SOME NOTES FOR THE COURSE "ANALYSIS FOR PHD STUDENTS" (2020)

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Contents

1. Sums and integrals

1.1. **Introductory examples.** Integration and summation are very closely related. Indeed, integrals are *defined* using sums. Furthermore, the general integral (cf., e.g., Folland Ch. 2) is a *generalization* of the concept of a sum; the latter is obtained from the former when the measure of integration is taken to be a counting measure. However in this first lecture I'd like to focus on some explicit connections between sums and the "elementary, first-year-calculus integral $\int f(x)\,dx$". Our focus will be on using integrals to estimate sums, since integrals are often easier to work with.

A well-known explicit connection between sums and integrals is the following:

Example 1.1. Let $M < N$ be integers and let f be any *increasing* function $[M-1, N+1] \to \mathbb{R}$. Then

$$\int_{M-1}^{N} f(x)\,dx \leq \sum_{n=M}^{N} f(n) \leq \int_{M}^{N+1} f(x)\,dx.$$

Indeed, "draw a picture"! A similar example: Suppose that f is any *convex* function $[M-\frac{1}{2}, N+\frac{1}{2}] \to \mathbb{R}$. Then

$$\sum_{n=M}^{N} f(n) \leq \int_{M-\frac{1}{2}}^{N+\frac{1}{2}} f(x)\,dx.$$

Indeed, again "draw a picture"!

Another familiar way in which integrals can sometimes be used to estimate sums is if the sum can be recognized as a *Riemann sum* (we will recall the definition of a Riemann integral using Riemann sums below; see Section 1.2). For example this method can be applied to the following question:

Example 1.2. Given a fixed number $\alpha > -1$, what is the asymptotic behavior of the sum $\sum_{n=1}^{N} n^\alpha$ as $N \to \infty$?

One solution is to rewrite the sum as

$$\sum_{n=1}^{N} n^\alpha = N^{\alpha+1} \sum_{n=1}^{N} \Big(\frac{n}{N}\Big)^\alpha \frac{1}{N}.$$

Here the right hand side can be recognized as a Riemann sum for the integral $\int_0^1 x^\alpha\,dx$, and from this we conclude that the sum tends to the value of $\int_0^1 x^\alpha\,dx$ as $N \to \infty$. Hence:

$$N^{-\alpha-1} \sum_{n=1}^{N} n^\alpha \to \int_0^1 x^\alpha\,dx = \frac{1}{\alpha+1}, \quad \text{as } N \to \infty.$$

The answer can be expressed:

$$\sum_{n=1}^{N} n^{\alpha} \sim \frac{N^{\alpha+1}}{\alpha+1} \qquad \text{as } N \to \infty. \tag{1.1}$$

Here we used the relation "$\sim$", which is defined as follows:

Definition 1.1. We write "$f(x) \sim g(x)$ as $x \to a$" to denote that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. Here a can be any real number, or $\pm\infty$. Note that this notation can only be used when $g(x) \neq 0$ for all x sufficiently near a.

Note that the same answer (1.1) could also be obtained, actually in a more precise form, using the technique of Example 1.1. Namely, let's *assume* $\alpha \geq 0$ so that the function $f(x) = x^{\alpha}$ is increasing (the other case $-1 < \alpha < 0$ can be treated similarly). Then

$$\int_0^N x^{\alpha}\, dx \leq \sum_{n=1}^{N} n^{\alpha} \leq \int_1^{N+1} x^{\alpha}\, dx,$$

that is:

$$\frac{N^{\alpha+1}}{\alpha+1} \leq \sum_{n=1}^{N} n^{\alpha} \leq \frac{(N+1)^{\alpha+1}-1}{\alpha+1} \qquad (\forall N \in \mathbb{N}). \tag{1.2}$$

This is clearly a more precise result than (1.1). We can deduce from (1.2) that $\sum_{n=1}^{N} n^{\alpha}$ equals $\frac{N^{\alpha+1}}{\alpha+1}$ plus a "lower order error", namely:

$$\sum_{n=1}^{N} n^{\alpha} = \frac{N^{\alpha+1}}{\alpha+1} + O(N^{\alpha}), \quad \forall N \in \mathbb{N} \quad (\text{for fixed } \alpha \geq 0). \tag{1.3}$$

Here the symbol "$O(\cdots)$" ("Big O") is defined as follows:

Definition 1.2. If a is a *non-negative* number, the symbol "$O(a)$" is used to denote any number b for which $|b| \leq Ca$, where C is a positive "constant", called *the implied constant.* We write "constant" within quotation marks since C is often allowed to depend on certain parameters.

We will discuss the "Big O" symbol and the implied constant more thoroughly in later lectures; for now we just give an exercise:

Exercise 1.1. Deduce (1.3) from (1.2).

(Note that we have to allow the implied constant in (1.3) to depend on α. But of course the implied constant is independent of N — this is the whole point of the statement (1.3)!)

We turn to a slightly different example:

Example 1.3. Assume that we are given an increasing sequence of positive numbers, $0 < \omega_1 \leq \omega_2 \leq \cdots$, which satisfy

$$\#\{n \in \mathbb{N} : \omega_n \leq T\} \sim cT^2 \qquad \text{as } T \to \infty, \tag{1.4}$$

where $c > 0$ is some constant. Then for which real numbers α do the series $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ converge? When convergence holds, can we estimate $\sum_{\omega_n > T} \omega_n^{-\alpha}$ as a function of T as $T \to \infty$?

(Notation: "$\sum_{\omega_n > T}$" means that we add over all n which satisfy the condition $\omega_n > T$.)

(To motivate the example, let us point out that the ω_n's may e.g. be the square roots of the non-zero eigenvalues of the Dirichlet problem for some bounded domain $\Omega \subset \mathbb{R}^2$ — in other words the eigenfrequencies of vibration of a given idealized "drum" in the plane. Then (1.4) is known to hold, with $c = (4\pi)^{-1}$, by the famous *Weyl's law.* In the study of such systems, sums like $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ are often of interest.)

Note that the sum $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ is a *positive* sum; each term is positive. If we only care about "order of magnitude", viz. if we are willing to sacrifice a numerical constant in our bounds, then questions about the asymptotic size of positive sums can often be answered using *dyadic decomposition.* We illustrate this for the first question in Example 1.3:

Clearly $\sum_{n=1}^{\infty} \omega_n^{-\alpha}$ *diverges* if $\alpha \leq 0$; hence from now on we may assume $\alpha > 0$. Our sum can be decomposed as:

$$\sum_{n=1}^{\infty} \omega_n^{-\alpha} = \sum_{\omega_n \leq 1} \omega_n^{-\alpha} + \sum_{m=0}^{\infty} \Big(\sum_{2^m < \omega_n \leq 2^{m+1}} \omega_n^{-\alpha} \Big). \tag{1.5}$$

(This is a *dyadic decomposition.*) Using $\alpha > 0$ we see that (1.5) is

$$\geq \sum_{m=0}^{\infty} \#\{2^m < \omega_n \leq 2^{m+1}\} \cdot 2^{-(m+1)\alpha}$$

and[1]

$$\leq \sum_{\omega_n \leq 1} \omega_n^{-\alpha} + \sum_{m=0}^{\infty} \#\{2^m < \omega_n \leq 2^{m+1}\} \cdot 2^{-m\alpha}.$$

The cardinalities appearing in these two bounds are precisely the cardinalities which (1.4) gives us information about! Namely, if we set

$$A(T) = \#\{\omega_n \leq T\} \qquad \text{for } T > 0,$$

[1]We here use the shorthand notation "$\{a < \omega_n \leq b\}$" for "$\{n \in \mathbb{N} : a < \omega_n \leq b\}$".

then the bounds which we pointed out above read:

$$\sum_{m=0}^{\infty}\big(A(2^{m+1})-A(2^m)\big)\cdot 2^{-(m+1)\alpha} \le \sum_{n=1}^{\infty}\omega_n^{-\alpha}$$
$$\le \sum_{\omega_n\le 1}\omega_n^{-\alpha}+\sum_{m=0}^{\infty}\big(A(2^{m+1})-A(2^m)\big)\cdot 2^{-m\alpha}, \tag{1.6}$$

and (1.4) says that $A(T)\sim cT^2$ as $T\to\infty$.

Let us note that apart from the sum $\sum_{\omega_n\le 1}\omega_n^{-\alpha}$ (which is finite since $A(1)$ is finite, by (1.4)), the lower and the upper bound in (1.6) only differ by the constant factor $2^{-\alpha}$. This is the central point of dyadic decomposition: In favorable situations the total contribution from each individual "dyadic interval" can be estimated from above and below by some simple expressions which only differ up to a multiplicative constant! (We could get rid of the sum $\sum_{\omega_n\le 1}\omega_n^{-\alpha}$ by applying dyadic decomposition also to the interval $0<\lambda\le 1$, i.e. writing the sum in (1.5) as $\sum_{m=-\infty}^{\infty}\sum_{2^m<\omega_n\le 2^{m+1}}\omega_n^{-\alpha}$; however we won't need this in the present discussion.)

Continuing, we note that $A(2^{m+1})-A(2^m)\ge 0$ for each $m\ge 0$, and from (1.4) it follows that

$$\#\big(A(2^{m+1})-A(2^m)\big)\sim 3c\cdot 2^{2m} \qquad \text{as } m\to\infty.$$

Using this and the bounds in (1.6), the convergence/divergence of $\sum_{n=1}^{\infty}\omega_n^{-\alpha}$ is seen to be equivalent to the convergence/divergence of the sum $\sum_{m=0}^{\infty}2^{2m}\cdot 2^{-\alpha m}$, and we thus conclude that $\sum_{n=1}^{\infty}\omega_n^{-\alpha}$ *converges when $\alpha>2$, and diverges when $\alpha\le 2$.*

Remark 1.3. Another quick way to get this answer goes via noticing that (1.4) actually implies $\omega_n\sim\sqrt{n/c}$ as $n\to\infty$.

We now move on to the second question in Example 1.3: For $\alpha>2$ we know that $\sum_{n=1}^{\infty}\omega_n^{-\alpha}$ converges, and hence $\sum_{\omega_n>T}\omega_n^{-\alpha}$ is a well-defined function of T (which is clearly positive, and decreasing). We now wish to give an asymptotic estimate of this sum as $T\to\infty$. For this we will use another very important method for the asymptotic study of sums: Consider the following way to rewrite $\sum_{\omega_n>T}\omega_n^{-\alpha}$ as an integral over the counting function $A(x)$ for $x\ge T$. Using $\omega_n^{-\alpha}=\int_{\omega_n}^{\infty}\alpha x^{-\alpha-1}\,dx$ we have

$$\sum_{\omega_n>T}\omega_n^{-\alpha}=\sum_{\omega_n>T}\int_{\omega_n}^{\infty}\alpha x^{-\alpha-1}\,dx=\int_T^{\infty}\sum_{\substack{n=1\\(T<\omega_n\le x)}}^{\infty}\alpha x^{-\alpha-1}\,dx$$
$$=\int_T^{\infty}\big(A(x)-A(T)\big)\alpha x^{-\alpha-1}\,dx. \tag{1.7}$$

(The change of order of summation here is permitted since all functions involved are nonnegative; indeed, write $\sum_{\omega_n > T} \int_{\omega_n}^{\infty} \alpha x^{-\alpha-1}\, dx$ as $\sum_{\omega_n > T} \int_{T}^{\infty} I(x > \omega_n)\, \alpha x^{-\alpha-1}\, dx$ and apply Folland's Theorem 2.15. [2])

Continuing from (1.7) we get:

$$\sum_{\omega_n > T} \omega_n^{-\alpha} = \int_T^{\infty} A(x) \alpha x^{-\alpha-1}\, dx - A(T) T^{-\alpha}. \tag{1.8}$$

Using here (1.4) we have:

$$\int_T^{\infty} A(x) \alpha x^{-\alpha-1}\, dx \sim \int_T^{\infty} cx^2 \cdot \alpha x^{-\alpha-1}\, dx = c \frac{\alpha T^{2-\alpha}}{\alpha - 2} \qquad \text{as } T \to \infty.$$

[Detailed proof of the last "$\sim$" relation: We know that $A(T) \sim cT^2$; hence given any $\varepsilon > 0$ there exists some $T_0 > 0$ such that $(c-\varepsilon)T^2 < A(T) < (c+\varepsilon)T^2$ for all $T \geq T_0$; hence for all $T \geq T_0$ we have $\int_T^{\infty} A(x) \alpha x^{-\alpha-1}\, dx \leq \int_T^{\infty} (c+\varepsilon) x^2 \cdot \alpha x^{-\alpha-1}\, dx = (c+\varepsilon) \frac{\alpha T^{2-\alpha}}{\alpha-2}$ and similarly $\int_T^{\infty} A(x) \alpha x^{-\alpha-1}\, dx \geq (c-\varepsilon) \frac{\alpha T^{2-\alpha}}{\alpha-2}$. The fact that this can be achieved for each $\varepsilon > 0$ leads to the desired conclusion.]

Furthermore in (1.8) we have $A(T)T^{-\alpha} \sim cT^{2-\alpha}$. Hence, since $\frac{\alpha}{\alpha-2} > 1$ and $\frac{\alpha}{\alpha-2} - 1 = \frac{2}{\alpha-2}$, we conclude:

$$\sum_{\omega_n > T} \omega_n^{-\alpha} \sim \frac{2c}{\alpha - 2} T^{2-\alpha} \qquad \text{as } T \to \infty.$$

This holds for any fixed $\alpha > 2$, and we have thus answered the second question in Example 1.3.

The computation in (1.7), (1.8) is very reminiscent of *integration by parts*, and in the next section will show that it is indeed a special case of integration by parts when viewed in the framework of the *Riemann-Stieltjes integral.* Namely, $\sum_{\omega_n > T} \omega_n^{-\alpha}$ can be expressed as $\int_T^{\infty} x^{-\alpha}\, dA(x)$, and integrating by parts we get $[A(x)x^{-\alpha}]_{x=T}^{x=\infty} + \alpha \int_T^{\infty} x^{-\alpha-1} A(x)\, dx$, i.e. the formula in (1.8)!

1.2. **The Riemann-Stieltjes Integral.** In this section we loosely follow [13, Appendix A].

Let us first recall the definition of the *Riemann integral* over a bounded interval:

[2]This is if we view the integrals as Lebesgue integrals; it is of course also possible to justify the present computation using only the Riemann integral.

Definition 1.4. Let real numbers $A < B$ and a function $g : [A, B] \to \mathbb{C}$ be given. We call a finite sequence $\{x_n\}_{n=0}^N$ is a *partition*[3] of $[A, B]$ if

$$A = x_0 \leq x_1 \leq \ldots \leq x_N = B. \tag{1.9}$$

For any partition $\{x_n\}_{n=0}^N$ of $[A, B]$ and any choice of numbers $\xi_n \in [x_{n-1}, x_n]$ for $n = 1, 2, \ldots, N$, we form the sum

$$S(\{x_n\}, \{\xi_n\}) = \sum_{n=1}^{N} g(\xi_n)(x_n - x_{n-1}). \tag{1.10}$$

We say that *the Riemann integral* $\int_A^B g(x)\, dx$ exists if there is some $I \in \mathbb{C}$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\big|S(\{x_n\}, \{\xi_n\}) - I\big| < \varepsilon \tag{1.11}$$

holds whenever $\{x_n\}$ and $\{\xi_n\}$ are as above and

$$\text{mesh}\{x_n\} = \max_{1 \leq n \leq N} (x_n - x_{n-1}) \leq \delta. \tag{1.12}$$

If this holds, then we also say that $\int_A^B g(x)\, dx$ *equals* I, and the function g is said to be *Riemann-integrable* on $[A, B]$.

It will be convenient in our discussion to call any pair of finite sequences $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ such that $\{x_n\}_{n=0}^N$ is a partition of $[A, B]$ and $\xi_n \in [x_{n-1}, x_n]$ for $n = 1, 2, \ldots, N$ a *"tagged partition of* $[A, B]$*";* we also agree that the mesh of $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ equals the mesh of $\{x_n\}$.

We will later give a precise criterion for which functions are Riemann-integrable; however let us already now point out the following fundamental result. We write $C([A, B])$ for the space of continuous functions $[A, B] \to \mathbb{C}$.

Theorem 1.5. *If* $g \in C([A, B])$ *then* g *is Riemann-integrable on* $[A, B]$.

This is a special case of Theorem 1.10 which we will prove below.

Let us also note:

Proposition 1.6. *If* $g : [A, B] \to \mathbb{C}$ *is Riemann integrable then* g *is bounded (that is, there exists some number* $M > 0$ *such that* $|g(x)| \leq M$ *for all* $x \in [A, B]$*).*

[3] Of course, this is *not* the standard notion of partition! Recall that the standard notion of a partition of a set X is: A family of nonempty subsets of X such that every element $x \in X$ belongs to exactly one of these subsets. However the two different usages of the word "partition" will not cause any confusion. Note also that the two concepts are related: If $\{x_n\}_{n=0}^N$ is a partition of $[A, B]$ in the sense of (1.9), then (e.g.) $\{[x_0, x_1), [x_1, x_2), \ldots, [x_{N-1}, X_N]\}$ is a partition of $[A, B]$ in the more standard sense.

Proof. Assume that g is *not* bounded. We will then prove that for *every* tagged partition $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ of $[A, B]$ there exists another sequence $\{\xi'_n\}_{n=1}^N$ such that also $\langle \{x_n\}_{n=0}^N, \{\xi'_n\}_{n=1}^N \rangle$ (with the *same* $\{x_n\}$!) is a tagged partition of $[A, B]$, and

$$\big|S(\{x_n\}, \{\xi_n\}) - S(\{x_n\}, \{\xi'_n\})\big| \geq 1. \tag{1.13}$$

Clearly this implies that g is *not* Riemann-integrable on $[A, B]$.

To prove the above claim, let $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ be a given tagged partition of $[A, B]$. Note that since g is not bounded, there is some $m \in \{1, \ldots, N\}$ such that the restriction of g to $[x_{m-1}, x_m]$ is not bounded. This implies that $x_{m-1} < x_m$ and that there is some $y \in [x_{m-1}, x_m]$ such that

$$|g(y)| \geq |g(\xi_m)| + (x_m - x_{m-1})^{-1},$$

and therefore

$$|g(y) - g(\xi_m)|(x_m - x_{m-1}) \geq 1.$$

Now define $\{\xi'_n\}_{n=1}^N$ by $\xi'_n = \xi_n$ for $n \neq m$ and $\xi'_m = y$. Then clearly $\langle \{x_n\}_{n=0}^N, \{\xi'_n\}_{n=1}^N \rangle$ is a tagged partition of $[A, B]$, and

$$\big|S(\{x_n\}, \{\xi_n\}) - S(\{x_n\}, \{\xi'_n\})\big| = \big|(g(\xi_m) - g(y))(x_m - x_{m-1})\big| \geq 1,$$

i.e. (1.13) holds. □

We next turn to the *Riemann-Stieltjes Integral* $\int_A^B g(x)\, df(x)$, which is a generalization of the Riemann integral. Intuitively, this integral is meant to give "$\int_A^B g(x) f'(x)\, dx$" (see Theorem 1.13 below for an aposteriori justification), but the integral exists also in many cases when $f'(x)$ does not exist for all x.

Definition 1.7. Let real numbers $A < B$ and two functions $f, g : [A, B] \to \mathbb{C}$ be given. For any tagged partition $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ of $[A, B]$, we form the sum

$$S(\{x_n\}, \{\xi_n\}) = \sum_{n=1}^{N} g(\xi_n)\big(f(x_n) - f(x_{n-1})\big). \tag{1.14}$$

We say that *the Riemann-Stieltjes integral* $\int_A^B g\, df = \int_A^B g(x)\, df(x)$ exists and has the value I if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\big|S(\{x_n\}, \{\xi_n\}) - I\big| < \varepsilon \tag{1.15}$$

whenever $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ is a tagged partition of $[A, B]$ of mesh $\leq \delta$.

Note that in the special case $f(x) = x$, Definition 1.7 specializes to Definition 1.4; hence the Riemann-Stieltjes integral is indeed a generalization of the Riemann integral!

Example 1.4. For any $A < B$ and any function $f : [A, B] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_A^B df(x)$ (viz. “$g \equiv 1$” in Definition 1.7) exists and equals $f(B) - f(A)$. This is trivial, since in this case $S(\{x_n\}, \{\xi_n\}) = f(B) - f(A)$ holds for *all* tagged partitions $\langle \{x_n\}, \{\xi_n\} \rangle$ of $[A, B]$.

Example 1.5. Let $A < B$, $g \in C([A, B])$, and assume that $f : [A, B] \to \mathbb{C}$ is piecewise constant, that is, there are numbers $A = x_0 < x_1 < x_2 < \ldots < x_n = B$ such that f is constant on each open interval (x_j, x_{j+1}), $j = 0, 1, \ldots, n-1$. Then

$$\begin{aligned} \int_A^B g\, df = \big(f(A+) - f(A)\big)g(A) + \sum_{j=1}^{n-1} \big(f(x_j+) - f(x_j-)\big)g(x_j) \\ + \big(f(B) - f(B-)\big)g(B), \end{aligned} \tag{1.16}$$

where

$$f(x+) = \lim_{t \to x^+} f(t) \qquad \text{and} \qquad f(x-) = \lim_{t \to x^-} f(t). \tag{1.17}$$

The proof is a simple exercise.

Example 1.6. One has to be careful when working with the general Riemann-Stieltjes integral, since some rules which are familiar from ordinary integrals may fail to hold in general. For example, it is *not* always true that if $A < C < B$ then $\int_A^B g(x)\, df(x) = \int_A^C g(x)\, df(x) + \int_C^B g(x)\, df(x)$! An example of this is the following: Suppose that

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise;} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.18}$$

Then $\int_{-1}^0 g\, df$ and $\int_0^1 g\, df$ both exist, but $\int_{-1}^1 g\, df$ does not exist! We leave the proof as an exercise. On the positive side, note that $\int_A^B g(x)\, df(x) = \int_A^C g(x)\, df(x) + \int_C^B g(x)\, df(x)$ holds whenever $\int_A^B g(x)\, df(x)$ exists (this can for example be easily proved using Lemma 1.11).

Unpleasant behavior such as in Example 1.6 typically arises in cases when both f and g have a common point of discontinuity.

A natural assumption when working with the Riemann-Stieltjes integral $\int_A^B g(x)\, df(x)$ is that f is of *bounded variation*. This concept is defined as follows:

Definition 1.8. If f is a function $f : [A, B] \to \mathbb{C}$, then the *variation of f over* $[A, B]$, $\mathrm{Var}_{[A,B]}(f)$, is defined by

$$\mathrm{Var}_{[A,B]}(f) = \sup \sum_{n=1}^{N} |f(x_n) - f(x_{n-1})|, \tag{1.19}$$

where the supremum is taken over all partitions $\{x_n\}_{n=0}^N$ of $[A, B]$. Thus $\text{Var}_{[A,B]}(f)$ is a well-defined number in $[0, \infty]$ (cf. Folland, Sec. 0.5). The function f is said to be of *bounded variation* if $\text{Var}_{[a,b]}(f) < \infty$. The space of all functions $f : [A, B] \to \mathbb{C}$ of bounded variation is denoted $BV([A, B])$.

The reader should check that the above definition agrees with that in Folland [5, p. 102]. Let us give an intuitive motivation of the above definition of the variation of f, following Folland [5, p. 101]: If $f(t)$ represents the position of a particle moving along the real line (or more generally in the complex plane) at time t, the "total variation" of f over the interval $[A, B]$ is the total distance traveled from time A to time B, as shown on an odometer. If f has a continuous derivative, this is just the integral of the "speed", $\int_A^B |f'(t)|\,dt$. The above definition of $\text{Var}_{[A,B]}(f)$ is simply the natural extension of "$\int_A^B |f'(t)|\,dt$" to the case when we have no smoothness hypothesis on f.

The assertion of the last sentence can be proved rigorously.

Proposition 1.9. *If $f \in C^1([A, B])$* [4] *then*

$$\text{Var}_{[A,B]}(f) = \int_A^B |f'(x)|\,dx. \tag{1.20}$$

In particular every function in $C^1([A, B])$ is of bounded variation, i.e. $C^1([A, B]) \subset BV([A, B])$.

We will prove Proposition 1.9 after the proof of our first main theorem:

Theorem 1.10. *Let $g \in C([A, B])$ and $f \in BV([A, B])$. Then the Riemann-Stieltjes integral $\int_A^B g\,df$ exists.*

To prepare for the proof, let us note a simple reformulation of the criterion for existence of $\int_A^B g(x)\,df(x)$:

Lemma 1.11. *The Riemann-Stieltjes integral $\int_A^B g\,df$ exists if and only if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for any two tagged partitions $\langle\{x_n\}, \{\xi_n\}\rangle$ and $\langle\{x'_n\}, \{\xi'_n\}\rangle$ of $[A, B]$, both having mesh $\leq \delta$, we have $|S(\{x_n\}, \{\xi_n\}) - S(\{x'_n\}, \{\xi'_n\})| < \varepsilon$.*

Proof. One direction is trivial: Namely, assume that $\int_A^B g\,df$ exists and equals I. Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that $|S(\{x_n\}, \{\xi_n\}) - I| < \varepsilon/2$ holds for any tagged partition $\langle\{x_n\}, \{\xi_n\}\rangle$

[4] As usual, $C^k([A, B])$ denotes the space of functions $f : [A, B] \to \mathbb{C}$ which are k times continuously differentiable, where at $x = A$ we only consider the *right* derivative(s), and at $x = B$ we only consider the *left* derivative(s).

of $[A, B]$ with mesh $\leq \delta$. Then if $\langle \{x_n\}, \{\xi_n\} \rangle$ and $\langle \{x'_n\}, \{\xi'_n\} \rangle$ are any two tagged partitions of $[A, B]$ both having mesh $\leq \delta$, we have

$$\begin{aligned} &|S(\{x_n\}, \{\xi_n\}) - S(\{x'_n\}, \{\xi'_n\})| \\ &\qquad \leq |S(\{x_n\}, \{\xi_n\}) - I| + |S(\{x'_n\}, \{\xi'_n\}) - I| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, assume that the condition given in the lemma holds. For each $j \in \mathbb{N}$, let us fix once and for all a tagged partition $\langle \{x_n^{(j)}\}, \{\xi_n^{(j)}\} \rangle$ of $[A, B]$ having mesh $\leq j^{-1}$, and set

$$I_j = S(\{x_n^{(j)}\}, \{\xi_n^{(j)}\}).$$

Then our assumption implies that $\{I_j\}_{j=1}^{\infty}$ is a Cauchy sequence! Hence

$$I = \lim_{j \to \infty} I_j \in \mathbb{R}$$

exists. Now let $\varepsilon > 0$ be given. Because of our assumption there exists some $\delta > 0$ such that $|S(\{x_n\}, \{\xi_n\}) - S(\{x'_n\}, \{\xi'_n\})| < \varepsilon/2$ holds whenever $\langle \{x_n\}, \{\xi_n\} \rangle$ and $\langle \{x'_n\}, \{\xi'_n\} \rangle$ are tagged partitions of $[A, B]$ having mesh $\leq \delta$. Now fix j so large that both $j^{-1} \leq \delta$ and $|I_j - I| < \varepsilon/2$ hold, and take $\langle \{x'_n\}, \{\xi'_n\} \rangle$ equal to $\langle \{x_n^{(j)}\}, \{\xi_n^{(j)}\} \rangle$ in the previous statement. The conclusion is that $|S(\{x_n\}, \{\xi_n\}) - I_j| < \varepsilon/2$ holds for any tagged partition $\langle \{x_n\}, \{\xi_n\} \rangle$ of $[A, B]$ having mesh $\leq \delta$. Hence also

$$|S(\{x_n\}, \{\xi_n\}) - I| \leq |S(\{x_n\}, \{\xi_n\}) - I_j| + |I_j - I| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that $\int_A^B g \, df$ exists and equals I. □

Proof of Theorem 1.10. Let $\varepsilon > 0$ be given. Since g is continuous on the closed and bounded interval $[A, B]$, g is *uniformly* continuous on $[A, B]$; hence there exists $\delta > 0$ such that

$$|g(x) - g(x')| < \varepsilon \qquad \text{for all } x, x' \in [a, b] \text{ with } |x - x'| \leq \delta. \tag{1.21}$$

We now claim that for any two tagged partitions $\langle \{x_n\}, \{\xi_n\} \rangle$ and $\langle \{x'_n\}, \{\xi'_n\} \rangle$ of $[A, B]$, both having mesh $\leq \delta$, we have

$$\Big| S(\{x_n\}, \{\xi_n\}) - S(\{x'_n\}, \{\xi'_n\}) \Big| \leq 2\varepsilon \operatorname{Var}_{[A,B]}(f). \tag{1.22}$$

This suffices to prove the existence of $\int_A^B g \, df$, by Lemma 1.11.

