a linear combination, with nonnegative coefficients, of the point masses at $y, y+\frac{1}{m}, \ldots, y+\frac{m-1}{m}$ for some $m \in \mathbb{N}$ and $y \in \mathbb{T}$.

(15p)

Problem 3. [**Formulation corrected, Oct. 14.**] Prove that for any $k \geq 1$:

$$\int_{-\infty}^{\infty} \frac{dx}{(1+|x-a|)^k (1+|x|)^k} \asymp_k \begin{cases} (1+|a|)^{-k} & \text{if } \; k > 1 \\ (1+|a|)^{-1} \log(2+|a|) & \text{if } \; k = 1, \end{cases}$$

uniformly over all $a \in \mathbb{R}$.

(10p)

Problem 4. Let a_0 be a fixed number with $0 < a_0 < 1$. Prove that

$$\int_1^{\infty} e^{-bx} x^{ab}\, dx \sim \frac{e^{-b}}{(1-a)b}$$

as $b \to \infty$, uniformly over all $a \in [0, a_0]$.

(15p)

Problem 5. Prove that for any given $A > 0$ there exist an interval $[a, b]$ and a real-valued function $f \in C^{\infty}([a, b])$ such that $f'(x) \geq 1$ for all $x \in [a, b]$, and

$$\left| \int_a^b e(f(x))\, dx \right| \geq A.$$

[Comment: An outline proof is given in connection with Prop. 2 in Stein's Ch. 8; however please provide more details on (1) exactly how such a function is constructed and (2) how the desired inequality is proved.]

(15p)

Problem 6. Let $n \in \mathbb{Z}^+$. Find (with proof) the largest possible constant $\alpha > 0$ such that

$$\int_0^1 e(x_1 t + x_2 t^2 + \cdots + x_n t^n)\, dt \ll_n (1 + |x|)^{-\alpha}, \qquad \forall x \in \mathbb{R}^n.$$

(15p)

Problem 7. Using stationary phase, prove that for any fixed $\alpha > 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda(\alpha \sin x - x)}\, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\cos\Big(\lambda\big(\sqrt{\alpha^2 - 1} - \arccos(\alpha^{-1})\big) - \frac{\pi}{4}\Big)}{\sqrt[4]{\alpha^2 - 1}\sqrt{\lambda}} + O\Big(\frac{1}{\lambda}\Big)$$

as $\lambda \to +\infty$ (through real numbers), where the implied constant may depend on α.

[Comment: Specializing to $\lambda = n \in \mathbb{Z}^+$ this gives an asymptotic formula for $J_n(\alpha n)$, via (8.9) in the lecture notes. It is satisfactory to note that this asymptotic formula is consistent with Prop. 10.1 in the lecture notes.]

(15p)

To be returned: Thursday, November 12, before midnight. Please send your solutions by email, or put them in my mailbox.

Note: Delayed exercises will in general be ignored. Exceptions are possible, but this requires that you have given me an explanation in advance, which I have approved.