**Analysis for PhD students (2020); Assignment 3**

**Problem 1.** (a) Compute the distributional derivative of the function $x \mapsto |x|$ on $\mathbb{R}$, as an explicit element in $L^1_{\text{loc}}(\mathbb{R})$.
(b) Let $B$ be the open unit ball in $\mathbb{R}^n$ centered at the origin. Compute the distributional derivatives $f_j = \frac{\partial}{\partial x_j}\chi_B$, $j = 1, \ldots, n$, as explicit signed measures on $M(\mathbb{R}^n)$. Also compute the total variation of these measures.
(c) Explain why the functions $\sin x$ and $\cos x$ (on $\mathbb{R}$) can be viewed as tempered distributions, and compute their Fourier transforms. (15p)

**Problem 2.** A distribution $F$ on $\mathbb{R}^n$ is called *homogeneous* of degree $\lambda$ if $F \circ S_r = r^\lambda F$ for all $r > 0$, where $S_r$ is the linear map $S_r(x) = rx$ on $\mathbb{R}^n$.
(a). Prove that $\delta$ is homogeneous of degree $-n$.
(b). Prove that if $F$ is homogeneous of degree $\lambda$, then for any multi-index $\alpha$, $\partial^\alpha F$ is homogeneous of degree $\lambda - |\alpha|$.
(c). Let $L$ be the distribution corresponding to the function $x \mapsto \chi_{(0,\infty)}(x) \log x$. Prove that its derivative, $L'$ (which is discussed in Folland's book, p. 288), satisfies the relation $L' \circ S_r = r^{-1}(L' + (\log r)\delta)$ for all $r > 0$. Prove also that this implies that $L'$ is *not* homogeneous (of any degree). (15p)

**Problem 3.** Prove that if $\psi \in C^\infty(\mathbb{R}^n)$ is a slowly increasing function then $\phi \mapsto \psi\phi$ is a continuous map of $\mathcal{S}$ into $\mathcal{S}$ (15p)

**Problem 4.** (a). Let $f \in L^2(\mathbb{R})$. Prove that the $L^2$ derivative $f'$ (in the sense of Folland's Exercise 8 in Ch. 8.2) exists iff $\xi\widehat{f} \in L^2$, in which case $\widehat{f'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$.
(b). Let $k, n \geq 1$. Prove that $H_k$ is equal to the space of all $f \in L^2(\mathbb{R}^n)$ that possess strong $L^2$ derivatives $\partial^\alpha f$ (again in the sense of Folland's Exercise 8 in Ch. 8.2) for $|\alpha| \leq k$. Prove also that (for $f \in H_k$) all these derivatives coincide with the distribution derivatives. (15p)

**Problem 5.** Let $\alpha$ be an irrational number, let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the translation $T(x, y) = (x+\alpha, y+\alpha)$, and let $\mu$ be the Lebesgue measure on $\mathbb{T}^2$.
(a). Prove that for any two Borel sets $A, B \subset \mathbb{T}$, if $T^{-1}(A\times B) = A\times B$, then $\mu(A \times B) = 0$ or $\mu(A \times B) = 1$.
(b). Prove that, still, $T$ is *not* an ergodic transformation of $(\mathbb{T}^2, \mu)$. (10p)

**Problem 6.** A measure-preserving system $(X, \mathcal{M}, \mu, T)$ with $\mu(X) = 1$ is said to be *mixing of order $k$* if

$$(*)\qquad \begin{cases} \mu\big(A_0 \cap T^{-n_1} A_1 \cap \cdots \cap T^{-n_k} A_k\big) \to \prod_{j=0}^{k} \mu(A_j) \\ \qquad \text{as } n_1,\ n_2 - n_1,\ n_3 - n_2,\ \cdots, n_k - n_{k-1} \to \infty, \end{cases}$$

for any sets $A_0, \ldots, A_k \in \mathcal{M}$.

Now let $(X, \mathcal{M}, \mu, T)$ be a Bernoulli shift.

(a). Prove that (*) holds whenever $A_0, \ldots, A_k$ are finite unions of cylinder sets.

(b). Prove that (*) holds for arbitrary $A_0, \ldots, A_k \in \mathcal{M}$. (Thus every Bernoulli shift is mixing of order $k$ for every $k \geq 1$.)

(15p)

**Problem 7.** (a) Define the sequences $(u_n)$ and $(v_n)$ recursively by

$$u_1 = 1,\ u_2 = 2 \quad \text{and} \quad u_{n+2} = 2u_{n+1} + u_n \ \ (n = 1, 2, 3, \ldots);$$
$$v_1 = 1,\ v_2 = 3 \quad \text{and} \quad v_{n+2} = 2v_{n+1} + v_n \ \ (n = 1, 2, 3, \ldots).$$

Let $x$ be an arbitrary number in $Y = (0,1) \setminus \mathbb{Q}$ with continued fraction expansion $x = [a_1, a_2, \ldots]$, and let $n$ be a positive integer. Prove that the first $n$ coefficients $a_1, \ldots, a_n$ are all equal to 2 if and only if $x$ lies strictly between $\dfrac{u_n}{u_{n+1}}$ and $\dfrac{v_n}{v_{n+1}}$.

(b) Prove that for Lebesgue almost every $x \in Y$, the pattern $2, \ldots, 2$ ($n$ terms 2) appears in the continued fraction expansion $x = [a_1, a_2, \ldots]$ with frequency $(-1)^n \log\big(\frac{(v_n+v_{n+1})u_{n+1}}{(u_n+u_{n+1})v_{n+1}}\big) / \log 2$, that is:

$$\lim_{m \to \infty} \frac{1}{m} \#\big\{j \in \{1, \ldots, m\} \ :\ a_j = a_{j+1} = \cdots = a_{j+n-1} = 2\big\}$$

$\left\{\begin{array}{l}\textbf{7 Dec: corrected} \\ \textbf{this line.}\end{array}\right\}$
$$= (-1)^n \frac{\log\Big(\dfrac{(v_n + v_{n+1})u_{n+1}}{(u_n + u_{n+1})v_{n+1}}\Big)}{\log 2}.$$

(15p)

[Comments: • To get somewhat cleaner statements, you may also like to verify that $v_n = u_n + u_{n-1}$ for all $n \geq 2$; also $2u_n^2 = v_n^2 + (-1)^{n+1}$ for all $n \geq 1$, and (hence) $\frac{(v_n+v_{n+1})u_{n+1}}{(u_n+u_{n+1})v_{n+1}} = 1 + \frac{(-1)^n}{v_{n+1}^2}$ for all $n \geq 1$. • You may also enjoy proving for yourself the corresponding statements for the sequences $1, \ldots, 1$!]

**Submission deadline:** Thursday, December 17, before midnight. Please send your solutions by email, or put them in my mailbox.

Note: Delayed exercises will in general be ignored. Exceptions are possible, but this requires that you have given me an explanation in advance, which I have approved.