In order to prove (1.22), let us pick a tagged partition $\langle \{x''_n\}, \{\xi''_n\} \rangle$ of $[A, B]$ such that both $\{x_n\}$ and $\{x'_n\}$ are subsequences of $\{x''_n\}$. We will then prove that

$$\Big| S(\{x_n\}, \{\xi_n\}) - S(\{x''_n\}, \{\xi''_n\}) \Big| \leq \varepsilon \operatorname{Var}_{[A,B]}(f). \tag{1.23}$$

This will complete the proof, since exactly the same argument will also give (1.23) with $\langle\{x_n\},\{\xi_n\}\rangle$ replaced by $\langle\{x'_n\},\{\xi'_n\}\rangle$; and then (1.22) follows using the triangle inequality.

In order to prove (1.23), assume $\{x_n\}=\{x_n\}_{n=0}^N$ and $\{x''_n\}=\{x''_n\}_{n=0}^M$, and note that since $\{x_n\}_{n=0}^N$ is a subsequence of $\{x''_n\}_{n=0}^M$ there exist indices $0=k_0<k_1<\ldots<k_N=M$ such that $x_n=x''_{k_n}$ for $n=0,\ldots,N$. Now

$$\begin{aligned}
&S(\{x_n\},\{\xi_n\})-S(\{x''_n\},\{\xi''_n\})\\
&=\sum_{n=1}^N g(\xi_n)\big(f(x_n)-f(x_{n-1})\big)-\sum_{n=1}^M g(\xi''_n)\big(f(x''_n)-f(x''_{n-1})\big)\\
&=\sum_{n=1}^N\bigg(g(\xi_n)\big(f(x_n)-f(x_{n-1})\big)-\sum_{k=1+k_{n-1}}^{k_n} g(\xi''_k)\big(f(x''_k)-f(x''_{k-1})\big)\bigg)
\end{aligned}$$

(1.24)
$$=\sum_{n=1}^N\sum_{k=1+k_{n-1}}^{k_n}\big(g(\xi_n)-g(\xi''_k)\big)\big(f(x''_k)-f(x''_{k-1})\big),$$

where in the last equality we used the fact that for every $n\in\{1,\ldots,N\}$ we have

$$\sum_{k=1+k_{n-1}}^{k_n}(f(x''_k)-f(x''_{k-1}))=f(x''_{k_n})-f(x''_{k_{n-1}})=f(x_n)-f(x_{n-1}).$$

It follows from (1.24) that

$$\begin{aligned}
&\Big|S(\{x_n\},\{\xi_n\})-S(\{x''_n\},\{\xi''_n\})\Big|\\
&\leq\sum_{n=1}^N\sum_{k=1+k_{n-1}}^{k_n}\big|g(\xi_n)-g(\xi''_k)\big|\big|f(x''_k)-f(x''_{k-1})\big|.
\end{aligned}$$

Here for any pair $\langle n,k\rangle$ appearing in the sum we have $\xi_n\in[x_{n-1},x_n]$ and $\xi''_k\in[x''_{k-1},x''_k]\subset[x_{n-1},x_n]$, and hence

$$|\xi_n-\xi''_k|\leq|x_n-x_{n-1}|\leq\operatorname{mesh}\{x_n\}\leq\delta.$$

Hence by (1.21) we have $|g(\xi_n)-g(\xi''_k)|<\varepsilon$ for all $\langle n,k\rangle$ appearing in our sum, and we conclude

$$\begin{aligned}
\Big|S(\{x_n\},\{\xi_n\})-S(\{x''_n\},\{\xi''_n\})\Big|&\leq\varepsilon\sum_{n=1}^N\sum_{k=1+k_{n-1}}^{k_n}\big|f(x''_k)-f(x''_{k-1})\big|\\
&=\varepsilon\sum_{k=1}^M\big|f(x''_k)-f(x''_{k-1})\big|\leq\varepsilon\operatorname{Var}_{[A,B]}(f).
\end{aligned}$$

We have thus proved (1.23), and the proof of the theorem is complete. □

Proof of Proposition 1.9. Since $f \in C^1([A,B])$, the function $x \to |f'(x)|$ is continuous, and thus the Riemann integral $\int_A^B |f'(x)|\,dx$ exists by Theorem 1.5. Hence for any $\varepsilon > 0$ there is some $\delta > 0$ such that for any tagged partition $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N \rangle$ of $[A,B]$ of mesh $\leq \delta$ we have

$$\left| \sum_{n=1}^N (x_n - x_{n-1})|f'(\xi_n)| - \int_A^B |f'(x)|\,dx \right| < \varepsilon. \tag{1.25}$$

Furthermore since f' is uniformly continuous on $[A,B]$, by taking δ sufficiently small we can ensure that for any numbers $x \leq \xi \leq y$ in $[A,B]$ satisfying $y - x \leq \delta$ we have

$$|f(y) - f(x) - (y-x)f'(\xi)| \leq (y-x)\varepsilon. \tag{1.26}$$

((Let us recall a proof of the last statement: Since $f \in C^1([A,B])$ and $[A,B]$ is a closed and bounded interval, f' is uniformly continuous on $[A,B]$; thus we can take $\delta > 0$ so small that $|f'(\xi) - f'(\eta)| < \varepsilon/2$ for any $\xi, \eta \in [A,B]$ with $|\xi - \eta| \leq \delta$. Now let $x \leq \xi \leq y$ be arbitrary numbers in $[A,B]$ with $y - x \leq \delta$; then by the mean-value theorem applied to $\Re f$ and $\Im f$ there exist $\eta_1, \eta_2 \in [x,y]$ such that $\Re f(y) - \Re f(x) = (y-x)\Re f'(\eta_1)$ and $\Im f(y) - \Im f(x) = (y-x)\Im f'(\eta_2)$. But $|\eta_1 - \xi| \leq y - x \leq \delta$; thus $|f'(\eta_1) - f'(\xi)| < \varepsilon/2$; hence also $|\Re f'(\eta_1) - \Re f'(\xi)| < \varepsilon/2$ and $|\Re f(y) - \Re f(x) - (y-x)\Re f'(\xi)| \leq \frac{\varepsilon}{2}(y-x)$; similarly $|\Im f(y) - \Im f(x) - (y-x)\Im f'(\xi)| \leq \frac{\varepsilon}{2}(y-x)$; adding these two we obtain (1.26).))

By taking $\delta > 0$ so small that both the statements around (1.25) and (1.26) hold, then for any partition $\{x_n\}_{n=0}^N$ of $[A,B]$ of mesh $\leq \delta$,

$$\begin{aligned}
&\left| \sum_{n=1}^N |f(x_n) - f(x_{n-1})| - \int_A^B |f'(x)|\,dx \right| \\
&\qquad < \varepsilon + \left| \sum_{n=1}^N |f(x_n) - f(x_{n-1})| - \sum_{n=1}^N (x_n - x_{n-1})|f'(x_n)| \right| \\
&\qquad \leq \varepsilon + \sum_{n=1}^N \Big| f(x_n) - f(x_{n-1}) - (x_n - x_{n-1})f'(x_n) \Big| \\
&\qquad \leq \varepsilon + \sum_{n=1}^N (x_n - x_{n-1})\varepsilon = (1 + B - A)\varepsilon.
\end{aligned} \tag{1.27}$$

Such a $\delta > 0$ can be obtained for every $\varepsilon > 0$; this immediately implies that *the supremum in* (1.19) *is* $\geq \int_A^B |f'(x)|\,dx$. Now note also that if $\{x_n'\}_{n=0}^N$ is an arbitrary partition of $[A,B]$ then we can find another partition $\{x_n\}_{n=0}^M$ of $[A,B]$ of mesh $\leq \delta$ such that $\{x_n'\}_{n=0}^N$ is a subsequence of $\{x_n\}_{n=0}^M$. Then by the triangle inequality we have $\sum_{n=1}^N |f(x_n') - f(x_{n-1}')| \leq \sum_{n=1}^M |f(x_n) - f(x_{n-1})|$, and also (1.27) holds. Using this we conclude also that *the supremum in* (1.19) *is* $\leq \int_A^B |f'(x)|\,dx$, and the proposition is proved. $\square$

We next prove a formula for *integration by parts:*

Theorem 1.12. *For arbitrary functions f and $g : [A, B] \to \mathbb{C}$, if $\int_A^B g(x)\,df(x)$ exists then $\int_A^B f(x)\,dg(x)$ also exists, and*

$$\int_A^B g(x)\,df(x) = \Big(f(B)g(B) - f(A)g(A)\Big) - \int_A^B f(x)\,dg(x). \tag{1.28}$$

Proof. For any tagged partition $\langle \{x_n\}_{n=0}^N, \{\xi_n\}_{n=1}^N\rangle$ of $[A, B]$ we have the following identity, if we set $\xi_0 = A$ and $\xi_{N+1} = B$:

$$\sum_{n=1}^N g(\xi_n)\big(f(x_n) - f(x_{n-1})\big) = f(B)g(B) - f(A)g(A) - \sum_{n=1}^{N+1} f(x_{n-1})\big(g(\xi_n) - g(\xi_{n-1})\big).$$

Here note that $\langle \{\xi_n\}_{n=0}^{N+1}, \{x_{n-1}\}_{n=1}^{N+1}\rangle$ is also a tagged partition of $[A, B]$, since $x_{n-1} \in [\xi_{n-1}, \xi_n]$, and the sum on the right hand sum is a Riemann-Stieltjes sum $S(\{\xi_n\}, \{x_{n-1}\})$ approximating $\int_A^B f(x)\,dg(x)$, Moreover, $\text{mesh}\{\xi_n\} \leq 2\,\text{mesh}\{x_n\}$, so that the sum on the right tends to $\int_A^B f(x)\,dg(x)$ as $\text{mesh}\{x_n\}$ tends to 0. □

Recall that the intuition behind the definition of the Riemann-Stieltjes integral is that $\int_A^B g\,df$ should equal $\int_A^B g(x)f'(x)\,dx$ when g and f are nice functions. The following theorem shows that this holds in quite some generality:

Theorem 1.13. *Let $f \in C^1([A, B])$ and let $g : [A, B] \to \mathbb{C}$ be Riemann-integrable. Then the Riemann-Stieltjes integral $\int_A^B g(x)\,df(x)$ exists, the function $x \mapsto g(x)f'(x)$ is Riemann-integrable, and we have*

$$\int_A^B g(x)\,df(x) = \int_A^B g(x)f'(x)\,dx. \tag{1.29}$$

In order to prepare for the proof of Theorem 1.13 we first prove two propositions – which are also useful in their own right.

Proposition 1.14. *Let $A < B$ and let g be an arbitrary function $[A, B] \to \mathbb{C}$. Then g is Riemann integrable if and only if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for every partition $\{x_n\}_{n=0}^N$ of $[A, B]$ with $\text{mesh}\{x_n\} \leq \delta$ we have*

$$\sum_{n=1}^N (x_n - x_{n-1}) \cdot \sup\big\{|g(\xi) - g(\xi')| \;:\; \xi, \xi' \in [x_{n-1}, x_n]\big\} \leq \varepsilon. \tag{1.30}$$

(Note that for any every partition $\{x_n\}$ of $[A, B]$, the left hand side of (1.30) is a well-defined number in $[0, \infty]$; cf. Folland, Sec. 0.5.)

Proof. Assume first that the stated condition holds. Let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that (1.30) holds for all partitions $\{x_n\}_{n=0}^N$

of $[A, B]$ with $\text{mesh}\{x_n\} \leq \delta$. We then claim that $|S(\{x_n\}, \{\xi_n\}) - S(\{x''_n\}, \{\xi''_n\})| \leq \varepsilon$ holds whenever $\langle\{x_n\}, \{\xi_n\}\rangle$ and $\langle\{x''_n\}, \{\xi''_n\}\rangle$ are tagged partitions of $[A, B]$ with mesh $\leq \delta$ such that $\{x_n\}$ is a subsequence of $\{x''_n\}$. Note that this suffices to show that g is Riemann integrable, by the same argument as in the proof of Theorem 1.10.

To prove the claim, note that if $\langle\{x_n\}, \{\xi_n\}\rangle$ and $\langle\{x''_n\}, \{\xi''_n\}\rangle$ are as above then we have, using the same notation as in the proof of Theorem 1.10 (see (1.24), but now with $f(x) \equiv x$):

$$\begin{aligned}
&\big|S(\{x_n\}, \{\xi_n\}) - S(\{x''_n\}, \{\xi''_n\})\big| \\
&= \Big|\sum_{n=1}^{N} \sum_{k=1+k_{n-1}}^{k_n} \big(g(\xi_n) - g(\xi''_k)\big)\big(x''_k - x''_{k-1}\big)\Big| \\
&\leq \sum_{n=1}^{N} \sum_{k=1+k_{n-1}}^{k_n} \big(x''_k - x''_{k-1}\big) \cdot \sup\big\{|g(\xi) - g(\xi')| \,:\, \xi, \xi' \in [x_{n-1}, x_n]\big\} \\
&= \sum_{n=1}^{N} \big(x_n - x_{n-1}\big) \cdot \sup\big\{|g(\xi) - g(\xi')| \,:\, \xi, \xi' \in [x_{n-1}, x_n]\big\} \leq \varepsilon,
\end{aligned}$$

and the claim is proved.

Conversely, assume that g is Riemann-integrable. Let $\varepsilon > 0$ be given. Then by Lemma 1.11 there is some $\delta > 0$ such that for any two tagged partitions $\langle\{x_n\}, \{\xi_n\}\rangle$ and $\langle\{x'_n\}, \{\xi'_n\}\rangle$ of $[A, B]$, both having mesh $\leq \delta$, we have $|S(\{x_n\}, \{\xi_n\}) - S(\{x'_n\}, \{\xi'_n\})| < \varepsilon/2$. Applying this in particular when $\{x_n\} = \{x'_n\}$ and considering the real part, it follows that if $\{x_n\}_{n=0}^N$ is any partition of $[A, B]$ with mesh $\leq \delta$, then

$$\sum_{n=1}^{N} (x_n - x_{n-1}) \cdot \Re(g(\xi_n) - g(\xi'_n)) < \frac{\varepsilon}{2}$$

for all choices of $\{\xi_n\}_{n=1}^N$ and $\{\xi'_n\}_{n=1}^N$ with $\xi_n, \xi'_n \in [x_{n-1}, x_n]$. Hence also

$$\sum_{n=1}^{N} (x_n - x_{n-1}) \cdot \sup\big\{\Re(g(\xi) - g(\xi')) \,:\, \xi, \xi' \in [x_{n-1}, x_n]\big\} \leq \frac{\varepsilon}{2} \tag{1.31}$$

Similarly one proves

$$\sum_{n=1}^{N} (x_n - x_{n-1}) \cdot \sup\big\{\Im(g(\xi) - g(\xi')) \,:\, \xi, \xi' \in [x_{n-1}, x_n]\big\} \leq \frac{\varepsilon}{2} \tag{1.32}$$

Note also that if F is any set of complex numbers satisfying $z \in F \Rightarrow -z \in F$ then $\sup\{|z| \,:\, z \in F\} \leq \sup\{\Re z \,:\, z \in F\} + \sup\{\Im z \,:\, z \in F\}$. Applying this with $F = \{g(\xi) - g(\xi') \,:\, \xi, \xi' \in [x_{n-1}, x_n]\}$ for each n and using (1.31) and (1.32) we conclude that (1.30) holds. $\square$

Proposition 1.15. *Let $A < B$ and let $f, g : [A, B] \to \mathbb{C}$ be two Riemann-integrable functions. Then also the (pointwise) product function fg is Riemann-integrable on $[A, B]$.*

Proof. By Proposition 1.6 both f and g are bounded, i.e. there exists some $M > 0$ such that $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in [A, B]$. Now the Riemann-integrability of fg follows by using the criterion in Proposition 1.14 and the inequality

$$\begin{aligned}\big|f(\xi)g(\xi) - f(\xi')g(\xi')\big| &\leq \big|f(\xi) - f(\xi')\big||g(\xi)| + |f(\xi')|\big|g(\xi) - g(\xi')\big| \\ &\leq M\Big(\big|f(\xi) - f(\xi')\big| + \big|g(\xi) - g(\xi')\big|\Big).\end{aligned}$$

□

Proof of Theorem 1.13. Take $M > 0$ such that $|g(x)| \leq M$ for all $x \in [A, B]$ (this is possible by Proposition 1.6). The fact that $x \mapsto g(x)f'(x)$ is Riemann integrable follows from Theorem 1.5 and Proposition 1.15. Let us write $S_1(\{x_n\}, \{\xi_n\})$ for the Riemann sum corresponding to $\int_A^B g(x)\, df(x)$, and $S_2(\{x_n\}, \{\xi_n\})$ for the Riemann sum corresponding to $\int_A^B g(x)f'(x)\, dx$.

Let $\varepsilon > 0$ be given. We can now choose $\delta > 0$ so small that $|S_2(\{x_n\}, \{\xi_n\}) - \int_A^B g(x)f'(x)\, dx| < \varepsilon$ holds for any tagged partition $\langle\{x_n\}, \{\xi_n\}\rangle$ of $[A, B]$ of mesh $\leq \delta$, and also

$$|f(y) - f(x) - (y - x)f'(\xi)| \leq (y - x)\varepsilon$$

for any numbers $x \leq \xi \leq y$ in $[A, B]$ satisfying $y - x \leq \delta$ (see below (1.26) for a proof of the latter.) Now let $\langle\{x_n\}, \{\xi_n\}\rangle$ be an arbitrary tagged partition of $[A, B]$ of mesh $\leq \delta$. Then

$$\begin{aligned}&\big|S_1(\{x_n\}, \{\xi_n\}) - \textstyle\int_A^B g(x)f'(x)\, dx\big| \\ &\leq \Big|S_2(\{x_n\}, \{\xi_n\}) - \textstyle\int_A^B g(x)f'(x)\, dx\Big| + \Big|S_1(\{x_n\}, \{\xi_n\}) - S_2(\{x_n\}, \{\xi_n\})\Big| \\ &< \varepsilon + \left|\sum_{n=1}^N \Big(g(\xi_n)(f(x_n) - f(x_{n-1})) - g(\xi_n)f'(\xi_n)(x_n - x_{n-1})\Big)\right| \\ &\leq \varepsilon + \sum_{n=1}^N \big|g(\xi_n)\big| \cdot \Big|f(x_n) - f(x_{n-1}) - (x_n - x_{n-1})f'(\xi_n)\Big| \\ &< \varepsilon + \sum_{n=1}^N M(x_n - x_{n-1})\varepsilon = (1 + M(B - A))\varepsilon.\end{aligned}$$

This proves that the Riemann-Stieltjes integral $\int_A^B g(x)\, df(x)$ exists and equals $\int_A^B g(x)f'(x)\, dx$. □

Example 1.7. Assume $A < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq B$; let $c_1, c_2, \ldots, c_m \in \mathbb{C}$, and set

$$f(x) = \sum_{\lambda_n \leq x} c_n$$

(the notation indicates a summation over the finite set of those $n \in \mathbb{N}$ for which $\lambda_n \leq x$). Then, if $g \in C^1([A, B])$, we have

$$\sum_{n=1}^{m} c_n g(\lambda_n) = f(B)g(B) - \int_A^B f(x)g'(x)\, dx.$$

Indeed,

$$\sum_{n=1}^{m} c_n g(\lambda_n) = \int_A^B g\, df = f(B)g(B) - f(A)g(A) - \int_A^B f\, dg$$
$$= f(B)g(B) - \int_A^B f(x)g'(x)\, dx,$$

where the first equality holds by Example 1.5, the second by Theorem 1.12, and the last equality holds by Theorem 1.13 (using also $f(A) = 0$).

In order to make the notation really flexible we also need the following definition of generalized Riemann-Stieltjes integrals.

Definition 1.16. We define the *generalized Riemann-Stieltjes integral*

$$\int_{A+}^{B} g(x)\, df(x) := \lim_{a \to A^+} \int_a^B g(x)\, df(x), \tag{1.33}$$

provided that $\int_a^B g(x)\, df(x)$ exists for all $a > A$ sufficiently near A.

Similarly we define

$$\int_{A-}^{B} g(x)\, df(x) := \lim_{a \to A^-} \int_a^B g(x)\, df(x); \tag{1.34}$$

$$\int_{-\infty}^{B} g(x)\, df(x) := \lim_{a \to -\infty} \int_a^B g(x)\, df(x), \tag{1.35}$$

Also, the generalized Riemann-Stieltjes integrals $\int_A^{B-} g(x)\, df(x)$, $\int_A^{B+} g(x)\, df(x)$ and $\int_A^{\infty} g(x)\, df(x)$ are defined in the analogous way.

Finally generalized Riemann-Stieltjes integrals with limits on both end-points are defined in the natural way, i.e.

$$\int_{A-}^{B+} g(x)\, df(x) := \lim_{b \to B^+} \lim_{a \to A^-} \int_a^b g(x)\, df(x); \tag{1.36}$$

$$\int_{-\infty}^{B-} g(x)\, df(x) := \lim_{b \to B^-} \lim_{a \to -\infty} \int_a^b g(x)\, df(x), \tag{1.37}$$

etc.

Remark 1.17. In (1.36) (and similarly in any of the other cases with limits on both end-points) it does not matter if the limit is considered as an iterated limit (in either order) or as a simultanous limit in a, b; if one of these limits exist (as a finite real number) then so do the other ones. This follows by fixing an arbitrary number $C \in (A, B)$ and using $\int_a^b g(x)\,df(x) = \int_a^C g(x)\,df(x) + \int_C^b g(x)\,df(x)$ inside the limit.

Example 1.8. Let $a_1, a_2, \ldots$ be any sequence of complex numbers, and set $f(x) = \sum_{1\leq n<x} a_n$. Also let $g \in C(\mathbb{R}^+)$. We then have, for any integers $1 \leq M \leq N$:

$$\sum_{n=M}^{N} a_n g(n) = \int_M^{N+} g(x)\,df(x) = \int_M^{N+\frac{1}{2}} g(x)\,df(x). \tag{1.38}$$

Hence also

$$\sum_{n=M}^{\infty} a_n g(n) = \int_M^{\infty} g(x)\,df(x). \tag{1.39}$$

On the other hand, if we set $f_1(x) = \sum_{1\leq n\leq x} a_n$ (thus $f_1(x) = f(x)$ except when x is an integer) then

$$\sum_{n=M}^{N} a_n g(n) = \int_{M-}^{N} g(x)\,df_1(x) = \int_{M-\frac{1}{2}}^{N} g(x)\,df_1(x) \tag{1.40}$$

and

$$\sum_{n=M}^{\infty} a_n g(n) = \int_{M-}^{\infty} g(x)\,df_1(x). \tag{1.41}$$

1.3. **Example: Euler-MacLaurin summation.** (The following presentation is partly influenced by Olver, [16, Ch. 8].) We will now discuss how the Riemann-Stieltjes integral can be used together with integration by parts to give increasingly precise estimates of a sum $\sum_{n=M}^{N} f(n)$ where M and N are integers, $M < N$, and f is a given (nice, not wildly oscillating) function on $[M, N]$. Actually let us consider instead the sum

$$\sum_{A<n\leq B} f(n),$$

where $A < B$ are arbitrary *real* numbers (and it is understood that the sum is taken over all *integers* n satisfying $A < n \leq B$).[5]

Referring to Example 1.5 we see that this sum can be expressed as

$$\int_A^B f(x)\,d\lfloor x\rfloor, \tag{1.42}$$

[5]This sum is of course not at all more general than "$\sum_{n=M}^{N} f(n)$" but the treatment becomes, in my opinion, slightly clearer in this way.

where $\lfloor x \rfloor$ is the "floor function", i.e. $\lfloor x \rfloor$ is the largest integer $\leq x$. (Make sure to think this through; in particular check that we do get the correct contributions at $x = A$ and $x = B$, if A or B happen to be an integer.)

Applying integration by parts (Theorem 1.12 and then Theorem 1.13) we get, assuming $f \in C^1([A, B])$:

(1.43)
$$\sum_{A<n\leq B} f(n) = \int_A^B f(x)\, d\lfloor x \rfloor = \Big[f(x) \lfloor x \rfloor \Big]_{x=A}^{x=B} - \int_A^B f'(x) \lfloor x \rfloor\, dx.$$

It is natural to compare $\sum_{A<n\leq B} f(n)$ with $\int_A^B f(x)\, dx$. Applying the analogous integration by parts for $\int_A^B f(x)\, dx$ we have

$$\int_A^B f(x)\, dx = \Big[f(x)(x-K) \Big]_A^B - \int_A^B f'(x)(x-K)\, dx, \tag{1.44}$$

for any constant K. (We used the fact that the most general primitive function of "1" is "$x - K$".) Combining (1.44) and (1.43) we conclude

$$\sum_{A<n\leq B} f(n) = \int_A^B f(x)\, dx - \Big[f(x)\big(x - \lfloor x \rfloor - K\big) \Big]_{x=A}^{x=B}$$
$$+ \int_A^B f'(x)\big(x - \lfloor x \rfloor - K\big)\, dx. \tag{1.45}$$

In order to make use of this formula we have to understand the last term, $\int_A^B f'(x)(x - \lfloor x \rfloor - K)\, dx$. Note that the function $x \mapsto x - \lfloor x \rfloor - K$ is oscillating around the mean value $\frac{1}{2} - K$. Now there's a general principle that *when dealing with an integral $\int_A^B h(x) g(x)\, dx$, where $h(x)$ is "slowly varying" while $g(x)$ is oscillating with mean value 0, it is often advantageous to* ***integrate by parts****: $\int_A^B h(x) g(x)\, dx = [h(x) G(x)]_{x=A}^{x=B} - \int_A^B h'(x) G(x)\, dx$ (where G is a primitive function of x); the point is that $\int_A^B h'(x) G(x)\, dx$ can here typically be expected to be* ***comparatively small!***

Applying this principle to $\int_A^B f'(x)\big(x - \lfloor x \rfloor - K\big)\, dx$ we see that we should take $K = \frac{1}{2}$ and then integrate by parts, assuming now $f \in C^2([A, B])$. We then need to compute the primitive function of $x - \lfloor x \rfloor - \frac{1}{2}$. (This function is sometimes called the *saw-tooth* function; draw a picture!) It is convenient to set

$$\omega_1(x) = x - \tfrac{1}{2} \quad \text{and} \quad \widetilde{\omega}_1(x) = \omega_1(x - \lfloor x \rfloor) = x - \lfloor x \rfloor - \tfrac{1}{2}.$$

(Thus $\widetilde{\omega}_1(x)$ is the periodic function with period one which agrees with $\omega_1(x)$ for $x \in [0, 1)$.) We note that for $x \in [0, 1]$ we have $\int_0^x \widetilde{\omega}_1(x_1)\, dx_1 = \int_0^x \omega_1(x_1)\, dx_1 = \frac{1}{2}(x^2 - x)$. Since $\widetilde{\omega}_1(x_1)$ is periodic with period one and $\int_0^1 \widetilde{\omega}_1(x_1)\,dx_1 = 0$, it follows that for general $x \in \mathbb{R}$, $\int_0^x \widetilde{\omega}_1(x_1)\,dx_1$ equals the periodic function with period one which agrees with $\frac{1}{2}(x^2-x)$ for $x \in [0,1)$; thus $\int_0^x \widetilde{\omega}_1(x_1)\,dx_1 = \widetilde{\tau}(x)$, where

$$\tau(x) = \tfrac{1}{2}(x^2 - x) \quad \text{and} \quad \widetilde{\tau}(x) = \tau(x - \lfloor x \rfloor).$$

Hence we have

$$\begin{aligned} \int_A^B f'(x)\big(x - \lfloor x \rfloor - \tfrac{1}{2}\big)\,dx &= \int_A^B \widetilde{\omega}_1(x) f'(x)\,dx \\ &= \Big[\big(\widetilde{\tau}(x) - K\big) f'(x)\Big]_{x=A}^{x=B} - \int_A^B (\widetilde{\tau}(x) - K) f''(x)\,dx, \end{aligned} \tag{1.46}$$

where K is an arbitrary constant (it does not have to be the same as our previous K).

This procedure can now be repeated: In order for the periodic function $\widetilde{\tau}(x) - K$ to have mean-value zero we should take $K = \int_0^1 \tau(x)\,dx = -\frac{1}{12}$; thus we set

$$\omega_2(x) = \tfrac{1}{2}(x^2 - x + \tfrac{1}{6}) \quad \text{and} \quad \widetilde{\omega}_2(x) = \omega_2(x - \lfloor x \rfloor).$$

Then $\widetilde{\omega}_2(x)$ is periodic with period one and $\int_0^1 \widetilde{\omega}_2(x)\,dx = 0$; the above formula reads

$$\int_A^B \widetilde{\omega}_1(x) f'(x)\,dx = \Big[\widetilde{\omega}_2(x) f'(x)\Big]_{x=A}^{x=B} - \int_A^B \widetilde{\omega}_2(x) f''(x)\,dx.$$

The rth step of this procedure ($r \in \mathbb{N}$) is to let $\omega_{r+1}(x)$ be that primitive function of $\omega_r(x)$ which satisfies $\int_0^1 \omega_{r+1}(x)\,dx = 0$; then set $\widetilde{\omega}_{r+1}(x) = \omega_{r+1}(x - \lfloor x \rfloor)$, and note that (if $f \in C^{r+1}([A,B])$):

$$\int_A^B \widetilde{\omega}_r(x) f^{(r)}(x)\,dx = \Big[\widetilde{\omega}_{r+1}(x) f^{(r)}(x)\Big]_{x=A}^{x=B} - \int_A^B \widetilde{\omega}_{r+1}(x) f^{(r+1)}(x)\,dx.$$

The result can be collected as follows: If $h \in \mathbb{N}$ and $f \in C^h([A,B])$, then

$$\begin{aligned} \sum_{A<n\leq B} f(n) = \int_A^B f(x)\,dx + \sum_{r=1}^{h} (-1)^r \Big[\widetilde{\omega}_r(x) f^{(r-1)}(x)\Big]_{x=A}^{x=B} \\ +(-1)^{h-1} \int_A^B \widetilde{\omega}_h(x) f^{(h)}(x)\,dx. \end{aligned} \tag{1.47}$$

Here from the above recursion formula we see that $\omega_r(x)$ is a polynomial of degree r (with x^r-coefficient $= r!^{-1}$). We compute:

$$\begin{aligned} \omega_1(x) &= x - \tfrac{1}{2} \\ \omega_2(x) &= \tfrac{1}{2}(x^2 - x + \tfrac{1}{6}) \\ \omega_3(x) &= \tfrac{1}{6}(x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x). \\ \omega_4(x) &= \tfrac{1}{24}(x^4 - 2x^3 + x^2 - \tfrac{1}{30}). \end{aligned}$$

It is customary to use a slightly different normalization: The *rth Bernoulli polynomial* is given by $B_r(x) = r! \cdot \omega_r(x)$. Thus from the above discussion we see that we can define $B_r(x)$ as follows (we extend to the case $r = 0$ in a natural way).

Definition 1.18. The *Bernoulli polynomials* $B_0(x)$, $B_1(x)$, $B_2(x)$, ..., are defined by $B_0(x) = 1$ and recursively by the relations $B_r'(x) = rB_{r-1}(x)$ and $\int_0^1 B_r(x)\,dx = 0$ for $r = 1, 2, 3, \ldots$. The rth *Bernoulli number* is defined by $B_r = B_r(0)$.

We have now proved (see (1.47)):

Theorem 1.19. ***The Euler-MacLaurin summation formula.*** *Let $A < B$ be real numbers, $h \in \mathbb{N}$ and $f \in C^h([A, B])$. Then*

$$\sum_{A<n\leq B} f(n) = \int_A^B f(x)\,dx + \sum_{r=1}^{h} \frac{(-1)^r}{r!}\Big[\widetilde{B}_r(x) f^{(r-1)}(x)\Big]_{x=A}^{x=B}$$

$$+(-1)^{h-1}\int_A^B \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx, \tag{1.48}$$

where $\widetilde{B}_r(x) = B_r(x - \lfloor x \rfloor)$.

The first Bernoulli polynomials are:

$$\begin{aligned}
B_0(x) &= 1\\
B_1(x) &= x - \tfrac{1}{2}\\
B_2(x) &= x^2 - x + \tfrac{1}{6}\\
B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x.\\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30}.
\end{aligned}$$

It follows immediately from the recursion formula that $B_r(1 - x) = B_r(x)$ for all even r and $B_r(1-x) = -B_r(x)$ for all odd r; also $B_r(0) = B_r(1) = 0$ for all odd $r \geq 3$. Furthermore, the periodized function $\widetilde{B}_r(x)$ is continuous for all $r \geq 2$.

The Euler-MacLaurin summation formula is very useful for obtaining asymptotic expansions of sums. For example we will see later how it is used to derive Stirling's formula for the Γ-function $\Gamma(z)$, with an error term with arbitrary power rate decay as $|z| \to \infty$. At present we content ourselves by giving a single example:

Example 1.9. Recall Example 1.2, the question about the asymptotic behavior of the sum $\sum_{n=1}^{N} n^\alpha$ for fixed $\alpha > -1$. We can use the Euler-MacLaurin summation formula to attack this question for an *arbitrary complex* α.

Indeed, by Theorem 1.19 applied with $f(x) = x^\alpha$, $A < 1$ tending to 1 and $B = N$, we have (using $\widetilde{B}_r(N) = B_r$ for $r \geq 1$ and $\widetilde{B}_r(1-) = B_r$ for $r \geq 2$ while $\widetilde{B}_1(1-) = \frac{1}{2} = 1 + B_1$):

$$\sum_{n=1}^{N} n^\alpha = \int_1^N x^\alpha \, dx + 1 + \sum_{r=1}^{h} \frac{(-1)^r B_r}{r!} \Big(f^{(r-1)}(N) - f^{(r-1)}(1) \Big) + (-1)^{h-1} \int_1^N \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x) \, dx.$$

Using $f^{(r)}(x) = \alpha(\alpha-1)\cdots(\alpha-r+1)x^{\alpha-r}$ we see that for $\alpha \neq -1$ the above can be expressed as

$$\sum_{n=1}^{N} n^\alpha = \frac{1}{\alpha+1}\Big(N^{\alpha+1} - 1\Big) + 1 + \frac{1}{\alpha+1} \sum_{r=1}^{h} (-1)^r B_r \binom{\alpha+1}{r} \Big(N^{\alpha+1-r} - 1 \Big) + (-1)^{h-1} \binom{\alpha}{h} \int_1^N \widetilde{B}_h(x) x^{\alpha-h} \, dx. \tag{1.49}$$

Since $|\widetilde{B}_h(x)|$ is bounded above by a constant which only depends on h, we see that the last integral is $O(N^{\Re\alpha - h+1})$ if $\Re\alpha > h-1$, $O(\log N)$ if $\Re\alpha = h-1$, and $O(1)$ if $\Re\alpha < h-1$ (the implied constant may depend on α and h but not on N). In particular if $\Re\alpha > 0$ then this leads to a more precise asymptotic formula than (1.3)! For a concrete example, say $\alpha = \frac{3}{2}$; then taking $h = 3$ above we get:

$$\sum_{n=1}^{N} n^{\frac{3}{2}} = \frac{2}{5} N^{\frac{5}{2}} + \frac{1}{2} N^{\frac{3}{2}} + \frac{1}{8} N^{\frac{1}{2}} + O(1).$$

Numerical example: For $N = 1000$ the left hand side equals $S = 12664925.95633\ldots$ and we find that $S - \frac{2}{5}N^{\frac{5}{2}} = 15815.3\ldots$, $S - (\frac{2}{5}N^{\frac{5}{2}} + \frac{1}{2}N^{\frac{3}{2}}) = 3.927\ldots$ and $S - (\frac{2}{5}N^{\frac{5}{2}} + \frac{1}{2}N^{\frac{3}{2}} + \frac{1}{8}N^{\frac{1}{2}}) = -0.0254\ldots$. In fact trying also $N = 10^4, 10^5, 10^6, \ldots$ it seems as if the difference $\sum_{n=1}^{N} n^\alpha - (\frac{2}{5}N^{\frac{5}{2}} + \frac{1}{2}N^{\frac{3}{2}} + \frac{1}{8}N^{\frac{1}{2}})$ tends to a number $-0.025485\ldots$ as $N \to \infty$. This will be explained below.

Note that once $h > \Re\alpha + 1$, we do not get any better power of N in the error term by increasing h further! This is easy to fix: If $h > \Re\alpha + 1$ then the integral $\int_1^\infty \widetilde{B}_h(x) x^{\alpha-h} \, dx$ is absolutely convergent and hence we can express the last term in (1.49) as

$$(-1)^{h-1} \int_1^\infty \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x) \, dx - (-1)^{h-1} \int_N^\infty \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x) \, dx.$$

Here the first integral is a constant independent of N, and to the second integral we can now apply repeated integration by parts just as in the proof of the Euler-MacLaurin formula to obtain, for an arbitrary integer

$k \geq h$:

$$= (-1)^{h-1} \int_1^\infty \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx + \sum_{r=h+1}^{k} \frac{(-1)^r B_r}{r!} f^{(r-1)}(N) \\ + (-1)^k \int_N^\infty \frac{\widetilde{B}_k(x)}{k!} f^{(k)}(x)\,dx,$$

and the good thing is that the last integral is $O(N^{\Re\alpha-k+1})$! (This is since $|\widetilde{B}_k(x)| = O(1)$ and $|f^{(k)}(x)| = O(x^{\Re\alpha-k})$ for all $x \geq 1$, and $k \geq h > \Re\alpha + 1$.)

Using this in (1.49), we obtain:

$$\sum_{n=1}^{N} n^\alpha = \frac{1}{\alpha+1} \sum_{r=0}^{k} (-1)^r B_r \binom{\alpha+1}{r} N^{\alpha+1-r} + C(\alpha) \\ + (-1)^k \binom{\alpha}{k} \int_N^\infty \widetilde{B}_k(x) x^{\alpha-k}\,dx, \tag{1.50}$$

where

$$C(\alpha) = 1 - \frac{1}{\alpha+1} \sum_{r=0}^{h} (-1)^r B_r \binom{\alpha+1}{r} + (-1)^{h-1} \binom{\alpha}{h} \int_1^\infty \widetilde{B}_h(x) x^{\alpha-h}\,dx. \tag{1.51}$$

Recall that this formula is valid for any complex $\alpha \neq -1$ and any $k, h \in \mathbb{N}$ satisfying $k \geq h > \Re\alpha+1$. The point of separating out the term $C(\alpha)$ is that this term does not depend on N, i.e. it appears as a *constant* in our asymptotic expansion as $N \to \infty$ for fixed α! Note that $C(\alpha)$ is independent of h since all the other terms in (1.50) are independent of h; this can of course also be seen easily by using integration by parts in (1.51). Note also that in (1.50) we have incorporated the term $\frac{1}{\alpha+1} N^{\alpha+1}$ in the r-sum by letting it start at $r = 0$.

The constant $C(\alpha)$ can easily be computed in practice (with rigorous error bounds) by evaluating the two sums in (1.50) for a modest value of N and an appropriate k, and bounding the last integral using simply $|\widetilde{B}_k(x)| \leq \sup_{x\in[0,1]} |B_k(x)|$. As a concrete example, for $\alpha = \frac{3}{2}$ and $k = 5$, (1.50) gives

$$\sum_{n=1}^{N} n^{\frac{3}{2}} = C(\tfrac{3}{2}) + \tfrac{2}{5} N^{\frac{5}{2}} + \tfrac{1}{2} N^{\frac{3}{2}} + \tfrac{1}{8} N^{\frac{1}{2}} + \tfrac{1}{1920} N^{-\frac{3}{2}} + O(N^{-\frac{5}{2}}),$$

and numerical evaluation for $N = 10^4, 10^5, 10^6$ strongly suggests that $C(\alpha) = -0.02548520188983303\ldots$.

Analytically, we can relate $C(\alpha)$ to the *Riemann zeta function!*[6] Namely, (recall that) the Riemann zeta function is defined by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } s \in \mathbb{C},\ \Re s > 1.$$

This sum is absolutely convergent, uniformly on compact subsets of $\{s \in \mathbb{C} \,:\, \Re s > 1\}$; hence $\zeta(s)$ is an analytic function in the region $\{s \in \mathbb{C} \,:\, \Re s > 1\}$. To see the connection, take $N \to \infty$ in (1.50) to conclude

$$C(\alpha) = \lim_{N\to\infty} \biggl(\sum_{n=1}^{N} n^\alpha - \frac{1}{\alpha+1} \sum_{r=0}^{k} (-1)^r B_r \binom{\alpha+1}{r} N^{\alpha+1-r} \biggr),$$

since the last term in (1.50) tends to zero. If $\Re\alpha < -1$ then each term in $\sum_{r=0}^{k}(-1)^r B_r \binom{\alpha+1}{r} N^{\alpha+1-r}$ tends to zero as $N \to \infty$ and therefore $C(\alpha) = \lim_{N\to\infty} \sum_{n=1}^{N} n^\alpha = \sum_{n=1}^{\infty} n^\alpha = \zeta(-\alpha)$. On the other hand, for an arbitrary fixed $h \in \mathbb{N}$, the formula (1.51) can be seen to define $C(\alpha)$ as an analytic function of α in the *larger* region $\{\alpha \in \mathbb{C} \,:\, \alpha \neq -1,\ \Re\alpha < h-1\}$ (because of uniform convergence on compacta). Hence, by uniqueness of analytic continuation, our formula for $C(\alpha)$ provides *the* analytic continuation of $\zeta(-\alpha)$ to this region! In particular, since h is arbitrary, we have proved that $\zeta(s)$ has an analytic continuation to all of $s \in \mathbb{C} \setminus \{1\}$!

(Connecting with our previous example: In Maple, typing `Digits:=30:` and then `evalf(Zeta(-3/2));` indeed gives "$-0.025485201889833\ldots$".)

Finally, it is interesting to consider the special case of α being a nonnegative integer: $\alpha \in \mathbb{Z}_{\geq 0}$. In this case we have $\binom{\alpha}{k} = \binom{\alpha}{h} = 0$ (since $k \geq h > \alpha + 1$) and $\binom{\alpha+1}{r} = 0$ for all integers $r \geq \alpha + 2$; hence (1.50) says

$$\sum_{n=1}^{N} n^\alpha = 1 + \frac{1}{\alpha+1} \sum_{r=0}^{\alpha+1} (-1)^r \binom{\alpha+1}{r} B_r \bigl(N^{\alpha+1-r} - 1\bigr) \tag{1.52}$$

This has been proved for all $N \in \mathbb{N}$, but one easily convinces oneself that this identity must also be valid at $N = 0$, [7] and this implies

$$\sum_{r=0}^{\alpha} (-1)^r \binom{\alpha+1}{r} B_r = \alpha + 1. \tag{1.53}$$

[6] In this paragraph we assume knowledge of some complex analysis, and we don't give as many details as mostly elsewhere.

[7] A completely elementary way of seeing this without going back and generalizing the earlier discussion is to note that the right hand equals $P(N)$, where $P(X)$ is a polynomial of degree $\leq \alpha + 1$; and the identity implies that $P(X+1) - P(X) - (X+1)^\alpha = 0$ for all $X \in \mathbb{N}$; but both sides of the last relations are polynomials, hence the last identity in fact holds for all $X \in \mathbb{R}$, and taking $X = 0$ and using $P(1) = 1$ we get the desired claim.

Using this, (1.52) can be simplified somewhat, into

$$\sum_{n=1}^{N} n^{\alpha} = \frac{1}{\alpha+1} \sum_{r=0}^{\alpha} (-1)^r \binom{\alpha+1}{r} B_r N^{\alpha+1-r}.$$

This is the so-called *Faulhaber's formula.* Furthermore, from (1.53) and (1.51) we obtain:

$$\zeta(-\alpha) = C(\alpha) = \frac{(-1)^{\alpha} B_{\alpha+1}}{\alpha+1}.$$

In particular $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, $\zeta(-2) = 0$, $\zeta(-3) = \frac{1}{120}$.

1.4. **Some more examples.**

Example 1.10. A counting function of fundamental importance in number theory is

$$\pi(x) = \#\{p \,:\, p \text{ is a prime number} \le x\}.$$

The *Prime Number Theorem* (PNT) gives an asymptotic formula for $\pi(x)$:

$$\pi(x) \sim \frac{x}{\log x} \qquad \text{as } x \to \infty.$$

The PNT was proved independently by Hadamard and de la Vallée-Poussin (1896); much of the work was based on a celebrated memoir by Riemann 1859. The starting point for the proof is the *Euler product formula* for the Riemann zeta function:

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}, \qquad \forall s \in \mathbb{C} \text{ with } \Re s > 1, \tag{1.54}$$

where the product is taken over all primes p. This formula is simply a rephrasing of the fundamental theorem of arithmetic (the fact that each positive integer has a unique factorization into primes) in terms of generating functions. Indeed, on a formal level unique factorization implies that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Big(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\Big) = \prod_p \frac{1}{1-p^{-s}},$$

and this calculation can easily be made rigorous for all s with $\Re s > 1$. (Similarly in the remainder of this example we will present calculations in a rather formal style, but they can all be made rigorous.)

The starting idea for the (standard) proof of the PNT is to try to *invert* the formula (1.54) to extract information about the primes, or more specifically about $\pi(x)$. An obvious first step is to take the logarithm in (1.54) so as to transform the product into a sum; in fact it turns out to be slightly more convenient to deal with the *derivative* of the logarithm; i.e.

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\log\prod_p \frac{1}{1-p^{-s}} = \sum_p \frac{d}{ds}\log\frac{1}{1-p^{-s}} = -\sum_p \frac{p^{-s}\log p}{1-p^{-s}}$$
$$= -\sum_p \Big(p^{-s}+p^{-2s}+p^{-3s}+\ldots\Big)\log p = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r \text{ for some prime } p \text{ and } r \in \mathbb{N} \\ 0 & \text{otherwise.}\end{cases}$$

Writing

$$\Psi(x) = \sum_{1\leq n\leq x}\Lambda(n),$$

the last formula can be expressed as

$$\frac{\zeta'}{\zeta}(s) = -\int_1^\infty x^{-s}\,d\Psi(x) = -s\int_1^\infty x^{-s-1}\Psi(x)\,dx,$$

where we integrated by parts in the last step. This means that $\frac{\zeta'}{\zeta}(s)$ *is a kind of Fourier transform of* $\Psi(x)$. Indeed, writing $s=\sigma+it$ $(\sigma,t\in\mathbb{R})$ and substituting $x=e^u$ we get $\frac{\zeta'}{\zeta}(\sigma+it) = -s\int_0^\infty e^{-\sigma u}\Psi(e^u)e^{-itu}\,du$, i.e. the function $t\mapsto\frac{\zeta'}{\zeta}(\sigma+it)$ is the Fourier transform of the function $u\mapsto -se^{-\sigma u}\Psi(e^u)$. I hope to discuss, in a later lecture, as an example on the inverse Fourier transformation and methods of asymptotic expansions, how the last formula can be inverted and used, in combination with the very important fact that $\zeta(s)$ has no zeros for $\Re s\geq 1$, to deduce

$$\Psi(x)\sim x \qquad \text{as } x\to\infty,$$

which can be seen in an elementary way to imply the PNT.

Example 1.11. The *Gauss circle problem* is about estimating the number of integer points in a circle of radius r centered at the origin, for large r, that is

$$A(r) = \#\big\{n=(n_1,n_2)\in\mathbb{Z}^2 \,:\, |n|^2 = n_1^2+n_2^2\leq r^2\big\}.$$

Gauss made the first progress on this problem by proving

$$A(r) = \pi r^2 + O(r), \qquad \forall r\geq 1. \tag{1.55}$$

This can be proved by estimating $A(r)$ from above and below using circles of slightly larger/smaller radius. In precise terms: Let us write $B_r\subset\mathbb{R}^2$ for the open disc with center at the origin and radius r. Let $M_r\subset\mathbb{R}^2$ be the union of all squares $n+[-\frac12,\frac12]^2$ for $n\in\mathbb{Z}^2$, $|n|\leq r$; then the area of M_r equals $A(r)$. Now $B_{r-\sqrt{1/2}}\subset M_r\subset\overline{B_{r+\sqrt{1/2}}}$ for all $r \geq 1$; this is easily seen by drawing a picture! (The detailed proof uses the triangle inequality and the fact that every point in a square $n+[-\frac{1}{2},\frac{1}{2}]^2$ has distance $\leq \sqrt{1/2}$ to its center n.) Hence by comparing areas we conclude:

$$\pi\big(r-\sqrt{1/2}\big)^2 \leq A(r) \leq \pi\big(r+\sqrt{1/2}\big)^2, \qquad \forall r \geq 1,$$

and this implies (1.55).

The error bound in (1.55) has been successively improved over the years; Sierpinski (1906) improved it to $O(r^{2/3})$, and the best known bound today is due to Bourgain and Watt (2017) [4] who proved $A(r) = \pi r^2 + O(r^{\frac{517}{824}+\varepsilon})$. [8] (Note that $\frac{517}{1648} \approx 0.6274\ldots$.) It has been conjectured that $A(r) = \pi r^2 + O(r^{\frac{1}{2}+\varepsilon})$. This bound would be optimal; it is know that $A(r) = \pi r^2 + O(r^\theta)$ *cannot* hold with any $\theta \leq \frac{1}{2}$.

One way to attack the the Gauss circle problem, which we will discuss in a later lecture, is by using the *Poisson summation formula.* This formula says that for any sufficiently nice function $f : \mathbb{R}^m \to \mathbb{C}$, if $\widehat{f}(\xi) = \int_{\mathbb{R}^m} f(x)e^{-2\pi i \xi\cdot x}\,dx$ is the Fourier transform of f then

$$\sum_{n\in\mathbb{Z}^m} f(n) = \sum_{\xi\in\mathbb{Z}^m} \widehat{f}(\xi). \tag{1.56}$$

“Sufficiently nice” here means that f has to be both sufficiently smooth and decay sufficiently fast at infinity; cf., e.g., Folland Theorem 8.32 for one precise statement. For the circle problem, one *would like to* take $f(x) = I(|x| \leq r)$, i.e. $f(x) = 1$ when $|x| \leq r$ and $f(x) = 0$ when $|x| > r$. With this choice the left hand side of (1.56) would equal $A(r)$ exactly! The problem is that this function f is far fram smooth; it is even discontinuous, and correspondingly (1.56) is not absolutely convergent and one has to do some work before one can make sense out of the right hand side in (1.56). In a later lecture we will see how to modify this approach and use it to prove the Sierpinski estimate $A(r) = \pi r^2 + O(r^{\frac{2}{3}})$.

The Gauss circle problem is only one very special case of the general problem of *counting the lattice points in a given (large) region.* This general problem has applications in many areas of mathematics and we will come back to it several times in later lectures.

[8]This statement should be understood as: For any fixed $\varepsilon > 0$ one has $A(r) = \pi r^2 + O(r^{\frac{517}{824}+\varepsilon})$ as $r \to \infty$ (or equivalently: for all $r \geq 1$). The implied constant is allowed to depend on ε but not on r.

2. Push-forward of measures and substitution in integrals

We here prove a very basic fact about substitution in integrals, which as far as I could see is not explicitly discussed in Folland's book. (Cf. Wikipedia.)

Definition 2.1. If $T : X \to Y$ is a measurable map from one measurable space $(X, \mathcal{M})$ to another measurable space $(Y, \mathcal{N})$, and μ is a measure on $(X, \mathcal{M})$, then the *push-forward* $T_*\mu : \mathcal{N} \to [0, +\infty]$ is defined by the formula $T_*\mu(E) = \mu(T^{-1}(E))$, $\forall E \in \mathcal{N}$. [9] One checks immediately that $T_*\mu$ is a measure on $(Y, \mathcal{N})$.

Now we have the following natural integration formula:

Proposition 2.2. *Let T, $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N})$ be as above. Then for any $f \in L^+(Y, \mathcal{N})$ we have $f \circ T \in L^+(X, \mathcal{M})$ and $\int_X (f \circ T)\, d\mu = \int_Y f\, d(T_*\mu)$. Similarly for any $f \in L^1(Y, T_*\mu)$ we have $f \circ T \in L^1(X, \mu)$ and, again, $\int_X (f \circ T)\, d\mu = \int_Y f\, d(T_*\mu)$.*

The proof is completely standard:

Proof. If ϕ is a simple function in $L^+(Y, \mathcal{N})$ with standard representation $\phi = \sum_{j=1}^n z_j \chi_{E_j}$ (thus $E_1, \ldots, E_n \in \mathcal{N}$ and these sets form a partition of Y), then $\phi \circ T = \sum_{j=1}^n z_j \chi_{T^{-1}(E_j)}$; this is a simple function in $L^+(X, \mathcal{M})$ in its standard representation, and

(2.1)
$$\int_X (\phi \circ T)\, d\mu = \sum_{j=1}^n z_j\, \mu(T^{-1}(E_j)) = \sum_{j=1}^n z_j\, [T_*\mu](E_j) = \int_Y \phi\, d(T_*\mu).$$

Now let f be an arbitrary function in $L^+(Y, \mathcal{N})$. Then $f \circ T$ is the composition of two measurable functions, hence $f \circ T$ is an $\mathcal{M}$-measurable function $X \to [0, +\infty]$, thus $f \circ T \in L^+(X, \mathcal{M})$. Let $\phi_1, \phi_2, \ldots$ be an increasing sequence of simple functions in $L^+(Y, \mathcal{N})$ such that $\phi_j \to f$ pointwise. Such a sequence exists by Folland's Theorem 2.10. Then $\phi_1 \circ T, \phi_2 \circ T, \ldots$ is an increasing sequence of simple functions in $L^+(X, \mathcal{M})$, and $\phi_j \circ T \to f \circ T$ pointwise. Hence

$$\int_X (f \circ T)\, d\mu = \lim_{j \to \infty} \int_X (\phi_j \circ T)\, d\mu = \lim_{j \to \infty} \int_X \phi_j\, d(T_*\mu) = \int_X f\, d(T_*\mu).$$

[The first equality holds by the Monotone Convergence Theorem; the second by (2.1), and the third by the Monotone Convergence Theorem.]

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[9] In this formula, recall that " $T^{-1}(E)$ " denotes the set $\{x \in X : T(x) \in E\}$, and this set is in $\mathcal{M}$ for every $E \in \mathcal{N}$, since T is measurable. Thus: "T^{-1}" is really a map from $\mathcal{N}$ to $\mathcal{M}$, and the push-forward measure $T_*\mu$ is the same as the composition $\mu \circ T^{-1}$; certain authors prefer to use the latter notation. (Cf., e.g., Kallenberg [12], where our Prop. 2.2 appears as [12, Lemma 1.22].)

Finally let f be an arbitrary function in $L^1(Y, T_*\mu)$. Then $f \circ T$ is an $\mathcal{M}$-measurable function $X \to \mathbb{C}$ and $|f \circ T| = |f| \circ T$ (where $|f| \in L^+(X, \mathcal{M})$) so that $\int_X |f \circ T|\, d\mu = \int_Y |f|\, d(T_*\mu) < \infty$ by what we have already proved; thus $f \circ T \in L^1(X, \mu)$. Finally $\int_X (f \circ T)\, d\mu = \int_Y f\, d(T_*\mu)$ follows by splitting f into its real and imaginary part, and the positive and negative parts of these (viz., using the definition of integrals of complex functions, Folland p. 53), and using the result which we have already proved for L^+-functions. $\square$

Example 2.1. Consider Folland's Theorem 2.44. Let us note that in view of the above Proposition 2.2, the two parts of that theorem are “obviously” equivalent! Similarly, also the two parts of Folland's Theorem 2.42 are “obviously” equivalent.

[Details: First assume that Thm. 2.44(a) holds. Given any $E \in \mathcal{L}^n$, the function χ_E is a Lebesgue measurable function on $\mathbb{R}^n$; hence Thm 2.44(a) (applied for T^{-1}) says that $\chi_E \circ T^{-1} = \chi_{T(E)}$ is Lebesgue measurable, and $\int \chi_E\, dm = |\det T^{-1}| \int \chi_{T(E)}\, dm$. In other words $T(E) \in \mathcal{L}^n$ and $m(E) = |\det T|^{-1} m(T(E))$, i.e. we have proved Thm 2.44(b). *Conversely*, now assume that Thm. 2.44(b) holds. Applying this for T^{-1}, we have that $T : \mathbb{R}^n \to \mathbb{R}^n$ is $(\mathcal{L}^n, \mathcal{L}^n)$-measurable, and that the push-forward measure T_*m equals $|\det T^{-1}| m = |\det T|^{-1} m$ (equality of measures on $(\mathbb{R}^n, \mathcal{L}^n)$). Hence if f is any Lebesgue measurable function f on $\mathbb{R}^n$, so is $f \circ T$, and furthermore by Prop. 2.2, if $f \geq 0$ or $f \in L^1(m)$ then

$$\int_{\mathbb{R}^n} (f \circ T)\, dm = \int_{\mathbb{R}^n} f\, d(T_*m) = |\det T|^{-1} \int_{\mathbb{R}^n} f\, dm.$$

In other words, we have proved Thm 2.44(a)!]

3. Conditional expectation and conditional probability

We here discuss how the concepts of conditional expectation and conditional probability arise as special cases of the Radon-Nikodym Theorem, using the set-up of Folland's book (cf. also Folland's Exercise 17, p. 93). For a more thorough presentation and development you should consult any standard book on probability theory; cf., e.g., Billingsley, [2, Sections 33–34].

We start by giving a solution to Folland's Exercise 17: Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$. Then $(X, \mathcal{N}, \mu_{|\mathcal{N}})$ is also a finite measure space. Now let $f \in L^1(\mu)$ and let $\lambda \in M(\mathcal{M})$ [10] be given by $d\lambda = f\,d\mu$; then $\lambda \ll \mu$ and hence $\lambda_{|\mathcal{N}}$, which is clearly a complex measure on $\mathcal{N}$, satisfies $\lambda_{|\mathcal{N}} \ll \mu_{|\mathcal{N}}$. Hence by the Radon-Nikodym Theorem there exists a unique function $g \in L^1(\mu_{|\mathcal{N}})$ such that

$$\forall E \in \mathcal{N}: \qquad \int_E f\,d\mu = \int_E g\,d\mu_{|\mathcal{N}}. \tag{3.1}$$

Recall that the uniqueness is understood in the usual sense of L^1 that we identify any two functions that agree a.e. (for us: $\mu_{|\mathcal{N}}$-a.e.). Thus, to be precise, the uniqueness says that if $g' \in L^1(\mu_{|\mathcal{N}})$ is another function satisfying (3.1) then $g = g'$ $\mu_{|\mathcal{N}}$-a.e.

This completes the solution of Folland's Exercise 17.

Let us note that the property (3.1) may equivalently be expressed as:

$$\forall E \in \mathcal{N}: \qquad \int_E f\,d\mu = \int_E g\,d\mu. \tag{3.2}$$

This follows from the fact (definition) that $\int_E g = \int \chi_E g$ (where $\chi_E g$ is $\mathcal{N}$-measurable) and the following lemma:

Lemma 3.1. *If $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, then $\int_X h\,d\mu_{|\mathcal{N}} = \int_X h\,d\mu$ for all $h \in L^1(\mu_{|\mathcal{N}})$.*

Proof. (Cf. Billingsley, [2, Ex. 16.4].) One easily checks (via the definitions in Folland's Sec. 2.3) that it suffices to prove the claim for all $h \in L^+(\mu_{|\mathcal{N}})$. Given such an h, by Folland's Theorem 2.10 there is a sequence $\{h_n\}$ of simple $\mathcal{N}$-measurable functions such that $0 \leq h_1 \leq h_2 \leq \cdots$ and $h_n \to h$ pointwise; and then by the Monotone Convergence Theorem we have $\int_X h\,d\mu_{|\mathcal{N}} = \lim_{n\to\infty} \int_X h_n\,d\mu_{|\mathcal{N}}$. Note that h and each h_n is also $\mathcal{M}$-measurable (since $\mathcal{N} \subset \mathcal{M}$), and by another application of the Monotone Convergence Theorem we have $\int_X h\,d\mu = \lim_{n\to\infty} \int_X h_n\,d\mu$. Hence it now suffices to prove $\int_X h_n\,d\mu_{|\mathcal{N}} = \int_X h_n\,d\mu$

[10] As in lecture #4 we write $M(\mathcal{M})$ for the set of all complex measures on $\mathcal{M}$.

for each n, and thus it suffices to prove that $\int_X h\,d\mu_{|\mathcal{N}} = \int_X h\,d\mu$ whenever h is a simple $\mathcal{N}$-measurable (nonnegative) function. By linearity we then reduce to the case when h is a characteristic function: $h = \chi_A$ for some $A \in \mathcal{N}$. But then $\int_X h\,d\mu_{|\mathcal{N}} = \mu_{|\mathcal{N}}(A) = \mu(A) = \int_X h\,d\mu$, and we are done. $\square$

To connect with probability theory, let us now assume that μ *is a probability measure*, i.e. $\mu(X) = 1$. In other words, $(X, \mathcal{M}, \mu)$ is a *probability space.* (A more common notation in probability theory would be to write Ω for X and P for μ; however we will continue writing X and μ.) A μ-measurable function on X is now called a *random variable;* in particular our $f \in L^1(\mu)$ is a random variable. The function $g \in L^1(\mu_{|\mathcal{N}})$ whose existence we proved above and which satisfies (3.1) and (3.2) is now called the *conditional expectation* of f given $\mathcal{N}$, and denoted by $\mathbb{E}[f\|\mathcal{N}]$. Thus, to recapitulate: $\mathbb{E}[f\|\mathcal{N}]$ is the unique function in $L^1(\mu_{|\mathcal{N}})$ satisfying

$$\int_E f\,d\mu = \int_E \mathbb{E}[f\|\mathcal{N}]\,d\mu, \qquad \forall E \in \mathcal{N}.$$

In the special case when f is the characteristic function of a set $A \in \mathcal{M}$; $f = \chi_A$, then $\mathbb{E}[f\|\mathcal{N}]$ is called the *conditional probability of A given $\mathcal{N}$*, and denoted by $\mu[A\|\mathcal{N}]$. Thus:

$$\mu[A\|\mathcal{N}] = \mathbb{E}[\chi_A\|\mathcal{N}],$$

and the defining property (3.2) reads:

$$\mu(A \cap E) = \int_E \mu[A\|\mathcal{N}]\,d\mu, \qquad \forall E \in \mathcal{N}. \tag{3.3}$$

Note that $\mu[A\|\mathcal{N}]$ is a *function*, and not just a number in $[0, 1]$. Informally, $\mu[A\|\mathcal{N}](x)$ may be interpreted as (at least for $\mu_{|\mathcal{N}}$-a.e. x): "The conditional probability that a μ-random element $\omega \in X$ happens to lie in A, *given* that ω is contained in exactly the same $\mathcal{N}$-sets as x [11]".

We note:

Lemma 3.2. *For any given $A \in \mathcal{M}$, we have $0 \leq \mu[A\|\mathcal{N}](x) \leq 1$ for $\mu_{|\mathcal{N}}$-a.e. x.*

Proof. Let $A \in \mathcal{M}$ be given. Note that the function $\Re\mu[A\|\mathcal{N}]$ satisfies the same defining property as $\mu[A\|\mathcal{N}]$; hence $\mu[A\|\mathcal{N}](x) \in \mathbb{R}$ must hold for $\mu_{|\mathcal{N}}$-a.e. x, and replacing $\mu[A\|\mathcal{N}]$ by $\Re\mu[A\|\mathcal{N}]$ we may assume $\mu[A\|\mathcal{N}](x) \in \mathbb{R}$ for all $x \in X$.

Now for any given $\varepsilon > 0$, let us set $E = \{x \in X \,:\, \mu[A\|\mathcal{N}](x) \geq 1 + \varepsilon\}$. Then $E \in \mathcal{N}$, and (3.3) gives $\mu(A \cap E) \geq (1 + \varepsilon)\mu(E)$. But

[11] in other words: given that $[\forall E \in \mathcal{N}\colon\ \omega \in E \Leftrightarrow x \in E]$.

$\mu(A \cap E) \leq \mu(E)$; hence $\mu(E) \geq (1+\varepsilon)\mu(E)$, which forces $\mu(E) = 0$. Letting $\varepsilon = 1/n$ and $n \to \infty$ this implies (using continuity from below for μ; cf. Folland's Thm. 1.8(c)) $\mu(\{x \in X : \mu[A\|\mathcal{N}](x) > 1\}) = 0$. By an entirely similar argument we also have $\mu(\{x \in X : \mu[A\|\mathcal{N}](x) < 0\}) = 0$, and this completes the proof. $\square$

We conclude by giving two examples to show how the above concept connects with the elementary or intuitive notion of "conditional probability". In these examples we will write P in place of μ and Ω in place of X.

Example 3.1. Let X and Y be two integer valued random variables whose joint distribution is given by probabilities

$$p_{ij} = P(X = i, Y = j), \qquad \forall i, j \in \mathbb{Z}.$$

Thus P is the probability measure on $(\mathbb{Z}^2, \mathcal{P}(\mathbb{Z}^2))$ determined by

$$P(A) = \sum_{\langle i,j \rangle} p_{ij},$$

and we of course have $p_{ij} \geq 0$ for all $i, j \in \mathbb{Z}$, and $\sum_{i,j\in\mathbb{Z}} p_{ij} = P(\mathbb{Z}^2) = 1$. Now let $\mathcal{N}$ be the sub-σ-algebra of $\mathcal{P}(\mathbb{Z}^2)$ given by

$$\mathcal{N} = \{B \times \mathbb{Z} \,:\, B \subset \mathbb{Z}\}.$$

For an arbitrary $A \subset \mathbb{Z}$, letting $A' := \mathbb{Z} \times A$ we wish to determine $P[A'\|\mathcal{N}]$, the conditional probability of the event A' (i.e., "the event $Y \in A$") given $\mathcal{N}$. The fact that $P[A'\|\mathcal{N}]$ is an $\mathcal{N}$-measurable function means that there is a function $g : \mathbb{Z} \to \mathbb{C}$ such that

$$P[A'\|\mathcal{N}](i,j) = g(i), \qquad \forall \langle i,j \rangle \in \mathbb{Z}^2.$$

Also the defining property (3.3) says that for every $E \in \mathcal{N}$,

$$P(E \cap A') = \sum_{\langle i,j \rangle \in E} g(i) p_{ij}.$$

In particular taking $E = \{i_0\} \times \mathbb{Z}$ for any $i_0 \in \mathbb{Z}$ (note that this E indeed lies in $\mathcal{N}$) we conclude:

$$P((\{i_0\} \times \mathbb{Z}) \cap A') = g(i_0) \cdot P(\{i_0\} \times \mathbb{Z}).$$

Using our random variables X and Y the same relation may be expressed as:

$$P(Y \in A \text{ and } X = i_0) = g(i_0) P(X = i_0).$$

Hence for any $i_0 \in \mathbb{Z}$ with $P(X = i_0) > 0$, and any $j \in \mathbb{Z}$, we have

$$P[A'\|\mathcal{N}](i_0, j) = g(i_0) = \frac{P(Y \in A \text{ and } X = i_0)}{P(X = i_0)}.$$

This is exactly the classical, elementary definition of the conditional probability of $Y \in A$ *given* that $X = i_0$.

Example 3.2. Let X and Y be real-valued random variables taking values in $[0,1]$, whose joint distribution is given by a probability density function $f \in C([0,1]^2)$ which is everywhere positive. Thus our probability space is $(\Omega, \mathcal{B}_\Omega, P)$, where $\Omega = [0,1]^2$, $\mathcal{B}_\Omega$ is the Borel σ-algebra on Ω, and P is the probability measure given by

$$P(A) = \int_{[0,1]^2} f(x)\,dx = \int_0^1 \int_0^1 f(x_1,x_2)\,dx_1\,dx_2, \qquad \forall A \in \mathcal{B}_\Omega,$$

where $dx = dx_1\,dx_2$ is Lebesgue measure. Now let $\mathcal{N}$ be the sub-σ-algebra of $\mathcal{B}_\Omega$ given by

$$\mathcal{N} = \{B \times [0,1] \,:\, B \in \mathcal{B}_{[0,1]}\}.$$

Given any $A \in \mathcal{B}_{[0,1]}$, letting $A' := [0,1] \times A \in \mathcal{B}_\Omega$, we wish to determine $P[A' \| \mathcal{N}] \in L^1(P_{|\mathcal{N}})$. The fact that this function is $\mathcal{N}$-measurable means that there is a Borel measurable function $g : [0,1] \to \mathbb{R}$ such that

$$P[A' \| \mathcal{N}](x_1,x_2) = g(x_1), \qquad \forall (x_1,x_2) \in \Omega.$$

The defining property (3.3) says that for every $E \in \mathcal{N}$,

$$P(E \cap A') = \int_E g(x_1)\,dP(x),$$

i.e.,

$$\int_{E \cap A'} f(x_1,x_2)\,dx_1\,dx_2 = \int_E g(x_1) f(x_1,x_2)\,dx_1\,dx_2.$$

But $E \in \mathcal{N}$ means that $E = B \times [0,1]$ for some $B \in \mathcal{B}_{[0,1]}$; hence the requirement is that the following should hold for every $B \in \mathcal{B}_{[0,1]}$:

$$\int_B \int_0^1 I((x_1,x_2) \in A') f(x_1,x_2)\,dx_2\,dx_1 = \int_B g(x_1) \int_0^1 f(x_1,x_2)\,dx_2\,dx_1.$$

This implies that the following must hold for (Lebesgue-)almost every x_1:

$$\int_0^1 I(x_2 \in A) f(x_1,x_2)\,dx_2 = g(x_1) \int_0^1 f(x_1,x_2)\,dx_2.$$

Hence since f is continuous and everywhere positive:

$$g(x_1) = \frac{\int_0^1 I(x_2 \in A) f(x_1,x_2)\,dx_2}{\int_0^1 f(x_1,x_2)\,dx_2}$$

for almost every $x_1 \in [0,1]$. This agrees with the "undergraduate formula" for the conditional probability of $Y \in A$ "given that $X = x_1$"!

4. Computing the Poisson kernel

We wish to calculate the inverse Fourier transform $\phi(x)$ of

$$\Phi(\xi) = e^{-2\pi|\xi|}.$$

This function $\phi(x)$ is called the *Poisson kernel;* see Folland p. 260. Thus:

$$\phi(x) = \Phi^\vee(x) = \int_{\mathbb{R}^n} e^{-2\pi|\xi|} e^{2\pi i x\cdot\xi}\, d\xi.$$

Folland outlines a proof of the explicit formula

$$\phi(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+|x|^2)^{-\frac{n+1}{2}}, \qquad \forall x \in \mathbb{R}^n. \tag{4.1}$$

in his Exercise 26, p. 262. This proof goes via expressing $\Phi(\xi)$ as a superposition of dilated Gauss kernels, and then using the fact that we already know the inverse Fourier transform of these (Prop 8.24). It is a very elegant and fairly short computation! However here we wish to give an alternative proof of (4.1), by pushing through the method which to me seems like the most natural/naive method possible. It turns out that this computation is not at all as nice as the one which Folland outlines in his Exercise 26 (at least not in the way which I carry it out below); however it provides an opportunity to illustrate several important points which are often useful in computations (namely: the fact that polar coordinates can certainly be useful for integration even if the integrand is not radial, and some tips on how to deal with complicated looking integrals and special functions).

Let S_1^{n-1} be the $n-1$ dimensional sphere, which we will always take to be concretely realized as $\mathrm{S}_1^{n-1} = \{x \in \mathbb{R}^n \,:\, |x| = 1\}$, just as in Folland, p. 78. Let σ be the unique Borel measure on S_1^{n-1} described in Folland's Theorem 2.49; this is the natural "$n-1$ dimensional volume measure" on S_1^{n-1}. Then by Theorem 2.49,

$$\begin{aligned}\phi(x) &= \int_0^\infty \int_{\mathrm{S}_1^{n-1}} e^{-2\pi r} e^{2\pi i x\cdot r\omega} r^{n-1}\, d\sigma(\omega)\, dr \\ &= \int_{\mathrm{S}_1^{n-1}} \int_0^\infty e^{2\pi(-1+ix\cdot\omega)r} r^{n-1}\, dr\, d\sigma(\omega).\end{aligned}$$

The inner integral can be evaluated (for any fixed $x \in \mathbb{R}^n$ and $\omega \in \mathrm{S}_1^{n-1}$) by substituting $r = \dfrac{u}{2\pi(1-ix\cdot\omega)}$. This gives

$$\int_0^\infty e^{2\pi(-1+ix\cdot\omega)r} r^{n-1}\, dr = \frac{1}{(2\pi)^n(1-ix\cdot\omega)^n}\int_C e^{-u}u^{n-1}\, du,$$

where C is the infinite ray in the complex plane which starts at 0 and goes through the point $1 - ix\cdot\omega$. For $R > 0$, let C_R be the part of the ray C which starts at 0 and ends at $z \in C$ with $|z| = R$. Also let D_R

be the contour which goes in the circle $\{|z| = R\}$ from the end-point of C_R to $z = R \in \mathbb{R}_{>0}$. Then by the Cauchy integral theorem,

$$\int_{C_R} e^{-u} u^{n-1}\, du + \int_{D_R} e^{-u} u^{n-1}\, du = \int_0^R e^{-u} u^{n-1}\, du.$$

Furthermore using $|e^{-u}u^{n-1}| = e^{-\Re u}|u|^{n-1} = e^{-R\cos(\arg(u))}R^{n-1}$ for all $u \in D_R$ and the fact that $\arg(1 - ix\cdot\omega) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we see that, with $c := \cos(\arg(1 - ix\cdot\omega)) = (1 + (x\cdot\omega)^2)^{-\frac{1}{2}}$:

$$\left|\int_{D_R} e^{-u} u^{n-1}\, du\right| \leq \int_{D_R} |e^{-u}u^{n-1}|\,|du| \leq \frac{\pi}{2} R^n e^{-cR} \to 0, \quad \text{as } R \to \infty.$$

Hence

$$\int_C e^{-u} u^{n-1}\, du = \lim_{R\to\infty} \int_{C_R} e^{-u} u^{n-1}\, du = \lim_{R\to\infty} \int_0^R e^{-u} u^{n-1} = \int_0^\infty e^{-u} u^{n-1}.$$

The last integral can be computed by repeated integration by parts, and is seen to equal $(n-1)!$, which can also be expressed as $\Gamma(n)$, the *gamma function* at n. (We discuss the gamma function more in detail in Sec. 7.1 below.) Thus we conclude:

$$\int_0^\infty e^{2\pi(-1+ix\cdot\omega)r} r^{n-1}\, dr = \frac{\Gamma(n)}{(2\pi)^n(1 - ix\cdot\omega)^n}.$$

Hence:

$$\phi(x) = \frac{\Gamma(n)}{(2\pi)^n} \int_{\mathrm{S}_1^{n-1}} (1 - ix\cdot\omega)^{-n}\, d\sigma(\omega).$$

We can use the fact that the measure σ is invariant under rotations (this is Folland's Exercise 62 on p. 80; it can be solved using his Theorems 2.49 and 2.44) to see that $\phi(x)$ is invariant under rotations: If $R : \mathbb{R}^n \to \mathbb{R}^n$ is any rotation about the origin then

$$\begin{aligned}\phi(Rx) &= \frac{\Gamma(n)}{(2\pi)^n} \int_{\mathrm{S}_1^{n-1}} (1 - i(Rx)\cdot\omega)^{-n}\, d\sigma(\omega) \\ &= \frac{\Gamma(n)}{(2\pi)^n} \int_{\mathrm{S}_1^{n-1}} (1 - ix\cdot(R^{-1}\omega))^{-n}\, d\sigma(\omega) \\ &= \frac{\Gamma(n)}{(2\pi)^n} \int_{\mathrm{S}_1^{n-1}} (1 - ix\cdot\varpi)^{-n}\, d\sigma(\varpi) = \phi(x),\end{aligned}$$

where in the third equality we substituted $\omega = R(\varpi)$ and used the fact that R is a bijection of S_1^{n-1} onto itself preserving the measure σ. (To be more precise, we used the integration formula for push-forwards of measures, Proposition 2.2 above, together with the fact that $R_*\sigma = \sigma$.) Of course, the fact that $\phi(x)$ is invariant under rotations can alternatively be seen from the very start, using $\phi(x) = \check{\Phi}(x) = \widehat{\widehat{\Phi}}(x)$

and Folland's Theorem 8.22(b) together with the fact that $\widetilde{\Phi} = \Phi$ is invariant under rotations.

Since $\phi(x)$ is invariant under rotations, it suffices to evaluate $\phi(x)$ when $x = (x_1, 0, \ldots, 0)$, $x_1 \geq 0$. In this case we have, writing $\omega = (\omega_1, \ldots, \omega_n)$:

$$\phi((x_1, 0, \ldots, 0)) = \frac{\Gamma(n)}{(2\pi)^n} \int_{\mathrm{S}_1^{n-1}} (1 - ix_1\omega_1)^{-n}\, d\sigma(\omega).$$

Note that as ω varies over S_1^{n-1}, ω_1 varies over the interval $[-1, 1]$, and it seems clear that the above integral over S_1^{n-1} should be expressible as an integral simply over $\omega_1 \in [-1, 1]$. Indeed, by Proposition 2.2 applied with the map T being $T : \mathrm{S}_1^{n-1} \to [-1, 1]$; $T(\omega) := \omega_1$, we have

$$\phi((x_1, 0, \ldots, 0)) = \frac{\Gamma(n)}{(2\pi)^n} \int_{[-1,1]} (1 - ix_1\omega_1)^{-n}\, d(T_*\sigma)(\omega_1).$$

The question is thus: *What is the push-forward $T_*\sigma$ of the measure σ under the projection $T : \mathrm{S}_1^{n-1} \to [-1, 1]$?* The answer is easily found e.g. using spherical coordinates; cf. Folland's exercise 65 on p. 80.[12]

$$d(T_*\sigma)(\omega_1) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}(1 - \omega_1^2)^{\frac{n-3}{2}}\, d\omega_1. \tag{4.2}$$

(Here $d\omega_1$ is Lebesgue measure, as usual.) Using this we have

$$\phi((x_1, 0, \ldots, 0)) = \frac{\Gamma(n)}{2^{n-1}\pi^{\frac{n+1}{2}}\Gamma(\frac{n-1}{2})} \int_{[-1,1]} (1 - ix_1\omega_1)^{-n}\, (1 - \omega_1^2)^{\frac{n-3}{2}}\, d\omega_1.$$

This explicit integral is perhaps not entirely simple to compute. I present one (dirty!) way to compute it below: The two main points I want to make are (1) it is often useful to use a computer algebra package, e.g. Maple, both to get the answer and to learn about e.g. special functions involved, and (2) it is often convenient to use handbooks of mathematical formulas, such as [6]; also google, Wikipedia, [14, http://dlmf.nist.gov/], etc, can be useful.

This integral can be computed using Maple: Typing

```
> simplify(int((1-omega1^2)^((n-3)/2)*(1-I*x1*omega1)^(-n),omega1=-1..1));
```

gives the answer

[12] Some details: By Folland's Exercise 65 we have for any Borel set $E \subset [-1, 1]$: $(T_*\sigma)(E) = \int_X I(\cos\phi_1 \in E) \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdots \sin\phi_{n-2}\, d\phi_1 \cdots d\phi_{n-2}\, d\theta$, where $X = (0, \pi)^{n-2} \times (0, 2\pi)$. Substituting $\omega_1 = \cos\phi_1$ we have $\int_0^\pi I(\cos\phi_1 \in E) \sin^{n-2}\phi_1\, d\phi_1 = \int_E (1 - \omega_1^2)^{\frac{n-3}{2}}\, d\omega_1$; also the integral over the remaining variables is recognized as $\sigma_{n-2}(S^{n-2})$, by applying the same exercise 65 with $n - 1$ in place of n. But $\sigma_{n-2}(S^{n-2}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ by Folland's Prop. 2.54. Hence the formula (4.2) follows.

```
  1/2     2      (- 1/2 - n/2)
Pi     (x1  + 1)                GAMMA(n/2 - 1/2)
------------------------------------------------
                 GAMMA(n/2)
```

Let us try to check where Maple's answers come from. In this case, typing the above without the "`simplify`" we see that the integral is related to the hypergeometric function (a fact which perhaps the more experienced readers could see from start without help). In fact, substituting $\omega_1 = 2u - 1$ we have

$$\int_{-1}^{1} (1 - ix_1\omega_1)^{-n} \, (1 - \omega_1^2)^{\frac{n-3}{2}} \, d\omega_1$$

$$= 2^{n-2} \int_0^1 u^{\frac{n-3}{2}} (1-u)^{\frac{n-3}{2}} (1 + ix_1 - 2ix_1 u)^{-n} \, du$$

$$= 2^{n-2}(1 + ix_1)^{-n} \int_0^1 u^{\frac{n-3}{2}} (1-u)^{\frac{n-3}{2}} \Big(1 - \frac{2ix_1}{1+ix_1} u\Big)^{-n} \, du$$

By [6, 9.111] and [6, 8.384] (cf. also wikipedia) we get

$$2^{n-2}(1 + ix_1)^{-n} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} F\Big(n, \frac{n-1}{2}; n-1; \frac{2ix_1}{1+ix_1}\Big),$$

where F is the (Gauss') hypergeometric function (often also denoted by ${}_2F_1$). Next using [6, 9.134.1] we get

$$= 2^{n-2} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} F\Big(\frac{n}{2}, \frac{n+1}{2}; \frac{n}{2}; -x_1^2\Big),$$

and by [6, 9.100–9.102] this is, assuming $|x_1| < 1$:

$$= 2^{n-2} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} \sum_{j=0}^{\infty} \binom{-(n+1)/2}{j} (-1)^j (-x_1^2)^j$$

$$= 2^{n-2} \frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)} (1 + x_1^2)^{-\frac{n+1}{2}}.$$

Using also the doubling formula for the Gamma function (see [6, 8.335.1]), $\Gamma(n-1) = \pi^{-\frac{1}{2}} 2^{n-2} \Gamma(\frac{n-1}{2}) \Gamma(\frac{n}{2})$; we conclude:

$$\int_{-1}^{1} (1 - ix_1\omega_1)^{-n} \, (1 - \omega_1^2)^{\frac{n-3}{2}} \, d\omega_1 = \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} (1 + x_1^2)^{-\frac{n+1}{2}}.$$

We have proved this for all $x_1 \in (-1, 1)$, but since both the left and the right sides in the last identity are clearly holomorphic functions in the open connected region

$$x_1 \in \mathbb{C} \setminus i\Big((-\infty, -1] \cup [1, \infty)\Big),$$

the identity must hold for all these x_1 by analytic continuation, and in particular the identity holds for all $x_1 \in \mathbb{R}$. This validates Maple's answer! Using this we conclude

$$\begin{aligned}\phi((x_1,0,\ldots,0)) &= \frac{\Gamma(n)}{2^{n-1}\pi^{\frac{n+1}{2}}\Gamma(\frac{n-1}{2})}\cdot\frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}(1+x_1^2)^{-\frac{n+1}{2}}\\ &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+x_1^2)^{-\frac{n+1}{2}},\end{aligned}$$

where in the last step we again used the doubling formula for Γ, i.e. $\Gamma(n) = \pi^{-\frac{1}{2}}2^{n-1}\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})$. (Note that our two applications of the doubling formula cancel each other; we only used it to check agreement with the Maple output.) Hence, using the fact that ϕ is invariant under rotations, we have

$$\phi(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+|x|^2)^{-\frac{n+1}{2}}, \qquad \forall x \in \mathbb{R}^n.$$

$\square$

5. Notation: "big O", "little o", "$\ll$", "$\gg$", "$\asymp$" and "$\sim$"

Note: The exact conventions regarding these symbols differ in the literature. The following are the conventions which we will use throughout these notes.

"Big O": If a is a *non-negative* number, the symbol "$O(a)$" is used to denote any number b for which $|b| \leq Ca$, where C is a positive "constant", called *the implied constant.* We write "constant" within quotation marks since C is often allowed to depend on certain parameters: When using the big-O notation it is very important to always be clear about *which* parameters C is allowed to depend on. Furthermore, it must always be clear for which variable ranges the bound holds. For example: "$f(x) = O(x^3)$ as $x \to \infty$" means that there is some constant $C > 0$ such that *for all sufficiently large* x we have $|f(x)| \leq Cx^3$. On the other hand, "$f(x) = O(x^3)$ for $x \geq 1$" means that there is some constant $C > 0$ such that $|f(x)| \leq Cx^3$ *holds for all* $x \geq 1$.

If the implied constant can be taken to be independent of *all* parameters present in the problem, then the implied constant is said to be *absolute.*

Note that whenever we use the notation "$O(a)$" we require that $a \geq 0$.

"little o": We write "$f(x) = o(g(x))$ as $x \to a$" to denote that $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$; we will only use this notation when $g(x) > 0$ for all x sufficiently near a! Thus for example if we write "$\sum_{n=1}^{N} a_n = \frac{2}{3}N^{\frac{3}{2}} - \frac{6}{7}N^{\frac{7}{6}}(1 + o(1))$ as $N \to \infty$" then "$o(1)$" denotes some function $f(N)$ which satisfies $\lim_{N\to\infty} f(N) = 0$. Note that, unlike the "big O"-notation *the "little o"-notation can only be used when we are taking a limit.*

"$\ll$": "$b \ll a$" means the same as $b = O(a)$.

"$\gg$": "$b \gg a$" means that there is a constant $C > 0$ (again called the *implied constant*) such that $|b| \geq Ca \geq 0$.

Thus note that "$a \ll b$" is in general not equivalent with "$b \gg a$" – but they are equivalent whenever both a and b are nonnegative.

"$\asymp$": "$b \asymp a$" means [$b \ll a$ and $b \gg a$].

"$\sim$": We write "$f(x) \sim g(x)$ as $x \to a$" to denote that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. Thus this notation can only be used when $g(x) \neq 0$ for all x sufficiently near a. We may note that *if* $g(x) \neq 0$ for all x sufficiently near a, then "$f(x) \sim g(x)$ as $x \to a$" is equivalent with "$f(x) = g(x)(1 + o(1))$ as $x \to a$".

6. On counting integer points in large convex sets

Given a set $E \subset \mathbb{R}^n$ we are interested in the number of integer points in E, i.e. $\#(\mathbb{Z}^n \cap E)$. If E is "large" and "nice" it seems clear that $\#(\mathbb{Z}^n \cap E)$ should be approximately equal to the *volume* of E. The following result gives a precise error bound for this approximation, when E is replaced by the rescaled set RE and we let $R \to \infty$.

Theorem 6.1. *Assume that E is a bounded open convex set in $\mathbb{R}^n$ ($n \geq 2$) and that there is a constant $C > 0$ such that*

$$|\widehat{\chi_E}(\xi)| \leq C(1 + |\xi|)^{-\frac{n+1}{2}}, \qquad \forall \xi \in \mathbb{R}^n. \tag{6.1}$$

Then there is a constant $C' > 0$ such that

$$\big|\#(\mathbb{Z}^n \cap RE) - \mathrm{vol}(RE)\big| \leq C' R^{\frac{(n-1)n}{n+1}}, \qquad \forall R \geq 1. \tag{6.2}$$

(Here "vol" denotes volume, i.e. Lebesgue measure; thus $\mathrm{vol}(RE) = R^n \, \mathrm{vol}(E)$.)

Here are some remarks to put the result in context:

Remark 6.2. It follows from results which we will prove later (see Ex. 8.1 and Sec. 9.3), that the bound (6.1) holds when C is a *ball*. But in fact (6.1) holds whenever the convex set C has a boundary ∂C which is sufficiently smooth and has everywhere positive gaussian curvature. Cf. Hlawka [10], [9], and Herz, [7].

Remark 6.3. The bound (6.2) with $R^{\frac{(n-1)n}{n+1}}$ replaced by R^{n-1} is "trivial" (note that R^{n-1} is the order of magnitude of the $(n-1)$-volume of the boundary $\partial(RE)$). We proved this in a special case in (1.55), and the proof in the general case is similar.

Remark 6.4. For $n = 2$ the bound in (6.2) is $C' R^{2/3}$; this gives the Sierpinski (1906) bound on the Gauss' circle problem. For $n = 3$ the bound in (6.2) is $C' R^{3/2}$.

Remark 6.5. Herz 1962, [8] proves a result similar to Theorem 6.1 but with a more precise discussion on the implied constant, C'. We will essentially follow Herz' proof below.

Remark 6.6. It will be immediate from the proof that the bound (6.2) is in fact *uniform over all translates of RE*, i.e. C' can be taken so that

$$\big|\#(\mathbb{Z}^n \cap (x + RE)) - \mathrm{vol}(RE)\big| \leq C' R^{\frac{(n-1)n}{n+1}}, \qquad \forall R \geq 1,\ x \in \mathbb{R}^n.$$

Proof of Theorem 6.1. The starting idea is to try to apply the Poisson summation formula to χ_{RE}, i.e.

$$\text{“} \sum_{k\in\mathbb{Z}^n} \chi_{RE}(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\chi_{RE}}(k) \text{ ”},$$

the point being that the left hand side in this relation equals $\#(\mathbb{Z}^n \cap RE)$, the number of integer points in RE. However some modification of is necessary since the sum $\sum_{k\in\mathbb{Z}^n} \widehat{\chi_{RE}}(k)$ typically does not converge!

A standard way to make the Fourier transform decay more rapidly would be to convolve χ_{RE} with a smooth bump function acting like an approximate identity, i.e. convolving with ϕ_δ, for some fixed $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying $\phi \geq 0$, $\int \phi = 1$, and with $\delta > 0$ chosen in an optimal way depending on R.[13] This method can be used to prove Theorem 6.1 (I carried this out in my 2013 version of these lecture notes).

Here we will instead follow the method of proof of Herz, [8, Sec. 1], convolving χ_{RE} with χ_{hE} with a suitably chosen $h \in \mathbb{R}$. More precisely, we consider the convolution $\chi_{R'E} * \chi_{hE}$ with appropriate $R' \approx R$ and $h \in \mathbb{R}$; we wish to bound χ_{RE} from above and below by such convolutions! In one direction this is quite easy: The bound from below is perhaps easiest: For any $0 < h < R$, we have

$$\operatorname{vol}(hE) \cdot \chi_{RE} \geq \chi_{(R-h)E} * \chi_{hE}. \tag{6.3}$$

[Proof: By Folland, [5, Prop. 8.6(d)], the support of $\chi_{(R-h)E} * \chi_{hE}$ is contained in the closure of $(R-h)E + hE$ [14], and using the fact that E is convex one easily proves $(R-h)E + hE = RE$; hence we have $(\chi_{(R-h)E} * \chi_{hE})(x) = 0$ whenever $x \notin RE$. Also for *all* $x \in \mathbb{R}^n$ we have $\left|(\chi_{(R-h)E} * \chi_{hE})(x)\right| \leq \|\chi_{(R-h)E}\|_\infty \|\chi_{hE}\|_1 = \operatorname{vol}(hE)$. Hence (6.3) is proved.]

The bound from above looks as follows: For any $R, h > 0$ we have:[15]

$$\operatorname{vol}(hE) \cdot \chi_{RE} \leq \chi_{(R+h)E} * \chi_{-hE}. \tag{6.4}$$

[Proof: We have

$$(\chi_{(R+h)E} * \chi_{-hE})(x) = \int_{\mathbb{R}^n} \chi_{(R+h)E}(x-y)\chi_{-hE}(y)\,dy, \tag{6.5}$$

and if $x \in RE$ then for every $y \in -hE$ we have $x - y \in RE + hE = (R+h)E$; i.e. the integrand in (6.5) equals one for all $y \in -hE$; and clearly it vanishes for all other y in $\mathbb{R}^n$. Hence for every $x \in RE$ we have $(\chi_{(R+h)E} * \chi_{-hE})(x) = \operatorname{vol}(-hE) = \operatorname{vol}(hE)$. Also, obviously, $\chi_{(R+h)E} * \chi_{-hE} \geq 0$ everywhere. Hence (6.4) holds.]

---

[13] One has to balance between two effects: Increasing δ leads to a better decay for the Fourier coefficients of $\chi_{R'E} * \phi_\delta$, but also forces us to choose R_1 and R_2 further away from R in order to ensure that $\chi_{R_1E} * \phi_\delta \leq \chi_{RE} \leq \chi_{R_2E} * \phi_\delta$.

[14] Notation: For any subsets $A, B \subset \mathbb{R}^n$ we write $A+B := \{a+b : a \in A, b \in B\}$.

[15] Note: We write $rE := \{rx : x \in E\}$ for *any* $r \in \mathbb{R}$ (not only for $r > 0$).

For any $a \in \mathbb{R} \setminus \{0\}$ we have $\widehat{\chi_{aE}}(\xi) = |a|^n \widehat{\chi_E}(a\xi)$; hence the Fourier transform of $\chi_{(R-h)E} * \chi_{hE}$ is

$$\xi \mapsto (R-h)^n h^n \widehat{\chi_E}((R-h)\xi)\widehat{\chi_E}(h\xi),$$

and the Fourier transform of $\chi_{(R+h)E} * \chi_{-hE}$ is

$$\xi \mapsto (R+h)^n h^n \widehat{\chi_E}((R+h)\xi)\widehat{\chi_E}(-h\xi).$$

Note that both these are $\ll (1+|\xi|)^{-n-1}$, because of the assumption on $\widehat{\chi_E}$ in Theorem 6.1. Furthermore both the functions $\chi_{(R-h)E} * \chi_{hE}$ and $\chi_{(R+h)E} * \chi_{-hE}$, by [5, Prop. 8.8] are continuous, and so are their Fourier transforms. Hence both $\chi_{(R-h)E} * \chi_{hE}$ and $\chi_{(R+h)E} * \chi_{-hE}$ satisfy all the necessary assumptions in the Posson Summation Formula, [5, Theorem 8.32]. Applying the resulting Poisson formulas together with the inequalities (6.3) and (6.4), we obtain, for any $0 < h < R$:

$$\frac{(R-h)^n h^n}{\mathrm{vol}(hE)} \sum_{k \in \mathbb{Z}^n} \widehat{\chi_E}((R-h)k)\widehat{\chi_E}(hk) \leq \#(\mathbb{Z}^n \cap RE)$$
$$\leq \frac{(R+h)^n h^n}{\mathrm{vol}(hE)} \sum_{k \in \mathbb{Z}^n} \widehat{\chi_E}((R+h)k)\widehat{\chi_E}(-hk). \tag{6.6}$$

Using $\widehat{\chi_E}(0) = \mathrm{vol}(E)$, we see that the contribution from the $k = 0$ term in the right sum in (6.6) is $(R+h)^n \mathrm{vol}(E)$. Note also that $(R+h)^n = R^n + O(R^{n-1}h)$, since $0 < h < R$. Hence we obtain

$$\#(\mathbb{Z}^n \cap RE) - \mathrm{vol}(RE) \leq O\biggl(R^{n-1}h + (R+h)^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \Bigl|\widehat{\chi_E}((R+h)k)\Bigr| \Bigl|\widehat{\chi_E}(-hk)\Bigr|\biggr),$$

where the implied constant depends on E, but not on R or h. Using also (6.1) and $R + h \asymp R$, this implies

$$\#(\mathbb{Z}^n \cap RE) - \mathrm{vol}(RE) \leq O\biggl(R^{n-1}h + R^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} (1+|Rk|)^{-\frac{n+1}{2}}(1+|hk|)^{-\frac{n+1}{2}}\biggr).$$

By a similar argument, if we *from now on assume* $0 < h \leq \frac{1}{2}R$ (so that $R - h \asymp R$), we obtain from the left inequality in (6.6):

$$\mathrm{vol}(RE) - \#(\mathbb{Z}^n \cap RE) \leq O\biggl(R^{n-1}h + R^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} (1+|Rk|)^{-\frac{n+1}{2}}(1+|hk|)^{-\frac{n+1}{2}}\biggr).$$

Hence we have proved:

$$\bigl|\#(\mathbb{Z}^n \cap RE) - \mathrm{vol}(RE)\bigr| \ll R^{n-1}h + R^n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} (1+|Rk|)^{-\frac{n+1}{2}}(1+|hk|)^{-\frac{n+1}{2}}. \tag{6.7}$$

It remains to bound the sum in (6.7) in an optimal way. This is a quite standard problem and a nice illustration of the method of dyadic decomposition. First of all, we have $R \geq 1$ and hence for all $k \in \mathbb{Z}^n \setminus \{0\}$ we have $|Rk| \geq 1$ and $1 + |Rk| \asymp |Rk| = R|k|$. Hence[16]

(6.8)
$$\sum_{k \in \mathbb{Z}^n \setminus \{0\}} (1 + |Rk|)^{-\frac{n+1}{2}} (1 + |hk|)^{-\frac{n+1}{2}} \asymp R^{-\frac{n+1}{2}} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{-\frac{n+1}{2}} (1 + |hk|)^{-\frac{n+1}{2}}.$$

If $h \geq 1$ then we similarly have $1 + |hk| \asymp h|k|$ and so we obtain that the above is

$$(Rh)^{-\frac{n+1}{2}} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{-(n+1)} \asymp (Rh)^{-\frac{n+1}{2}}.$$

(The fact that $\sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{-(n+1)} < \infty$ is well-known and it will also be seen in the discussion below.) However it will turn out that the case $0 < h < 1$ is more relevant for us. When this holds, we have two regimes for the factor $(1+|hk|)^{-\frac{n+1}{2}}$: If $|k| \leq h^{-1}$ then $(1+|hk|)^{-\frac{n+1}{2}} \asymp 1$, while if $|k| > h^{-1}$ then $(1+|hk|)^{-\frac{n+1}{2}} \asymp h^{-\frac{n+1}{2}} |k|^{-\frac{n+1}{2}}$. Hence we obtain, when $0 < h < 1$, that the expression in (6.8) is

$$\asymp R^{-\frac{n+1}{2}} \sum_{\substack{k \in \mathbb{Z}^n \\ 1 \leq |k| \leq h^{-1}}} |k|^{-\frac{n+1}{2}} + (Rh)^{-\frac{n+1}{2}} \sum_{\substack{k \in \mathbb{Z}^n \\ |k| > h^{-1}}} |k|^{-(n+1)} \tag{6.9}$$

We here apply dyadic decomposition: We note that the set

$$\{k \in \mathbb{Z}^n \ : \ 1 \leq |k| \leq h^{-1}\}$$

is contained in the union of the pairwise disjoint sets

$$A_m := \{k \in \mathbb{Z}^n \ : \ 2^m \leq |k| < 2^{m+1}\}$$

for $m = 0, 1, \ldots, M$, where M is the minimal integer satisfying $2^{M+1} > h^{-1}$ (thus $M \geq 0$ since $0 < h < 1$). Note that for every $k \in A_m$ we have $|k|^{-\frac{n+1}{2}} \leq 2^{-\frac{m(n+1)}{2}}$. Furthermore by an elementary argument we have

$$\# A_m \leq \# \{k \in \mathbb{Z}^n \ : \ |k| < 2^{m+1}\} \ll (2^{m+1})^n \ll 2^{mn}.$$

Hence

$$\sum_{\substack{k \in \mathbb{Z}^n \\ 1 \leq |k| \leq h^{-1}}} |k|^{-\frac{n+1}{2}} \leq \sum_{m=0}^{M} \# A_m \cdot 2^{-\frac{m(n+1)}{2}} \ll \sum_{m=0}^{M} 2^{\frac{m(n-1)}{2}} \ll 2^{\frac{M(n-1)}{2}}$$

(in the last bound we used the fact that $2^{(n-1)/2} \geq \sqrt{2} > 1$, since $n \geq 2$). Also $2^M \leq h^{-1}$ by the definition of M; hence we conclude:

$$\sum_{\substack{k \in \mathbb{Z}^n \\ 1 \leq |k| \leq h^{-1}}} |k|^{-\frac{n+1}{2}} \ll h^{-\frac{n-1}{2}}. \tag{6.10}$$

[16]In all of the following bounds, up to and including (6.12), the implied constants *depend only on* n.

Similarly, to bound the last sum in (6.9), we express the set

$$\{k \in \mathbb{Z}^n \,:\, |k| > h^{-1}\}$$

as the union of the pairwise disjoint sets

$$A_m := \{k \in \mathbb{Z}^n \,:\, 2^m h^{-1} < |k| \leq 2^{m+1} h^{-1}\}$$

for $m = 0, 1, 2, \ldots$. Similarly as above, we have $|k|^{-(n+1)} < (2^m h^{-1})^{-(n+1)}$ for all $k \in A_m$, and $\#A_m \ll 2^{mn} h^{-n}$. Hence:

$$\sum_{\substack{k \in \mathbb{Z}^n \\ |k| > h^{-1}}} |k|^{-(n+1)} \leq \sum_{m=0}^{\infty} \#A_m \cdot (2^m h^{-1})^{-(n+1)} \ll \sum_{m=0}^{\infty} 2^{mn} h^{-n} \cdot 2^{-m(n+1)} h^{n+1}$$

$$= h \sum_{m=0}^{\infty} 2^{-m} \ll h. \tag{6.11}$$

Using the bounds in (6.10) and (6.11) we conclude that the expression in (6.9) is $\ll R^{-\frac{n+1}{2}} h^{-\frac{n-1}{2}}$ (both terms give the same contribution).

Incorporating also the case $h \geq 1$, we have now proved that

$$\sum_{k \in \mathbb{Z}^n \setminus \{0\}} (1 + |Rk|)^{-\frac{n+1}{2}} (1 + |hk|)^{-\frac{n+1}{2}} \ll \begin{cases} (Rh)^{-\frac{n+1}{2}} & \text{if } h \geq 1 \\ R^{-\frac{n+1}{2}} h^{-\frac{n-1}{2}} & \text{if } 0 < h < 1 \end{cases}$$

$$= R^{-\frac{n+1}{2}} h^{-\frac{n-1}{2}} \min(1, h^{-1}). \tag{6.12}$$

Plugging this into (6.7), we conclude:

$$\big|\mathrm{vol}(RE) - \#(\mathbb{Z}^n \cap RE)\big| \ll R^{n-1} h + R^{\frac{n-1}{2}} h^{-\frac{n-1}{2}} \min(1, h^{-1}). \tag{6.13}$$

We have proved that this holds for any $R \geq 1$ and any $0 < h \leq \frac{1}{2}R$. Now for given $R \geq 1$ we choose h so as to minimize (the order of magnitude of) the right hand side. A simple analysis shows that the correct choice is $h = R^{-\frac{n-1}{n+1}}$,[17] and this gives

$$\big|\#(\mathbb{Z}^n \cap RE) - \mathrm{vol}(RE)\big| \ll R^{\frac{(n-1)n}{n+1}},$$

i.e. we have proved Theorem 6.1. □

Remark 6.7. The bounds obtained above using dyadic decomposition, i.e. (6.10) and (6.11), are in fact *optimal* (as $h \to 0$). This is easily proved using the *same* dyadic decompositions, arguing similarly as above but with bounds from below.

[17]For this to be acceptable, we have to require $R \geq 2^{\frac{n+1}{2n}}$ so as to guarantee $0 < h \leq \frac{1}{2}R$; however this is not a problem, since the bound (6.2) is trivial when restricted to a fixed finite interval $1 \leq R \leq B$.

7. The gamma function

In this section we introduce the gamma function, and at the same time get an opportunity to discuss some techniques for the study of asymptotics, in particular techniques for estimating positive integrals.

7.1. The gamma function; basic facts.

For easy reference we here collect the basic facts about the gamma function, mostly without proofs. For more details, cf., e.g., Ahlfors [1, Ch. 6.2.4-5], Olver [16, Ch. 2.1] and Wikipedia.

The Γ-function is commonly defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad \text{for } z \in \mathbb{C} \text{ with } \Re z > 0, \tag{7.1}$$

together with the relation

$$\Gamma(z+1) = z\Gamma(z) \tag{7.2}$$

which can be used to extend $\Gamma(z)$ to a meromorphic function for all $z \in \mathbb{C}$, the only poles being at $z \in \{0, -1, -2, -3, \ldots\}$, and each pole being simple. (Note that (7.1) defines $\Gamma(z)$ as an analytic function in the region $\{\Re z > 0\}$, and using integration by parts one proves that (7.2) holds in this region; it is then easy to prove that if (7.2) is used to define $\Gamma(z)$ also when $\Re z \leq 0$, then we get a meromorphic function as claimed.)

We have

$$\Gamma(n) = (n-1)!, \qquad \forall n \in \mathbb{N}.$$

The Γ-function is also given by the following infinite product formula (which is sometimes used as a definition):

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \Big(1 + \frac{z}{n}\Big) e^{-z/n}, \tag{7.3}$$

where γ is *Euler's constant,* defined so that $\Gamma(1) = 1$, i.e.

$$\gamma := -\log\Big(\prod_{n=1}^{\infty} \Big(1 + \frac{1}{n}\Big) e^{-1/n}\Big) = \lim_{N\to\infty} \Big(-\log N + \sum_{n=1}^{N} \frac{1}{n}\Big) = 0.57722\ldots. \tag{7.4}$$

The product in (7.3) is a so called Weierstrass product (cf. Wikipedia), and since $\sum_{n=1}^{\infty} n^{-2} < \infty$ the product in (7.3) is uniformly absolutely convergent[18] on compact subsets of $\mathbb{C}$, and (thus) $\Gamma(z)^{-1}$ is an entire

[18] Recall that a product $\prod_{n=1}^{\infty}(1 + u_n(z))$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |u_n(z)| < \infty$. Hence in the present case we should set $u_n(z) = (1 + \frac{z}{n}\Big) e^{-z/n} - 1$, and the claim is that then $\sum_{n=1}^{\infty} |u_n(z)|$ converges, and converges uniformly with respect to z in any compact subset of $\mathbb{C}$.

function which has simple zeros at each point $z = 0, -1, -2, \ldots$, and no other zeros.

The Γ-function satisfies the following relations (identities between meromorphic functions of $z \in \mathbb{C}$):

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}; \tag{7.5}$$

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z)\Gamma(z+\tfrac{1}{2}). \tag{7.6}$$

(The relation (7.6) is called Legendre's duplication formula.)

An important formula involving the Γ-function is the following:

$$\int_0^1 x^{a-1}(1-x)^{b-1}\, dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{7.7}$$

true for all $a, b \in \mathbb{C}$ with $\Re a, \Re b > 0$. (Cf. Folland p. 77, Exercise 60.) The above function (as a function of a and b) is called the *beta function*, $B(a, b)$. Many other integrals can be transformed into a beta function – namely any convergent integral of the form $\int_A^B L_1(x)^\alpha L_2(x)^\beta\, dx$ where L_1 and L_2 are two affine linear forms of x such that $L_1(x)$ is 0 or ∞ at $x = A$ and $L_2(x)$ is 0 or ∞ at $x = B$ (here A or B may be $\pm\infty$).

Finally, we mention that the $(n-1)$-dimensional volume of the unit sphere S_1^{n-1} is

$$\sigma(S_1^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(\frac{1}{2}n)},$$

and (hence) the volume of the n-dimensional unit ball is

$$\mathrm{vol}(B_1^n) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)}.$$

Cf. Folland, Prop. 2.54 and Cor. 2.55. (When using these formulas for n odd, note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$; hence $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$, etc.)

7.2. **Stirling's formula.** We have the following asymptotic formula for $\Gamma(z)$ for z large:

Theorem 7.1. ***(Stirling's formula.)*** *For any fixed $\varepsilon > 0$ we have*

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right) \log z - z + \log \sqrt{2\pi} + O\left(|z|^{-1}\right), \tag{7.8}$$

for all z with $|z| \geq 1$ and $\left|\arg z\right| \leq \pi - \varepsilon$. (The implied constant depends on ε but of course not on z. Also in the right hand side we use the principal branch of the logarithm function.) In fact we have the following more precise asymptotic formula, for any $m \in \mathbb{Z}_{\geq 0}$:

(7.9)

$$\log \Gamma(z) = \log \sqrt{2\pi} + \left(z - \tfrac{1}{2}\right) \log z - z + \sum_{k=0}^{m} \frac{B_{2k+2}}{(2k+2)(2k+1)} z^{-2k-1} + O\left(|z|^{-2m-3}\right)$$

for all z with $|z| \geq 1$ and $\left|\arg z\right| \leq \pi - \varepsilon$. (The implied constant depends on m and ε but of course not on z.) Here B_r is the rth Bernoulli number; cf. Definition 1.18.

Exponentiating, (7.8) gives:

$$\Gamma(z) = \sqrt{2\pi} \cdot \frac{z^{z-\frac{1}{2}}}{e^z} \cdot e^{O(|z|^{-1})} = \sqrt{2\pi} \cdot \frac{z^{z-\frac{1}{2}}}{e^z} \cdot \Big(1 + O(|z|^{-1})\Big)$$

for all z with $|z| \geq 1$ and $\left|\arg z\right| \leq \pi - \varepsilon$. Here if z is general complex one has to remember that $z^{z-\frac{1}{2}}$ is by definition the same as $\exp((z-\frac{1}{2}) \log z)$ where the principal branch of the logarithm is used.

There is a slight modification of Stirling's formula which is often convenient to remember:

Corollary 7.2. *For any fixed $\varepsilon > 0$ and $\alpha \in \mathbb{C}$ we have*

$$\log \Gamma(z + \alpha) = \left(z + \alpha - \tfrac{1}{2}\right) \log z - z + \log \sqrt{2\pi} + O\left(|z|^{-1}\right), \tag{7.10}$$

for all z with $|z| \geq 1$, $|z + \alpha| \geq 1$ and $\left|\arg(z + \alpha)\right| \leq \pi - \varepsilon$. (The implied constant depends on ε and α but of course not on z.)

This corollary follows more or less immediately from (7.8) by using $\log(z+\alpha) = \log z + \frac{\alpha}{z} + O(|z|^2)$ for $|z|$ large (viz., the Taylor expansion of $\log(1 + \alpha z^{-1})$ for z large); there are some instructive technicalities involved[19] in the proof and therefore we write it out in an appendix below; see Sec. 7.5.

It is important to note that if we are interested in finer asymptotics as in (7.9) then the transformation from z to $z + \alpha$ is *not* as simple as in (7.10) – but it can of course be worked out, using Taylor expansions.

[19] which fall outside the main scope of the course.

As an example, taking $\alpha = 1$ in Corollary 7.10 gives

$$\Gamma(z+1) = \sqrt{2\pi}\, z^{z+\frac{1}{2}}\, e^{-z}(1 + O(|z|^{-1})), \tag{7.11}$$

and in particular:

$$n! = \sqrt{2\pi} \cdot \frac{n^{n+\frac{1}{2}}}{e^n}(1 + O(n^{-1})), \qquad \forall n \geq 1.$$

Let us now discuss the *proof* of Stirling's formula: One method of proof is to use the product formula, (7.3), take the logarithm, and then apply the Euler-MacLaurin summation formula, Theorem 1.19. Let us discuss this in some detail. First, since $\Gamma(z)$ is a meromorphic function of $z \in \mathbb{C}$ with no zeros, and simple poles at $z = 0, -1, -2, \ldots$ and no other points, we can define a branch of $\log \Gamma(z)$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ [20] (cf., e.g., [18, Thm. 13.11(b)⇔(h)]). This branch is uniquely determined by requiring that $\log \Gamma(z) > 0$ for all large $z \in \mathbb{R}_{>0}$. Now the product formula, (7.3), implies:

$$\log \Gamma(z) = -\log z - \gamma z + \sum_{n=1}^{\infty} \Big(\frac{z}{n} - \log\Big(1 + \frac{z}{n}\Big)\Big), \qquad \forall z \in \mathbb{C} \setminus (-\infty, 0], \tag{7.12}$$

where in the right hand side the principal branch of the logarithm is used throughout. (Outline of details: One checks that the sum in (7.12) is uniformly absolutely convergent for z in compact subsets of $\mathbb{C} \setminus (-\infty, 0]$; hence the right hand side defines an analytic function in the region $\mathbb{C} \setminus (-\infty, 0]$. This function is real for $z \in \mathbb{R}_{>0}$, and using (7.3) one verifies that the exponential of this function equals $\Gamma(z)$. Done!)

By writing $\sum_{n=1}^{\infty}$ as $\lim_{N\to\infty} \sum_{n=1}^{N}$ and then using the fact that $\sum_{n=1}^{N} \frac{z}{n} - z \log N = \gamma z$ (cf. (7.4)), the formula (7.12) can be rewritten as ($\forall z \in \mathbb{C} \setminus (-\infty, 0]$):

$$\log \Gamma(z) = \lim_{N \to \infty} \Big(z \log N - \sum_{n=0}^{N} \log(z+n) + \sum_{n=1}^{N} \log n \Big). \tag{7.13}$$

This is the formula to which we apply the Euler-MacLaurin summation formula, Theorem 1.19. For the details of how this leads to Theorem 7.1, cf., e.g., Olver [16, Ch. 8.4].

[20] That is, an analytic function $g : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ satisfying $e^{g(z)} = \Gamma(z)$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$.

7.3. **Some estimates of the incomplete gamma functions.** In computations one often encounters the following integrals:

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt$$

and

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt.$$

These are called (upper and lower) *incomplete gamma functions;* cf. Wikipedia. Obviously,

$$\gamma(s,x)+\Gamma(s,x)=\Gamma(s) \qquad (x\in\mathbb{R}_{>0},\ s\in\mathbb{C},\ \Re s>0). \tag{7.14}$$

In this section we give some basic estimates of the incomplete gamma function for s real; note that in this case the integrand, “$t^{s-1}e^{-t}$”, is positive; and all of the following results can be seen as examples of basic techniques for estimating positive integrals.

As a first example, let us consider the *(complementary) error function;*

$$\operatorname{erfc}(A) := \frac{2}{\sqrt{\pi}}\int_A^\infty e^{-t^2}\,dt. \tag{7.15}$$

(Cf. Wikipedia.) The name of this function is connected with the normal distribution in probability theory: If X is a random variable with mean 0 and variance $\frac12$ then $\operatorname{erfc}(A)$ is the probability of the event $|X|>A$. Note that the error function is indeed a special case of an incomplete gamma function, since we have, by substituting $t=\sqrt{u}$:

$$\int_A^\infty e^{-t^2}\,dt = \frac12\int_{A^2}^\infty e^{-u}\,\frac{du}{\sqrt{u}} = \frac12\,\Gamma(\tfrac12, A^2). \tag{7.16}$$

The following lemma gives the order of magnitude of the integral in (7.15) as $A\to\infty$:

Lemma 7.3.

$$\int_A^\infty e^{-t^2}\,dt \asymp \frac{e^{-A^2}}{A} \qquad \textit{for all } A\geq 1.$$

Proof. The upper bound follows from

$$\int_A^\infty e^{-t^2}\,dt = \int_0^\infty e^{-(A+u)^2}\,du \leq \int_0^\infty e^{-A^2-2Au}\,du = \frac{e^{-A^2}}{2A},$$

and the lower bound follows from

$$\int_A^\infty e^{-t^2}\,dt = \int_0^\infty e^{-(A+u)^2}\,du \geq \int_0^1 e^{-A^2-2Au-1}\,du = e^{-A^2-1}\left[\frac{e^{-2Au}}{2A}\right]_{u=0}^{u=1}$$

$$= e^{-A^2-1}\,\frac{1-e^{-2A}}{2A} \geq \frac{1-e^{-2}}{2e}\cdot\frac{e^{-A^2}}{A}.$$

(Note that we used the assumption "$A \geq 1$" in the last "$\geq$".) $\square$

Of course, using $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$, Lemma 7.3 immediately implies

Lemma 7.4.
$$\int_0^A e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} + O\Big(\frac{e^{-A^2}}{A}\Big) \qquad \text{for all } A \geq 1.$$

The following bound generalizes[21] Lemma 7.3:

Lemma 7.5. *For any fixed $s \in \mathbb{R}$ and $c > 0$, we have*
$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt \asymp x^{s-1}e^{-x}, \qquad \forall x \geq c.$$
(The implied constant depends on s and c, but not on x.)

Proof. The bound from below is very easy; one merely has to note that $t^{s-1} \gg x^{s-1}$ for all $t \in [x, 2x]$ [22]; hence
$$\int_x^\infty t^{s-1}e^{-t}\,dt \gg x^{s-1}\int_x^{2x} e^{-t}\,dt = x^{s-1}(e^{-x} - e^{-2x}) \geq (1-e^{-c})\cdot x^{s-1}e^{-x}.$$

The bound from above is *also* very easy if $s \leq 1$, since then t^{s-1} is a decreasing function of t, and so
$$\int_x^\infty t^{s-1}e^{-t}\,dt \leq x^{s-1}\int_x^\infty e^{-t}\,dt = x^{s-1}e^{-x}.$$
For larger s we may integrate by parts in order to lower the exponent of t in the integrand: Assume $k < s \leq k+1$ with $k \in \mathbb{Z}^+$. Integrating by parts k times gives
$$\begin{aligned}\int_x^\infty t^{s-1}e^{-t}\,dt &= \Big[-t^{s-1}e^{-t}\Big]_x^\infty + (s-1)\int_x^\infty t^{s-2}e^{-t}\,dt \\ &= x^{s-1}e^{-x} + (s-1)\int_x^\infty t^{s-2}e^{-t}\,dt \\ &\dots \\ &= \sum_{j=1}^{k}\Big(\prod_{m=1}^{j-1}(s-m)\Big)x^{s-j}e^{-x} + \Big(\prod_{m=1}^{k}(s-m)\Big)\int_x^\infty t^{s-k-1}e^{-t}\,dt.\end{aligned}$$
Here t^{s-k-1} is a decreasing function of t and hence $\int_x^\infty t^{s-k-1}e^{-t}\,dt \leq x^{s-k-1}\int_x^\infty e^{-t}\,dt = x^{s-k-1}e^{-x}$, implying that the above expression is
$$\leq \sum_{j=1}^{k+1}\Big(\prod_{m=1}^{j-1}(s-m)\Big)x^{s-j}e^{-x} \ll x^{s-1}e^{-x}.$$
$\square$

[21](via (7.16))

[22]Indeed, if $s \geq 1$ then $t^{s-1} \geq x^{s-1}$ for all $t \geq x$, while if $0 < s \leq 1$ then $t^{s-1} \geq 2^{s-1}x^{s-1}$ for all $t \in [x, 2x]$.

Remark 7.6. The computation in the above proof, where we integrated by parts k times, is valid for *any* $s > 0$ and $k \in \mathbb{Z}^+$, and it immediately implies the following *asymptotic expansion* of $\Gamma(s,x)$ for large x: For any fixed $s \in \mathbb{R}$, $c > 0$ and $k \in \mathbb{Z}^+$, we have

$$\Gamma(s,x) = \sum_{j=1}^{k} a_{s,j}\, x^{s-j} e^{-x} + O\Big(x^{s-k-1} e^{-x}\Big) \qquad \forall x \geq c, \tag{7.17}$$

with $a_{s,j} = \prod_{m=1}^{j-1}(s-m)$. [Details: When $k \geq s-1$, the expansion in (7.17) follows immediately by the arguments at the end of the proof of the lemma. But then one also notes that the claim in (7.17) for one fixed k immediately implies the corresponding claim for any smaller (positive) k. Hence (7.17) indeed holds for any fixed $k \in \mathbb{Z}^+$.]

Exercise 7.1. Deduce from (7.17) a precise asymptotic expansion of $\text{erfc}(A)$ for large A. Compare with Wikipedia.

Let us also note that, using (7.14), Lemma 7.5 immediately gives an asymptotic formula for $\gamma(s,x)$ valid for fixed $s > 0$ and large x:

Lemma 7.7. *For any fixed $s > 0$ and $c > 0$, we have*

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t}\, dt = \Gamma(s) + O\big(x^{s-1} e^{-x}\big), \qquad \forall x \geq c.$$

(The implied constant depends on s and c, but not on x.)

7.4. **Γ-asymptotics directly from the integral.** As a further, more involved, example of techniques for bounding and estimating positive integrals, we will here discuss the asymptotics of $\Gamma(s)$ as $s \to \infty$ along the real axis, using the formula $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$ (cf. (7.1)).

This present section should be compared with Olver [16, Ch. 3.8] and Wong [20, Ch. II.1].

We assume from start that $s > 0$, and we will focus on the case of s large. We consider the integral

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx,$$

and we note that the integrand is positive for all x. It turns out that our computations will be quite a bit cleaner if we work instead with

$$\lambda := s - 1.$$

Thus: We assume from start that $\lambda > -1$, and we consider the integral

$$\Gamma(\lambda + 1) = \int_0^\infty e^{-x} x^{\lambda}\, dx.$$

As a first step we determine the monotonicity properties of the integrand. Set

$$f(x) = e^{-x}x^{\lambda}.$$

We compute:

$$f'(x) = (\lambda - x)x^{\lambda-1}e^{-x}.$$

Already here we see that it is convenient to assume $\lambda > 0$, so let's assume this. Now we see that $f(x)$ is increasing for $0 < x < \lambda$ and decreasing for $x > \lambda$; thus $f(x)$ attains its maximum at $x = \lambda$. When studying how quickly $f(x)$ descends when x moves away from λ it is convenient to set $x = \lambda + y$, and consider the *logarithm* of $f(x)$:

$$\begin{aligned}\log f(\lambda + y) &= -(\lambda + y) + \lambda \log(\lambda + y)\\ &= \lambda\Big(-1 - \frac{y}{\lambda} + \log\lambda + \log\Big(1 + \frac{y}{\lambda}\Big)\Big).\end{aligned}$$

We see that it is convenient to take $u = \frac{y}{\lambda}$ as a new variable. Then:

$$\log f\Big(\lambda(1+u)\Big) = \lambda \log\Big(\frac{\lambda}{e}\Big) - \lambda\Big(u - \log(1+u)\Big),$$

or in other words:

$$f(\lambda(1+u)) = \Big(\frac{\lambda}{e}\Big)^{\lambda} e^{-\lambda(u - \log(1+u))}.$$

Substituting $x = \lambda(1+u)$ in our integral we get:

$$\Gamma(\lambda + 1) = \lambda^{\lambda+1}e^{-\lambda}\int_{-1}^{\infty} e^{-\lambda(u-\log(1+u))}\,du. \tag{7.18}$$

Set

$$g(u) = u - \log(1+u).$$

It is clear from the above analysis that $g(u)$ attains its minimum at $u = 0$ [23]. Using the Taylor expansion for $\log(1+u)$ we see that, for any fixed $0 < c_0 < 1$,

$$g(u) = \frac{1}{2}u^2 + O(u^3), \qquad \forall u \in [-c_0, c_0].$$

Hence for any $0 < \alpha \le c_0$:

$$\begin{aligned}\int_{-\alpha}^{\alpha} e^{-\lambda(u-\log(1+u))}\,du &= \int_{-\alpha}^{\alpha} e^{-\frac{1}{2}\lambda(u^2+O(u^3))}\,du \qquad \Big\{\text{Set } \ u = \big(\tfrac{2}{\lambda}\big)^{1/2}t\Big\}\\ &= \sqrt{\frac{2}{\lambda}}\int_{-\alpha(\lambda/2)^{1/2}}^{\alpha(\lambda/2)^{1/2}} e^{-t^2+O(\lambda^{-\frac{1}{2}}t^3)}\,dt.\end{aligned}$$

[23] This can of course also easily be checked at this point: We have $g'(u) = \frac{u}{1+u}$, thus $g'(u) < 0$ for $-1 < u < 0$ and $g'(u) > 0$ for $u > 0$.

If we assume that $\alpha \leq 10\lambda^{-\frac{1}{3}}$ (say) then $\lambda^{-\frac{1}{2}}t^3$ in the last integral always has a bounded absolute value, and hence we may continue as follows:

$$= \sqrt{\frac{2}{\lambda}} \int_{-\alpha(\lambda/2)^{1/2}}^{\alpha(\lambda/2)^{1/2}} e^{-t^2} \Big(1 + O(\lambda^{-\frac{1}{2}}t^3)\Big)\, dt$$

$$= \sqrt{\frac{2}{s-1}} \int_{-\alpha(\lambda/2)^{1/2}}^{\alpha(\lambda/2)^{1/2}} e^{-t^2}\, dt + O(\lambda^{-1}).$$

(In the last equality we simply used $\int_{-\infty}^{\infty} e^{-t^2}|t^3|\, dt = O(1)$.) Using Lemma 7.3 to estimate the explicit integral, we conclude:

$$\int_{-\alpha}^{\alpha} e^{-\lambda(u-\log(1+u))}\, du = \sqrt{\frac{2\pi}{\lambda}} + O\Big(\frac{e^{-\frac{1}{2}\lambda\alpha^2}}{\alpha\lambda} + \frac{1}{\lambda}\Big).$$

It remains to bound the integrals $\int_{-1}^{-\alpha}$ and $\int_{\alpha}^{\infty}$. The first of these is easily handled using the fact that $u - \log(1+u) \geq \frac{1}{2}u^2$ for all $u \in (-1, 0]$; this is clear e.g. from the Taylor expansion of $\log(1+u)$. Hence we get:

$$\int_{-1}^{-\alpha} e^{-\lambda(u-\log(1+u))}\, du \leq \int_{-1}^{-\alpha} e^{-\frac{1}{2}\lambda u^2}\, du$$
$$\leq \sqrt{\frac{2}{\lambda}} \int_{\alpha(\lambda/2)^{1/2}}^{\infty} e^{-t^2}\, dt \ll \frac{e^{-\frac{1}{2}\lambda\alpha^2}}{\alpha\lambda}.$$

(Cf. Lemma 7.3 regarding the last bound.) Finally to bound the $\int_{\alpha}^{\infty}$ we use the fact that there is an absolute constant $c_1 > 0$ such that $u - \log(1+u) \geq c_1 u^2$ for all $u \in [0, 1]$. (Prove this fact as an exercise! In fact the optimal choice of c_1 is $c_1 = 1 - \log 2 = 0.3068\ldots$.) Furthermore there is an absolute constant $c_2 > 0$ such that for all $u \geq 1$ we have $u - \log(1 + u) \geq c_2 u$. (Prove this fact as an exercise!) Hence

$$\int_{\alpha}^{\infty} e^{-\lambda(u-\log(1+u))}\, du \leq \int_{\alpha}^{1} e^{-c_1\lambda u^2}\, du + \int_{1}^{\infty} e^{-c_2\lambda u}\, du$$
$$\ll \frac{e^{-c_1\lambda\alpha^2}}{\alpha\lambda} + \frac{e^{-c_2\lambda}}{\lambda}.$$

Adding together the integrals we have now proved that (using the fact that $0 < c_1 \leq \frac{1}{2}$ and thus $e^{-\frac{1}{2}\lambda\alpha^2} \leq e^{-c_1\lambda\alpha^2}$):

$$\int_{-1}^{\infty} e^{-\lambda(u-\log(1+u))}\, du = \sqrt{\frac{2\pi}{\lambda}} + O\Big(\frac{e^{-c_1\lambda\alpha^2}}{\alpha\lambda} + \frac{1}{\lambda}\Big).$$

The last relation has been proved for any $\lambda > 0$ and any α satisfying $0 < \alpha \leq c_0 < 1$ and $\alpha \leq 10\lambda^{-\frac{1}{3}}$. Making now a definite choice of α, say $\alpha = \frac{1}{2}\lambda^{-\frac{1}{3}}$, and assuming $\lambda \geq 1$ so as to ensure $\alpha \leq \frac{1}{2}$, we trivially have that the first error term decays exponentially (with respect to λ)

and is subsumed by the second error term, $O(\lambda^{-1})$. Hence, recalling (7.18), we have proved:

$$\Gamma(\lambda+1) = \sqrt{2\pi}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda}\Big(1+O\big(\lambda^{-\frac{1}{2}}\big)\Big), \qquad \forall\lambda\geq 1. \tag{7.19}$$

Note that this agrees with (7.11) above, but with a worse error term.

In order to get a better error term, and even an asymptotic expansion of $\Gamma(\lambda+1)$, we modify the treatment of (7.18) as follows:[24] We have seen that $g(u) = u - \log(1+u)$ is strictly decreasing for $u \in (-1, 0]$ and strictly increasing for $u \in [0,\infty)$, since $g'(u) = \frac{u}{1+u}$. Hence the function g has continuous *inverses*, $h_1 : [0,\infty) \to (-1, 0]$ (a decreasing function with $h_1(0) = 0$, satisfying $g(h_1(v)) = v$, $\forall v \in \mathbb{R}_{\geq 0}$) and $h_2 : [0,\infty) \to [0,\infty)$ (an increasing function with $h_2(0) = 0$, satisfying $g(h_2(v)) = v$, $\forall v \in \mathbb{R}_{\geq 0}$). We note that h_1 and h_2 are C^∞ in $(0,\infty)$, and substituting $u = h_1(v)$ and $u = h_2(v)$ in the integral in (7.18) we obtain:

$$\begin{aligned}\int_{-1}^{\infty} e^{-\lambda(u-\log(1+u))}\,du &= \int_{-1}^{0}\cdots+\int_0^\infty\cdots \\ &= \int_0^\infty e^{-\lambda v}(-h_1'(v))\,dv + \int_0^\infty e^{-\lambda v}h_2'(v)\,dv. \end{aligned}\tag{7.20}$$

By implicit differentiation using $g(h_j(v)) = v$ we see that

$$h_j'(v) = 1 + h_j(v)^{-1}, \qquad j = 1, 2.$$

In particular, for any fixed constant $c_3 > 0$, both $-h_1'(v)$ and $h_2'(v)$ are *bounded* and positive for all $v \geq c_3$. (Namely: $0 < -h_1'(v) \leq -1 - h_1(c_3)^{-1}$ and $0 < h_2'(v) \leq 1 + h_2(c_3)^{-1}$.) Hence the contribution from $v \geq c_3$ to the integrals in (7.20) is:

$$\ll \int_{c_3}^\infty e^{-\lambda v}\,dv = \frac{e^{-c_3\lambda}}{\lambda}, \tag{7.21}$$

i.e. exponentially small.

It remains to treat the integrals for v near 0, and here we will use the power series expansion of $h_j(v)$: Since $g(u)$ is analytic for $u \in \mathbb{C}$ with $|u| < 1$, with the power series

$$g(u) = u - \log(1+u) = \frac{1}{2}u^2 - \frac{1}{3}u^3 + \frac{1}{4}u^4 - \cdots,$$

it follows that there exists some open disc $\Omega \subset \mathbb{C}$ centered at 0, and an analytic function $H : \Omega \to \mathbb{C}$, such that $H(0) = 0$, $H(w) > 0$ for

[24]The following discussion is in fact a special case of *Laplace's method* for obtaining the asymptotic expansion of an integral of the form $\int_a^b \varphi(t)e^{-\lambda h(t)}\,dt$ as $\lambda \to \infty$, with h a *real-valued* function. Again compare Olver [16, Ch. 3.8] and Wong [20, Ch. II.1].

$w \in \Omega \cap \mathbb{R}_{>0}$, and $g(H(w)) = w^2$ for all $w \in \Omega$. The power series for $H(w)$ can be found by substituting in $g(H(w)) = w^2$, and we compute:

$$H(w) = \sqrt{2}w + \frac{2}{3}w^2 + \frac{1}{9\sqrt{2}}w^3 + \ldots, \qquad \forall w \in \Omega. \tag{7.22}$$

Let $r > 0$ be the radius of Ω. Now since $v \mapsto H(\sqrt{v})$ for $v \in [0, r^2)$ is a continuous function with $H(\sqrt{0}) = 0$, increasing at least for small v, and whose composition with g is the identity function, we conclude that $h_2(v) = H(\sqrt{v})$ for all $v \in [0, r^2)$. Similarly $h_1(v) = H(-\sqrt{v})$ for all $v \in [0, r^2)$. Differentiating this relation gives

$$h_j'(v) = \frac{(-1)^j}{2\sqrt{v}} H'((-1)^j\sqrt{v}) = \frac{(-1)^j}{2\sqrt{v}}\Big(\sqrt{2} + \frac{4}{3}(-1)^j\sqrt{v} + \frac{1}{3\sqrt{2}}v + \ldots\Big),$$

for all $v \in (0, r^2)$. Hence if we choose $c_3 = \frac{1}{2}r^2$, say, then we have

$$\int_0^{c_3} e^{-\lambda v}(-h_1'(v))\,dv = \int_0^{c_3} e^{-\lambda v}\Big(\frac{1}{\sqrt{2v}} - \frac{2}{3} + \frac{\sqrt{v}}{6\sqrt{2}} + O(v)\Big)\,dv \tag{7.23}$$

Here note that for each fixed $\alpha > -1$, we have by Lemma 7.7 (assuming $s \geq 2$, say):

$$\int_0^{c_3} e^{-\lambda v} v^\alpha\,dv = \lambda^{-\alpha-1}\int_0^{c_3\lambda} e^{-u}u^\alpha\,du = \Gamma(\alpha+1)\lambda^{-(\alpha+1)} + O\big(\lambda^{-1}e^{-c_3\lambda}\big).$$

Using this for $\alpha = -\frac{1}{2}$, 0, $\frac{1}{2}$ and 1, we conclude from (7.23):

$$\int_0^{c_3} e^{-\lambda v}(-h_1'(v))\,dv = \frac{\Gamma(\frac{1}{2})}{\sqrt{2}}\lambda^{-\frac{1}{2}} - \frac{2}{3}\Gamma(1)\lambda^{-1} + \frac{\Gamma(\frac{3}{2})}{6\sqrt{2}}\lambda^{-\frac{3}{2}} + O\Big(\lambda^{-2}\Big)$$

(since the error term $\lambda^{-1}e^{-c_3\lambda}$ decays exponentially and is therefore subsumed by $O(\lambda^{-2})$). Similarly

$$\int_0^{c_3} e^{-\lambda v}h_2'(v)\,dv = \frac{\Gamma(\frac{1}{2})}{\sqrt{2}}\lambda^{-\frac{1}{2}} + \frac{2}{3}\Gamma(1)\lambda^{-1} + \frac{\Gamma(\frac{3}{2})}{6\sqrt{2}}\lambda^{-\frac{3}{2}} + O\Big(\lambda^{-2}\Big).$$

Adding these together, and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ and the fact that the contribution from $v \geq c_3$ is bounded by (7.21), we conclude:

$$\int_{-1}^{\infty} e^{-\lambda(u-\log(1+u))}\,du = \sqrt{2\pi}\Big(\lambda^{-\frac{1}{2}} + \frac{1}{12}\lambda^{-\frac{3}{2}} + O\big(\lambda^{-2}\big)\Big). \tag{7.24}$$

It is clear from the $\pm$ symmetry that if we keep track of one more term in (7.22), then the λ^{-2} cancels, i.e. we may actually improve the error in (7.24) to $O(\lambda^{-\frac{5}{2}})$. Hence, recalling (7.18), we have proved:

$$\Gamma(\lambda+1) = \sqrt{2\pi}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda}\Big(1 + \frac{1}{12}\lambda^{-1} + O(\lambda^{-2})\Big), \qquad \forall \lambda \geq 1. \tag{7.25}$$

Clearly by keeping track of more terms in (7.22) we can obtain an asymptotic expansion with an error $O(\lambda^{-N})$ for any fixed $N \in \mathbb{Z}^+$.

Exercise 7.2. By keeping track of a few more terms, verify that

(7.26)
$$\Gamma(\lambda+1)=\sqrt{2\pi}\lambda^{\lambda+\frac{1}{2}}e^{-\lambda}\Big(1+\frac{1}{12}\lambda^{-1}+\frac{1}{288}\lambda^{-2}+O(\lambda^{-3})\Big),\qquad \forall\lambda\geq 1.$$

Finally let us verify that (7.25) and (7.26) agree with Stirling's formula, Theorem 7.1 – that is, let us deduce from (7.26) an asymptotic expansion of $\Gamma(s)$ in terms of negative powers of s. This can be proved by using Taylor expansions in a similar manner as in Section 7.5. However, since the jump step is exactly one when passing from λ to $s=\lambda+1$, it is simpler to just use the formula (7.1)! Thus, (7.26) implies that for all $s\geq 1$:

$$\log\Gamma(s)=\log\big(s^{-1}\Gamma(s+1)\big)$$
$$=\log\sqrt{2\pi}+(s-\tfrac{1}{2})\log s-s+\log\bigg(1+\frac{1}{12}s^{-1}+\frac{1}{288}s^{-2}+O(s^{-3})\bigg),$$

and writing $\alpha:=\frac{1}{12}s^{-1}+\frac{1}{288}s^{-2}+O(s^{-3})$ we have, if s is sufficiently large: $\log(1+\alpha)=\alpha-\frac{1}{2}\alpha^2+O(\alpha^3)=\frac{1}{12}s^{-1}+\frac{1}{288}s^{-2}-\frac{1}{2}\cdot\frac{1}{12^2}s^{-2}+O(s^{-3})=\frac{1}{12}s^{-1}+O(s^{-3})$. Hence:

$$\log\Gamma(s)=\log\sqrt{2\pi}+(s-\tfrac{1}{2})\log s-s+\frac{1}{12}s^{-1}+O(s^{-3}).$$

This agrees with (7.9) for $m=0$, since $B_2=\frac{1}{6}$.

7.5. **Appendix: Proof of Corollary 7.2.**

By Stirling's formula, Theorem 7.1, we have

$$\log\Gamma(z+\alpha) = \big(z+\alpha-\tfrac{1}{2}\big)\log(z+\alpha) - (z+\alpha) + \log\sqrt{2\pi} + O\big(|z+\alpha|^{-1}\big), \tag{7.27}$$

for all z with $|z+\alpha| \geq 1$ and $\big|\arg(z+\alpha)\big| \leq \pi-\varepsilon$. Here and below, for definiteness, we consider the argument function to take its values in $(-\pi,\pi]$, i.e. $\arg : \mathbb{C}\setminus\{0\} \to (-\pi,\pi]$.

Let us fix a constant $C > 1$ so large that $\big|\arg(1+w)\big| < \frac{1}{2}\varepsilon$ for all $w \in \mathbb{C}$ with $|w| \leq C^{-1}$. Note that if $|z| \geq C|\alpha|$ and $|z| \geq 1$ then

$$\arg(z+\alpha) = \arg(z(1+\alpha/z)) \equiv \arg(z) + \arg(1+\alpha/z) \pmod{2\pi};$$

also $\big|\arg(z+\alpha)\big| \leq \pi-\varepsilon$ and $\big|\arg(1+\alpha/z)\big| < \frac{1}{2}\varepsilon$, and therefore $\big|\arg(z)\big| \leq \pi - \frac{1}{2}\varepsilon$ and

$$\arg(z+\alpha) = \arg(z) + \arg(1+\alpha/z).$$

Hence

$$\log(z+\alpha) = \log z + \log\Big(1+\frac{\alpha}{z}\Big),$$

where in all three places we use the principal branch of the logarithm function. Since $|\alpha/z| \leq C^{-1} < 1$ we can continue:

$$\log(z+\alpha) = \log z + \frac{\alpha}{z} + O\Big(\frac{\alpha^2}{z^2}\Big) = \log z + \frac{\alpha}{z} + O\big(|z|^{-2}\big)$$

(since we allow the implied constant to depend on α). Using this in (7.27) we get

$$\begin{aligned}\log\Gamma(z+\alpha) &= \big(z+\alpha-\tfrac{1}{2}\big)\Big(\log z + \frac{\alpha}{z} + O\big(|z|^{-2}\big)\Big) - (z+\alpha) + \log\sqrt{2\pi} + O\big(|z+\alpha|^{-1}\big)\\ &= \big(z+\alpha-\tfrac{1}{2}\big)\log z - z + \log\sqrt{2\pi} + O\big(|z|^{-1}\big) + O\big(|z+\alpha|^{-1}\big)\\ &= \big(z+\alpha-\tfrac{1}{2}\big)\log z - z + \log\sqrt{2\pi} + O\big(|z|^{-1}\big),\end{aligned}$$

where in the last step we used the fact that $|z+\alpha| \geq |z| - |\alpha| = |z|\big(1-|\alpha/z|\big) \geq (1-C^{-1})|z| \gg |z|$. Hence we have proved the desired formula for all z satisfying $|z| \geq 1$, $|z+\alpha| \geq 1$, $|\arg(z+\alpha)| \leq \pi-\varepsilon$ and $|z| \geq C|\alpha|$.

It remains to treat z satisfying $|z| \geq 1$, $|z+\alpha| \geq 1$, $|\arg(z+\alpha)| \leq \pi-\varepsilon$ and $|z| \leq C|\alpha|$. This is trivial: The set of such z is *compact* and $\log\Gamma(z+\alpha) - \big(z+\alpha-\frac{1}{2}\big)\log z + z - \log\sqrt{2\pi}$ is continuous on this set, hence bounded. Also $|z|$ is bounded on the set; hence $|z|^{-1}$ is bounded from below. Hence by adjusting the implied constant we have $\log\Gamma(z+\alpha) - \big(z+\alpha-\frac{1}{2}\big)\log z + z - \log\sqrt{2\pi} = O\big(|z|^{-1}\big)$ for all z in our compact set, as desired. $\square$

8. The J-Bessel function

In this section we give an introduction to the J-Bessel function.

8.1. **Introduction.**

The J-Bessel function can be defined by the following Taylor series expansion around $z = 0$:

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \Big(\frac{z}{2}\Big)^{2m}. \tag{8.1}$$

This definition works for any $\nu \in \mathbb{C}$ and any $z \in \mathbb{C} \setminus (-\infty, 0]$, say, and for fixed ν we see that $J_\nu(z)$ is an analytic function of $z \in \mathbb{C} \setminus (-\infty, 0]$. ($J_\nu(z)$ is also jointly analytic in the variables ν, z.) Note that if ν happens to be a nonnegative integer then $J_\nu(z)$ is in fact an *entire* function, i.e. an analytic function in $z \in \mathbb{C}$.

Note that $J_\nu(z)$ is real when ν and z are real. See Figure 3 for graphs of the functions $J_\nu(z)$ for $z > 0$, for $\nu = 0$, 1, 10 and 100.

For given $\nu \in \mathbb{C}$, the function $z \mapsto J_\nu(z)$ is a solution of the so called *Bessel differential equation*,

$$f''(z) + \frac{1}{z} f'(z) + \Big(1 - \frac{\nu^2}{z^2}\Big) f(z) = 0. \tag{8.2}$$

The solution (8.1) of (8.2) is easily obtained using the Frobenius method; that is, one makes the Ansatz $f(z) = z^\nu \sum_{n=0}^\infty a_n z^n$ and seeks possible values of ν and $a_0, a_1, \ldots \in \mathbb{C}$ ($a_0 \neq 0$) which make f satisfy (8.2). Note that also $J_{-\nu}(z)$ is a solution to (8.2) (since the equation (8.2) remains unchanged when replacing ν by $-\nu$) and in fact, if $\nu \notin \mathbb{Z}$, then $\{J_\nu(z), J_{-\nu}(z)\}$ form a *fundamental system* of solutions, i.e. any solution to (8.2) can be expressed as a linear combination of these two. In the (important!) special case $\nu = n \in \mathbb{Z}$ however, we have

$$J_{-n}(z) = (-1)^n J_n(z) \qquad (n \in \mathbb{Z}), \tag{8.3}$$

and another function is needed to obtain a fundamental system of solutions to (8.2).

Exercise 8.1. Verify (8.3) directly from (8.1).
[Hint: For $n \in \mathbb{Z}^+$, the first n terms in the sum defining $J_{-n}(z)$ *vanish.*]

Let us record some basic recurrence relations for the Bessel functions, both of which can be proved directly from (8.1):

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z); \tag{8.4}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z). \tag{8.5}$$

From these we also deduce:

$$J_{\nu+1}(z) = \frac{\nu}{z} J_\nu(z) - J'_\nu(z); \tag{8.6}$$

$$J_{\nu-1}(z) = \frac{\nu}{z} J_\nu(z) + J'_\nu(z). \tag{8.7}$$

An alternative formula which can be taken as the definition of the J-Bessel function when $\Re\nu > -\frac{1}{2}$ is the following (cf. [6, 8.411.10]):

$$\begin{aligned} J_\nu(z) &= \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\, dt \\ &= \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-\pi/2}^{\pi/2} e^{iz\sin\theta}(\cos\theta)^{2\nu}\, d\theta. \end{aligned} \tag{8.8}$$

Proof of (8.8) *from our definition* (8.1)*:* The second formula follows from the first by substituting $t = \sin\theta$. It remains to prove the first formula. Using $e^{izt} = \sum_{n=0}^{\infty} \frac{(izt)^n}{n!}$ (which is true for all z, t) we have

$$\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\, dt = \int_{-1}^{1} \sum_{n=0}^{\infty} \frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\, dt.$$

Here we may change order of summation and integration, since

$$\begin{aligned} \int_{-1}^{1} \sum_{n=0}^{\infty} \Big| \frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}} \Big|\, dt &= \int_{-1}^{1} \sum_{n=0}^{\infty} \frac{|izt|^n}{n!}(1-t^2)^{\Re\nu-\frac{1}{2}}\, dt \\ = \int_{-1}^{1} e^{|zt|}(1-t^2)^{\Re\nu-\frac{1}{2}}\, dt &\le e^{|z|} \int_{-1}^{1} (1-t^2)^{\Re\nu-\frac{1}{2}}\, dt < \infty, \end{aligned}$$

where the last step holds since $\Re\nu > -\frac{1}{2}$. Hence

$$\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\, dt = \sum_{n=0}^{\infty} \int_{-1}^{1} \frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\, dt.$$

In the last expression, for each odd n the integrand is an odd function of t and hence the integral vanishes. Thus we may continue:

$$= \sum_{m=0}^{\infty} \int_{-1}^{1} \frac{(izt)^{2m}}{(2m)!}(1-t^2)^{\nu-\frac{1}{2}}\, dt = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m)!} \cdot 2 \int_0^1 t^{2m}(1-t^2)^{\nu-\frac{1}{2}}\, dt.$$

Substituting $t = \sqrt{u}$ in the integral and then using (7.7) and (7.6) we get:

$$= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\Gamma(2m+1)} \cdot \int_0^1 u^{m-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}\,du$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\pi^{-\frac{1}{2}} 2^{2m}\Gamma(m+\frac{1}{2})\Gamma(m+1)} \frac{\Gamma(m+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(m+\nu+1)}$$
$$= \sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2}) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \Big(\frac{z}{2}\Big)^{2m}.$$

Hence, comparing with (8.1) and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have proved (8.8). □

For $\nu = n \in \mathbb{Z}$ we have the following alternative integral formula for $J_n(z)$ (cf. [6, 8.411.1]):

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin\theta - in\theta}\,d\theta \qquad (n \in \mathbb{Z}). \tag{8.9}$$

Before proving this formula, let us make some observations about the right hand side. Given $z \in \mathbb{C}$ we set $f(\theta) = e^{iz\sin\theta}$; note that f is periodic with period 2π, i.e. $f(\theta + 2\pi) = f(\theta)$ for all θ. It follows that the right hand side in (8.9) equals $\frac{1}{2\pi}\int_a^{a+2\pi} e^{iz\sin\theta - in\theta}\,d\theta$ for any $a \in \mathbb{R}$, and in fact the formula says that $J_n(z)$ is the nth Fourier coefficient of the periodic function f.[25] Furthermore, by substituting $\theta = \pi - \omega$ one verifies that the integral satisfies the following $n \leftrightarrow -n$ symmetry relation:

(8.10)
$$\int_{-\pi}^{\pi} e^{iz\sin\theta - in\theta}\,d\theta = \int_0^{2\pi} e^{iz\sin\theta - in\theta}\,d\theta = (-1)^n \int_{-\pi}^{\pi} e^{iz\sin\omega + in\omega}\,d\omega.$$

We now give the *proof of* (8.9)*:* Because of (8.10) and (8.3), it suffices to prove the formula for $n \in \mathbb{Z}_{\geq 0}$. We have $e^{iz\sin\theta} = \sum_{k=0}^{\infty} \frac{(iz\sin\theta)^k}{k!}$, with absolute convergence uniformly over $\theta \in [-\pi, \pi]$; hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\theta - in\theta}\,d\theta = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} \frac{1}{2\pi}\int_{-\pi}^{\pi} (\sin\theta)^k e^{-in\theta}\,d\theta. \tag{8.11}$$

Here use $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ to get $(\sin\theta)^k = (2i)^{-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{(k-2j)i\theta}$. From this we see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\sin\theta)^k e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{if } k < n \text{ or } k \not\equiv n \bmod 2; \\ (2i)^{-k}(-1)^{\frac{1}{2}(k-n)}\binom{k}{(k-n)/2} & \text{if } k \geq n \text{ and } n \equiv k \bmod 2. \end{cases}$$

[25] To connect with the notation in Folland [5, Ch. 8] one should of course consider the rescaled function $g(\theta) = f(2\pi\theta)$ which is periodic with period 1, i.e. a function on $\mathbb{T}$; and (8.9) says that $J_n(z) = \widehat{g}(n)$.

Writing $k = n + 2m$ we thus conclude that the expression in (8.11) is

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{n+2m}}{(n+2m)!} \binom{n+2m}{m} = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{n+2m}}{m!(n+m)!} = J_n(z),$$

where the last equality holds because of (8.1). □

Exercise 8.2. Give an alternative proof of (8.9), by using (8.8) instead of (8.1). [Cf. the brief outline in Stein, [19, p. 338 (below (16))].]

8.2. **Application: The Fourier transform of radial functions.** The J-Bessel function can be used to express the Fourier transform of the surface measure σ on the unit sphere S_1^{n-1}. [26] Indeed (cf. Folland's Exercise 22, p. 256), for any $\xi \in \mathbb{R}^n$,

$$\widehat{\sigma}(\xi) = \int_{\mathrm{S}_1^{n-1}} e^{-2\pi i \xi \cdot \omega} \, d\sigma(\omega) = \int_{\mathrm{S}_1^{n-1}} e^{-2\pi i |\xi| \omega_1} \, d\sigma(\omega)$$
$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} e^{-2\pi i |\xi| \omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} \, d\omega_1 = 2\pi |\xi|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi|\xi|),$$

where we first used the rotational symmetry, then used (4.2), and finally used (8.8).

It follows that if $F \in L^1(\mathbb{R}^n)$ is *any* radial function, i.e. a function such that $F(\xi)$ only depends on $|\xi|$; say $F(\xi) = f(|\xi|)$, then

$$\widehat{F}(\xi) = \int_{\mathbb{R}^n} F(x) e^{-2\pi i \xi \cdot x} \, dx = \int_0^\infty f(\rho) \int_{\mathrm{S}_1^{n-1}} e^{-2\pi i \xi \cdot \rho\omega} \, d\sigma(\omega) \, \rho^{n-1} \, d\rho$$

$$= \int_0^\infty f(\rho) \widehat{\sigma}(\rho\xi) \rho^{n-1} \, d\rho = 2\pi |\xi|^{1-\frac{n}{2}} \int_0^\infty f(\rho) \rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi\rho|\xi|) \, d\rho. \tag{8.12}$$

Example 8.1. Let us compute the Fourier transform of $\chi_{B_1^n}$, the characteristic function of the unit ball B_1^n in $\mathbb{R}^n$. We have $\chi_{B_1^n}(\xi) = f(|\xi|)$ with $f = \chi_{[0,1]}$. Hence by (8.12),

$$\widehat{\chi_{B_1^n}}(\xi) = 2\pi|\xi|^{1-\frac{n}{2}} \int_0^1 \rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi\rho|\xi|) \, d\rho \qquad \Big\{ \text{set } \ \rho = u/(2\pi|\xi|) \Big\}$$
$$= (2\pi)^{-\frac{n}{2}} |\xi|^{-n} \int_0^{2\pi|\xi|} u^{\frac{n}{2}} J_{\frac{n}{2}-1}(u) \, du.$$

However, using (8.7) we have $\frac{d}{du}(u^{\frac{n}{2}} J_{\frac{n}{2}}(u)) = u^{\frac{n}{2}}(\frac{n/2}{u} J_{\frac{n}{2}}(u) + J'_{\frac{n}{2}}(u)) = u^{\frac{n}{2}} J_{\frac{n}{2}-1}(u)$, i.e. $u^{\frac{n}{2}} J_{\frac{n}{2}}(u)$ is a primitive function of $u^{\frac{n}{2}} J_{\frac{n}{2}-1}(u)$. Also this function $u^{\frac{n}{2}} J_{\frac{n}{2}}(u)$ equals 0 at $u = 0$. Hence we conclude:

$$\widehat{\chi_{B_1^n}}(\xi) = |\xi|^{-\frac{n}{2}} J_{n/2}(2\pi|\xi|).$$

Returning to the general situation, note that the formula (8.12) gives an explicit expression for the Fourier transform of a radial function (which is again a radial function). Applying now *Fourier inversion*, we conclude that for any "nice" $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ [27], if we define

$$\widetilde{f}(r) = 2\pi r^{1-\frac{n}{2}} \int_0^\infty f(\rho) \rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi r\rho) \, d\rho,$$

[26] We view σ as a Borel measure on $\mathbb{R}^n$ in the usual way, i.e. $\sigma(E) = \sigma(E \cap \mathrm{S}_1^{n-1})$ for any Borel subset $E \subset \mathbb{R}^n$.

[27] More precisely, for any $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ such that both $\int_0^\infty |f(\rho)| \rho^{n-1} \, d\rho < \infty$ and $\int_0^\infty |\widetilde{f}(\rho)| \rho^{n-1} \, d\rho < \infty$.

then f can be recovered by applying exactly the same integral transform once more, i.e.:

$$f(\rho) = 2\pi\rho^{1-\frac{n}{2}} \int_0^\infty \widetilde{f}(r) r^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi\rho r)\, dr, \tag{8.13}$$

Exercise 8.3. The inversion formula (8.13) can be seen as a special case of the "Hankel inversion formula": For any fixed $\nu \geq -\frac{1}{2}$, the Hankel transform of a function $g : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is defined by

$$\mathcal{H}_\nu g(r) = \int_0^\infty g(\rho)\rho J_\nu(r\rho)\, d\rho,$$

and the Hankel inversion formula says that for any "nice" g we have $\mathcal{H}_\nu \mathcal{H}_\nu g = g$. Verify that the inversion formula (8.13) follows from $\mathcal{H}_\nu \mathcal{H}_\nu g = g$ applied with $\nu = \frac{n}{2} - 1$ and $g(\rho) = f(\rho)\rho^{\frac{n}{2}-1}$.

8.3. **Application: The Dirichlet eigenfunctions in a disk.** In this section we will see how the J-Bessel function shows up when seeking the *Dirichlet eigenfunctions and eigenvalues in a disk.* Specifically, let Ω be the disk $\Omega = B_a^2 = \{|x| < a\}$ in $\mathbb{R}^2$, and consider the following PDE (for real-valued $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $\lambda \geq 0$):

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in all } \Omega \\ u_{|\partial\Omega} = 0. \end{cases} \tag{8.14}$$

The first equation says that u is an eigenvalue of the Laplace operator $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ with eigenvalue $-\lambda$; the second equation says that u should satisfy the Dirichlet boundary conditions, i.e. u should vanish along the boundary of Ω.

Physically, the eigenvalues λ of the above problem corresponds to the eigenfrequencies of vibration of an idealized circular "drum" of radius a in the plane; and the eigenfunctions u give the corresponding "vibration patterns". Note also that solving problem (8.14) is a first step in solving e.g. the heat or wave equation in a *cylinder* domain, using separation of variables.

Recall that using Green's formula one easily sees that all eigenvalues λ to the above problem (as well as for the corresponding Dirichlet eigenvalue problem in any nice domain in $\mathbb{R}^n$) are *positive*. Hence in the following investigation we will assume $\lambda > 0$ from start.

Let us now try to solve (8.14) (i.e. to find all solution pairs u, λ) by expressing u in polar coordinates and separating variables.[28] Thus we write (by slight abuse of notation) $u(r, \theta)$ for the value of u at the points $(r\cos\theta, r\sin\theta) \in \mathbb{R}^2$. Recalling that the Laplacian in polar coordinates is given by $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$, we see that the task is to solve:

$$\begin{cases} \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u + \lambda u = 0, & 0 < r < a,\ 0 \leq \theta \leq 2\pi, \\ u(a, \theta) = 0, & 0 \leq \theta \leq 2\pi. \end{cases}$$

Separating the variables r and θ means making the Ansatz that u is of the form $u(r, \theta) = R(r)\phi(\theta)$. Then we get:

$$\begin{cases} \Big(R''(r) + \frac{1}{r}R'(r) + \lambda R(r)\Big)\phi(\theta) = -\frac{1}{r^2}R(r)\phi''(\theta); \\ R(a) = 0; \\ \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi). \end{cases}$$

If the first equation has a non-vanishing solution then there must exist a constant $\mu \in \mathbb{R}$ such that

$$R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = \frac{\mu}{r^2}R(r), \qquad 0 < r < a,$$

[28]We here follow Pinchover and Rubinstein, [17, Sec. 9.5.3], to which we refer for more details. Our exposition will be slightly sloppy regarding certain details.

and

$$\phi''(\theta) = -\mu\phi(\theta), \qquad \forall\theta \in [0, 2\pi].$$

The last equation has the general solution

$$\phi(\theta) = \begin{cases} Ae^{\sqrt{-\mu}\theta} + Be^{-\sqrt{-\mu}\theta} & \text{if } \mu < 0 \\ A + B\theta & \text{if } \mu = 0 \\ A\cos(\sqrt{\mu}\theta) + B\sin(\sqrt{\mu}\theta) & \text{if } \mu > 0. \end{cases}$$

One easily checks that if $\mu < 0$ then the there is no choice of A, B other than $A = B = 0$ which makes the boundary conditions $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$ hold; if $\mu = 0$ then the boundary conditions are satisfied iff $B = 0$ (i.e. ϕ is a constant function), and if $\mu > 0$ then the boundary conditions are satisfied iff $\sqrt{\mu} \in \mathbb{N}$. Thus we may – incorporating also the case $\mu = 0$ – from now on write $\mu = \mu_n = n^2$ ($n \in \mathbb{Z}_{\geq 0}$) and the general ϕ-solution is

$$\phi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \qquad (A_n, B_n \in \mathbb{R}).$$

(Here B_0 is "redundant" since $\sin(0 \cdot \theta) \equiv 0$, and this is in agreement with the fact that the constant functions are the only solutions when $\mu = 0$.) The equation for $R(r)$ now reads:

$$R''(r) + \frac{1}{r}R'(r) + \Big(\lambda - \frac{n^2}{r^2}\Big)R(r) = 0, \qquad 0 < r < a,$$

Applying the change of variables $s = \sqrt{\lambda}r$, i.e. writing

$$\psi(s) = R(s/\sqrt{\lambda}),$$

the equation takes the form

$$\psi''(s) + \frac{1}{s}\psi'(s) + \Big(1 - \frac{n^2}{s^2}\Big)\psi(s) = 0, \qquad 0 < s < \sqrt{\lambda}a.$$

This is the *Bessel differential equation!* Our boundary conditions are $\psi(\sqrt{\lambda}\,a) = 0$ and the requirement that $\lim_{s\to 0}\psi(s)$ should exist and be finite. For the present case, i.e. order $= n$, it can be shown that the only solution to the Bessel differential equation for which $\lim_{s\to 0^+}\psi(s)$ exists and is finite, is $\psi(s) = J_n(s)$ (up to multiplication with a constant). Remember that we also have the boundary condition $J_n(\sqrt{\lambda}\,a) = \psi(\sqrt{\lambda}\,a) = R(a) = 0$. That is, $\sqrt{\lambda}\,a$ must be a zero of $J_n(z)$. Let us write $0 < j_{n,1} < j_{n,2} < \ldots$ for the full set of positive zeros of $J_n(z)$. Then we conclude: *The general solution $\langle\lambda, u\rangle$ of our Dirichlet eigenvalue problem* (8.14) *with u being of the form $u(r,\theta) = R(r)\Phi(\theta)$, is:*

$$\lambda = (j_{n,m}/a)^2, \qquad u_{n,m}(r,\theta) = J_n\Big(\frac{j_{n,m}}{a}r\Big)\big(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta\big),$$

where $\langle n, m\rangle$ runs through $\mathbb{Z}_{\geq 0} \times \mathbb{N}$. (Recall that each $B_{0,m}$ is "redundant".)

9. Complements to Stein's Ch. 8.1 on stationary phase

9.1. **Regarding $\int_0^\infty e^{i\lambda x}\psi(x)x^\mu\,dx$.** The following lemma is Stein's [19, 8.5.1(d)], with the formula for a_j corrected:

Lemma 9.1. *For any fixed $\psi\in C_c^\infty(\mathbb{R})$, $\mu\in\mathbb{C}$ with $\Re\mu>-1$, and $N\in\mathbb{Z}^+$, we have*

$$\int_0^\infty e^{i\lambda x}\psi(x)x^\mu\,dx=\sum_{j=0}^{N-1}a_j\lambda^{-j-1-\mu}+O(\lambda^{-N-1-\Re\mu})\qquad \text{as }\ \lambda\to+\infty,$$

where $a_j=i^{j+\mu+1}\,\dfrac{\Gamma(j+\mu+1)}{j!}\psi^{(j)}(0)$.

Proof. (We follow the outline on [19, p. 356], but provide a few more details.)

To start with, we discuss a modified integral $\int_0^\infty e^{i\lambda x}x^\mu\,\widetilde{\psi}(x)\,dx$, where $\widetilde{\psi}$ is a fixed function in $C_c^\infty(\mathbb{R})$ which *equals* 1 *on the support of* ψ. Using $\Re\mu>-1$ and $\widetilde{\psi}\in C_c^\infty(\mathbb{R})$ we have

$$\int_0^\infty e^{i\lambda x}x^\mu\,\widetilde{\psi}(x)\,dx=\lim_{\varepsilon\to0+}\int_0^\infty e^{i\lambda x}e^{-\varepsilon x}x^\mu\,\widetilde{\psi}(x)\,dx.\tag{9.1}$$

On the other hand, for any $\varepsilon>0$ and $\lambda\in\mathbb{R}$ we have, by a change of contour argument very similar to [19, p. 335 (lines 5–9)]:

$$\int_0^\infty e^{i\lambda x}e^{-\varepsilon x}x^\mu\,dx=(\varepsilon-i\lambda)^{-1-\mu}\int_0^\infty e^{-y}y^\mu\,dy=\frac{\Gamma(\mu+1)}{(\varepsilon-i\lambda)^{\mu+1}}.$$

Hence, assuming $\lambda>0$, it follows that

$$\lim_{\varepsilon\to0+}\int_0^\infty e^{i\lambda x}e^{-\varepsilon x}x^\mu\,dx=\frac{\Gamma(\mu+1)}{\lambda^{\mu+1}}e^{\frac{\pi}{2}(\mu+1)i}=\frac{\Gamma(\mu+1)\,i^{\mu+1}}{\lambda^{\mu+1}}.\tag{9.2}$$

We wish to combine (9.1) and (9.2) to get an asymptotic formula for $\int_0^\infty e^{i\lambda x}x^\mu\,\widetilde{\psi}(x)\,dx$. To this end, we note that

$$\sup_{\varepsilon>0}\left|\int_0^\infty e^{i\lambda x}e^{-\varepsilon x}x^\mu\big(1-\widetilde{\psi}(x)\big)\,dx\right|=O(\lambda^{-K})\qquad\text{as }\ \lambda\to+\infty,\tag{9.3}$$

for any $K\geq0$. [Proof: Write the integral as $\int_0^\infty e^{(i\lambda-\varepsilon)x}u(x)\,dx$ with $u(x)=x^\mu\big(1-\widetilde{\psi}(x)\big)$. Note that $u\in C^\infty(\mathbb{R})$, with $u(x)=0$ for x near 0 and $u(x)=x^\mu$ for x large; hence for any $j\geq0$ we have $u^{(j)}(x)=0$ for x near 0 and $u^{(j)}(x)=\mu(\mu-1)\cdots(\mu-j+1)x^{\mu-j}$ for x large. Hence for any given $\lambda>0$ and $\varepsilon>0$ we may integrate by parts K times to get

$$\int_0^\infty e^{(i\lambda-\varepsilon)x}u(x)\,dx=(-1)^K(i\lambda-\varepsilon)^{-K}\int_0^\infty e^{(i\lambda-\varepsilon)x}u^{(K)}(x)\,dx$$

If $K > \Re\mu + 1$ then $\int_0^\infty |u^{(K)}(x)|\,dx < \infty$, and hence

$$\left|\int_0^\infty e^{(i\lambda-\varepsilon)x} u(x)\,dx\right| \leq |i\lambda-\varepsilon|^{-K}\int_0^\infty |u^{(K)}(x)|\,dx \ll \lambda^{-K}.$$

Here the implied constant depends on u and K, but not on λ and *not on* ε. Hence we have proved (9.3).[29]]

Subtracting the two limit expressions in (9.1) and (9.2), and using (9.3), we conclude that (now writing ν in place of μ):

$$\int_0^\infty e^{i\lambda x} x^\nu\,\widetilde{\psi}(x)\,dx = \frac{\Gamma(\nu+1)\,i^{\nu+1}}{\lambda^{\nu+1}} + O(\lambda^{-K}) \qquad \text{as } \lambda\to+\infty. \tag{9.4}$$

This has been proved for any fixed $K \geq 0$ and any fixed $\nu \in \mathbb{C}$ with $\Re\nu > -1$.

Finally we now turn to the integral $\int_0^\infty e^{i\lambda x}\psi(x)x^\mu\,dx$. By Taylor's formula, for any integer $M \geq 0$ there exists some $R_M \in C^\infty(\mathbb{R})$ such that

$$\psi(x) = \sum_{j=0}^M b_j x^j + x^{M+1}R_M(x) \qquad (\forall x\in\mathbb{R}),$$

where $b_j = \psi^{(j)}(0)/j!$. Using this expansion, and also the fact that $\widetilde{\psi} = 1$ on the support of ψ, we have:

$$\int_0^\infty e^{i\lambda x}\psi(x)x^\mu\,dx = \int_0^\infty e^{i\lambda x}\psi(x)x^\mu\widetilde{\psi}(x)\,dx$$

$$= \sum_{j=0}^M b_j\int_0^\infty e^{i\lambda x}x^{j+\mu}\widetilde{\psi}(x)\,dx + \int_0^\infty e^{i\lambda x}R_M(x)x^{M+1+\mu}\widetilde{\psi}(x)\,dx. \tag{9.5}$$

Here, because of the assumption $\Re\mu > -1$, the function $u(x) = R_M(x)x^{M+1+\mu}\widetilde{\psi}(x)$ is in $C^M(\mathbb{R})$, and for every $j = 0,1,\ldots,M$ we have $u^{(j)}(0) = 0$ and also $u^{(j)}(x) = 0$ for all large x (since $\widetilde{\psi}$ has compact support). Hence by integrating by parts M times,

$$\int_0^\infty e^{i\lambda x}R_M(x)x^{M+1+\mu}\widetilde{\psi}(x)\,dx = (-1)^M(i\lambda)^{-M}\int_0^\infty e^{i\lambda x}u^{(M)}(x)\,dx$$
$$= O(\lambda^{-M}).$$

We also apply (9.4) (with $\nu = j+\mu$ for $j = 0,1,\ldots,M$) to get an asymptotics for each term in the sum in (9.5). This gives:

$$\int_0^\infty e^{i\lambda x}\psi(x)x^\mu\,dx = \sum_{j=0}^M b_j\frac{\Gamma(j+\mu+1)i^{j+\mu+1}}{\lambda^{j+\mu+1}} + O(\lambda^{-K}+\lambda^{-M}).$$

[29]Pedantically, we have only proved (9.3) under the extra assumption that $K > \Re\mu + 1$; however this trivially implies that (9.3) also holds if $0 \leq K \leq \Re\mu + 1$.

Applying this with $K = M =$ [any fixed integer larger than $N+1+\Re\mu$], and using $b_j = \psi^{(j)}(0)/j!$, we obtain the statement of the lemma. $\square$

9.2. Explicit formula for the coefficients in Stein's Prop. 3.

Exercise 9.1. In the special case $\phi(x) = x^2$, prove that the coefficients (a_j) in Stein's [19, Prop. 8.3] (with $x_0 = 0$ and $k = 2$) are given explicitly by:

$$a_1 = a_3 = a_5 = \cdots = 0$$

and

$$a_{2\ell} = \sqrt{\pi}\, e^{\frac{\pi}{4}i} \frac{i^\ell}{2^{2\ell}\,\ell!}\, \psi^{(2\ell)}(0) \qquad (\ell = 0, 1, 2, \ldots). \tag{9.6}$$

[Hint: This can be proved by working explicitly from the proof of [19, Prop. 8.3]. Compare [19, 8.5.1(a)] for a slightly different proof which applies also in higher dimension; however note that the formula there should be corrected into[30] "$a_j = (i\pi)^{n/2} \frac{i^j}{2^{2j}\,j!} (\Delta^j \psi)(0)$".]

9.3. Example: Asymptotics for $J_m(r)$ as $r \to \infty$, using (8.8).

Using the formula (8.8),

$$J_m(r) = \frac{(\frac{r}{2})^m}{\Gamma(m+\frac12)\Gamma(\frac12)} \int_{-1}^{1} e^{irx} (1-x^2)^{m-\frac12}\, dx, \tag{9.7}$$

we will now seek an asymptotic formula for $J_m(r)$ for fixed $m \in \mathbb{C}$ with $\Re m > -\frac12$, as $r \to +\infty$ (through real numbers). This integral is of the general form "$\int_a^b e^{i\lambda\phi(x)} \psi(x)\, dx$" which is discussed in Stein's [19, Ch. 8.1] – with $\phi(x) = x$ and $\psi(x) = (1-x^2)^{m-\frac12}$ and $\lambda = r \to +\infty$. Since the function $\phi(x) = x$ has no critical points, the only contributions to the asymptotic formula will come from the endpoints of the interval of integration. The standard approach to separate the individual contributions is to choose smooth, real-valued functions $\Psi_1, \Psi_2, \Psi_3 \in C^\infty(\mathbb{R})$ such that

$$\Psi_1(x) + \Psi_2(x) + \Psi_3(x) = 1 \quad (\forall x \in [-1, 1])$$

and, say

$$\Psi_1(x) = 1 \ \ (\forall x \in [-1, -\tfrac{9}{10}]); \qquad \Psi_1(x) = 0 \ \ (\forall x \geq -\tfrac45);$$
$$\Psi_3(x) = 1 \ \ (\forall x \in [\tfrac{9}{10}, 1]); \qquad \Psi_3(x) = 0 \ \ (\forall x \leq \tfrac45);$$

(Note that it then follows that $\Psi_2(x) = 1$ for all $x \in [-\frac45, \frac45]$ and $\Psi_2(x) = 0$ for all x with $\frac{9}{10} \leq |x| \leq 1$.) We then split the integral

[30] The "a_j" there is not the same as "a_j" in Prop. 3, even in dimension $n = 1$.

which we are interested into a sum of three terms:

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\,dx = \sum_{j=1}^{3}\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\Psi_j(x)\,dx. \tag{9.8}$$

We will consider each term separately.

First of all, since the function $x \mapsto (1-x^2)^{m-\frac{1}{2}}\Psi_2(x)$ is smooth and has compact support in $(-1,1)$, [19, Prop. 8.1] implies that the Ψ_2-contribution in (9.8) is rapidly decaying, i.e.

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\Psi_2(x)\,dx = O(r^{-N}) \qquad \text{as } r \to +\infty,$$

for any $N \geq 0$. (The implied constant of course depends on N.)

Next, in the Ψ_1-contribution in (9.8) we replace x by $x-1$ and use $1-(x-1)^2 = x(2-x)$. This gives:

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\Psi_1(x)\,dx = e^{-ir}\int_{0}^{2} e^{irx}x^{m-\frac{1}{2}}(2-x)^{m-\frac{1}{2}}\Psi_1(x-1)\,dx. \tag{9.9}$$

The last integral can be expressed as $\int_0^\infty e^{irx}x^{m-\frac{1}{2}}\psi(x)\,dx$ with $\psi(x) = (2-x)^{m-\frac{1}{2}}\Psi_1(x-1)$ for $x \in [0,2]$ and $\psi(x) = 0$ for $x > 2$; [31] clearly then $\psi \in C_c^\infty([0,\infty))$ and we may extend ψ to a function in $C_c^\infty(\mathbb{R})$. Hence by Lemma 9.1 we conclude

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\Psi_1(x)\,dx = e^{-ir}\sum_{j=0}^{N-1} c_j r^{-j-m-\frac{1}{2}} + O(r^{-N-\frac{1}{2}-\Re m}) \tag{9.10}$$

as $r \to \infty$, for any fixed $N \geq 0$, where[32]

$$c_j = i^{j+m+\frac{1}{2}}\frac{\Gamma(j+m+\frac{1}{2})}{j!}\Big(\frac{d^j}{dx^j}(2-x)^{m-\frac{1}{2}}\Big)_{|x=0}.$$

We compute

$$\begin{aligned}\frac{d^j}{dx^j}(2-x)^{m-\frac{1}{2}} &= (-1)^j\left(\prod_{k=1}^{j}(m-k+\tfrac{1}{2})\right)(2-x)^{m-j-\frac{1}{2}}\\ &= (-1)^j\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m-j+\frac{1}{2})}\cdot(2-x)^{m-j-\frac{1}{2}};\end{aligned}$$

hence

$$c_j = e^{\frac{\pi}{2}i(j+m+\frac{1}{2})}(-1)^j\frac{\Gamma(m+\frac{1}{2})\Gamma(j+m+\frac{1}{2})}{j!\Gamma(m-j+\frac{1}{2})}2^{m-j-\frac{1}{2}}.$$

[31] thus in fact $\psi(x) = 0$ for all $x \geq \frac{1}{5}$.

[32] Using also the fact that since $\Psi_1(x) = 1$ for all x near -1, the function $\psi(x) = (2-x)^{m-\frac{1}{2}}\Psi_1(x-1)$ satisfies $\psi^{(j)}(0) = \big(\frac{d^j}{dx^j}(2-x)^{m-\frac{1}{2}}\big)_{|x=0}$.

Finally, the Ψ_3-contribution in (9.8) can be treated using symmetry. Indeed, note that we may take $\Psi_3(x) \equiv \Psi_1(-x)$, and with this choice we have

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\Psi_3(x)\,dx = \int_{-1}^{1} e^{-irx}(1-x^2)^{m-\frac{1}{2}}\Psi_1(x)\,dx$$
$$= \overline{\int_{-1}^{1} e^{irx}(1-x^2)^{\overline{m}-\frac{1}{2}}\Psi_1(x)\,dx}.$$

Hence, by (9.10) with $\overline{m}$ in place of m, we conclude that

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\Psi_3(x)\,dx = e^{ir}\sum_{j=0}^{N-1} c_j' r^{-j-m-\frac{1}{2}} + O(r^{-N-\frac{1}{2}-\Re m})$$

as $r \to \infty$, for any fixed $N \geq 0$, where

$$c_j' = e^{-\frac{\pi}{2}i(m+\frac{1}{2}+j)}(-1)^j \frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{1}{2}+j)}{j!\Gamma(m+\frac{1}{2}-j)} 2^{m-j-\frac{1}{2}}.$$

Now we can add up the three terms in (9.8). Note that

$$c_j e^{-ir} + c_j' e^{ir} = (-1)^j \frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{1}{2}+j)}{j!\Gamma(m+\frac{1}{2}-j)} 2^{m+\frac{1}{2}-j} \times \begin{cases} (-1)^{j/2}\cos\bigl(r-\frac{m\pi}{2}-\frac{\pi}{4}\bigr) & \text{if } 2 \mid j \\ (-1)^{(j-1)/2}\sin\bigl(r-\frac{m\pi}{2}-\frac{\pi}{4}\bigr) & \text{if } 2 \nmid j. \end{cases}$$

Hence we conclude:

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\,dx = \cos\Bigl(r-\frac{m\pi}{2}-\frac{\pi}{4}\Bigr)\sum_{0\leq k<N/2}\alpha_k r^{-m-2k-\frac{1}{2}}$$
$$+\sin\Bigl(r-\frac{m\pi}{2}-\frac{\pi}{4}\Bigr)\sum_{0\leq k<(N-1)/2}\beta_k r^{-m-2k-\frac{3}{2}}$$
$$+O(r^{-\Re m-N-\frac{1}{2}}),$$

where

$$\alpha_k = (-1)^k \frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{1}{2}+2k)}{(2k)!\,\Gamma(m+\frac{1}{2}-2k)} 2^{m+\frac{1}{2}-2k}$$

and

$$\beta_k = -(-1)^k \frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2}+2k)}{(2k+1)!\,\Gamma(m-\frac{1}{2}-2k)} 2^{m-\frac{1}{2}-2k}.$$

Plugging this into (9.7) we finally conclude:

$$\begin{aligned} J_m(r) = \Big(\frac{\pi r}{2}\Big)^{-\frac12} \cos\Big(r - \frac{m\pi}{2} - \frac{\pi}{4}\Big) \sum_{0\le k<N/2} a_k r^{-2k} \\ -\Big(\frac{\pi r}{2}\Big)^{-\frac12} \sin\Big(r - \frac{m\pi}{2} - \frac{\pi}{4}\Big) \sum_{0\le k<(N-1)/2} b_k r^{-2k-1} \\ +O(r^{-N-\frac12}), \end{aligned} \tag{9.11}$$

where

$$a_k = (-1)^k \frac{\Gamma(m+\frac12+2k)}{(2k)!\,\Gamma(m+\frac12-2k)} 2^{-2k} \tag{9.12}$$

and

$$b_k = (-1)^k \frac{\Gamma(m+\frac32+2k)}{(2k+1)!\,\Gamma(m-\frac12-2k)} 2^{-2k-1}. \tag{9.13}$$

(Note that this is equivalent with the formula which Stein states in [19, 8.5.2(a)], except that the “+” in the last line of p. 356 should be corrected to “−”; cf. (9.11).)

Remark 9.2. Note that $a_k = (-1)^k A_{m,2k}$ and $b_k = (-1)^k A_{m,2k+1}$, where

$$A_{m,n} = \frac{\Gamma(m+\frac12+n)}{n!\,\Gamma(m+\frac12-n)} 2^{-n} = \frac{\prod_{j=-n}^{n-1}(m+\frac12+j)}{2^n n!} \qquad (\forall n \in \mathbb{Z}_{\ge 0}). \tag{9.14}$$

In particular we have in the above expansion:

$$a_0 = 1; \qquad b_0 = \frac{(m-\frac12)(m+\frac12)}{2}; \qquad a_1 = -\frac{(m-\frac32)(m-\frac12)(m+\frac12)(m+\frac32)}{8}.$$

Exercise 9.2. In the above discussion, we assumed throughout that $\Re m > -\frac12$. However, by using the fact that (9.11) holds for any fixed $m \in \mathbb{C}$ with $\Re m > -\frac12$ together with the recursion formula (8.4), prove that (9.11) is in fact valid for *any* fixed $m \in \mathbb{C}$.

Exercise 9.3. Note that if $m \in \frac12 + \mathbb{Z}$ then (in (9.14)) $A_{m,n} = 0$ for all $n \ge |m| + \frac12$, and hence only finitely many a_k's and b_k's are non-zero. Prove that in this case, if $N = |m| + \frac12$ (or larger), then (9.11) is an *exact* formula, i.e. the error term “$O(r^{-N-\frac12})$” is in fact identically zero.

[Hint: When $\nu = m \in \{\frac12, \frac32, \ldots\}$, the integral in (8.8) can be evaluated by repeated integration by parts. In fact it suffices to use this for $m = \frac12$ and $m = \frac32$; then we can use the recursion formula (8.4) to reach all other $m \in \frac12 + \mathbb{Z}$.]

9.4. **Example: Asymptotics for $J_m(r)$ when $m \in \mathbb{Z}$, using (8.9).** This is discussed in Stein, [19, 8.1.4.1]. Of course the asymptotic formula obtained there is a special case of the asymptotic expansion obtained in Section 9.3 above; however it is still useful to study this as an example of the stationary phase method, since here the phase function has critical points in the domain of integration; this was not the case in Section 9.3.

Exercise 9.4. It is a fun[33] exercise to try to recover in this way the (explicit) asymptotic expansion, as in (9.11)–(9.13) (now for $m \in \mathbb{Z}$), and not just the leading order term as in [19, 8.1.4.1; eq. (14)]. (One can use the formula (9.6) above; however first one has to carry out the substitution as in [19, 8.1.3.2], to transform the phase function near the critical point(s) into the standard form $\phi(x) = x^2$.)

9.5. **Example: The Airy function.** The Airy function is defined by

$$\mathrm{Ai}(\xi) = \frac{1}{\pi} \int_0^\infty \cos(\tfrac{1}{3} w^3 + \xi w)\, dw, \qquad \xi \in \mathbb{R}.$$

Note that this integral should be considered as a generalized integral, i.e. $\mathrm{Ai}(\xi) = \lim_{A \to +\infty} \frac{1}{\pi} \int_0^A \cos(\frac{1}{3} w^3 + \xi w)\, dw$,[34] since we have $\int_0^\infty |\cos(\frac{1}{3} w^3 + \xi w)|\, dw = \infty$.

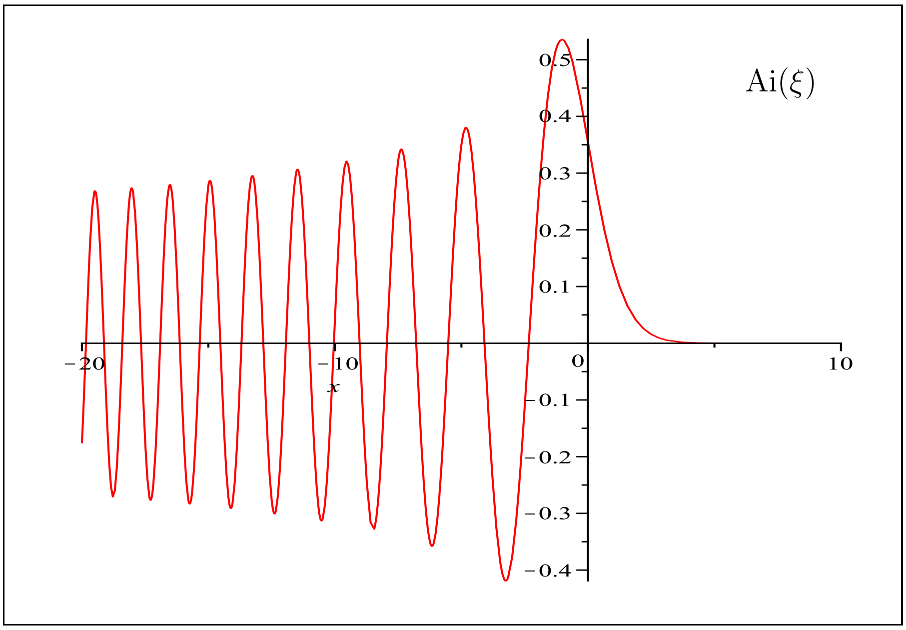

Figure 1 – The Airy function $\mathrm{Ai}(\xi)$.

[33]But quite complicated, it seems to me!

[34]We will see below that this limit exists and is finite.

The Airy function is a smooth function satisfying the differential equation

$$\mathrm{Ai}''(\xi) = \xi \cdot \mathrm{Ai}(\xi),$$

with

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} \quad \text{and} \quad \mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(\frac{1}{3})}.$$

It also satisfies the following asymptotic relations:

$$\mathrm{Ai}(\xi) = \frac{1}{2\sqrt{\pi}} \xi^{-1/4} e^{-\frac{2}{3}\xi^{3/2}} \left(1 + O\left(\xi^{-3/2}\right)\right) \qquad \text{as } \xi \to \infty \tag{9.15}$$

and

$$\mathrm{Ai}(\xi) = \frac{1}{\sqrt{\pi}} |\xi|^{-1/4} \left(\cos\left(\tfrac{2}{3}|\xi|^{3/2} - \tfrac{\pi}{4}\right) + O\left(|\xi|^{-3/2}\right)\right) \qquad \text{as } \xi \to -\infty. \tag{9.16}$$

Cf. [16, Ch. 11.1]. In this section we prove, as a further illustration of the method of stationary phase, a weaker version of the formula (9.16).

Thus we consider the function

$$\mathrm{Ai}(-u) = \frac{1}{\pi} \int_0^\infty \cos(\tfrac{1}{3}w^3 - uw)\, dw$$

for $u > 0$, and in particular as $u \to +\infty$.

The stationary points of the integrand are $w = \pm\sqrt{u}$, and only $w = \sqrt{u}$ lies in the range of integration. Substituting $w = \sqrt{u}(1 + x)$ we get

$$\begin{aligned} \mathrm{Ai}(-u) &= \frac{\sqrt{u}}{\pi} \int_{-1}^\infty \cos\left(u^{3/2}\left(-\tfrac{2}{3} + x^2 + \tfrac{1}{3}x^3\right)\right) dx \\ &= \frac{\sqrt{u}}{\pi} \Re\left(e^{-\frac{2}{3}u^{3/2}i} \int_{-1}^\infty e^{iu^{3/2}(x^2 + \frac{1}{3}x^3)}\, dx\right). \end{aligned} \tag{9.17}$$

The last integral is *almost* of the form studied in Stein, [19, Ch. 8.1]; namely, we consider

$$\int_{-1}^\infty e^{i\lambda\phi(x)}\, dx \qquad \text{as } \lambda \to +\infty,$$

where

$$\lambda := u^{3/2} \qquad \text{and} \qquad \phi(x) = x^2 + \tfrac{1}{3}x^3.$$

The only thing that makes the integral not fit completely into that framework is the fact mentioned above, that the integral is a generalized one. However as we will see below, this causes very little extra difficulty.

In order to apply the machinery developed in [19, Ch. 8.1], we write $1 = \Psi_1 + \Psi_2 + \Psi_3$, where Ψ_1, Ψ_2, Ψ_3 are smooth functions from $\mathbb{R}$ to $\mathbb{R}$, with $\mathrm{supp}(\Psi_1) \subset (-\infty, 0)$, $\mathrm{supp}(\Psi_3) \subset (0, \infty)$, while $\mathrm{supp}(\Psi_2)$ is contained in a small neighborhood of the critical point $x = 0$. Then we decompose

(9.18)
$$\int_{-1}^{\infty} e^{i\lambda\phi(x)}\,dx = \int_{-1}^{0} e^{i\lambda\phi(x)}\,\Psi_1(x)\,dx + \int_{\mathbb{R}} e^{i\lambda\phi(x)}\,\Psi_2(x)\,dx + \int_{0}^{\infty} e^{i\lambda\phi(x)}\,\Psi_3(x)\,dx.$$

The Ψ_1-term in (9.18) is $O(\lambda^{-1})$ as $\lambda \to +\infty$, by [19, Cor. to Prop. 8.2]. [Details: Take $a \in (-1, 0)$ so that $\mathrm{supp}(\Psi_1)$ is contained in $(-\infty, a]$; note that a is a completely absolute constant. Our integral equals $\int_{-1}^{a} e^{i\lambda\phi(x)}\Psi_1(x)\,dx$, and $\phi'(x) = x(2+x)$ is increasing and $\le \phi'(a) < 0$ for all $x \in [-1, a]$. Hence [19, Cor. to Prop. 8.2] applies with $k = 1$, $\phi_{\mathrm{new}}(x) := |\phi'(a)|^{-1}\phi(x)$ and $\lambda_{\mathrm{new}} := |\phi'(a)|\lambda$.]

Next we consider the Ψ_3-term in (9.18). Let us first verify that this term converges, as a generalized integral. Since $\Psi_3 = 1$ for all large x, this is equivalent to the statement that the integral $\int_{-1}^{\infty} e^{i\lambda\phi(x)}\,dx$ converges; and we will prove this by proving that

(9.19)
$$\forall\lambda > 0 : \forall\varepsilon > 0 : \exists A_0 > 0 : \forall B > A > A_0 : \qquad \left| \int_{A}^{B} e^{i\lambda\phi(x)}\,dx \right| < \varepsilon.$$

This is actually an immediate consequence of [19, Prop. 8.2]! Indeed, we have $\phi'(x) = x(2+x)$ and this function is increasing for $x > 0$ and satisfies $\phi'(x) \ge \phi'(A_0)$ whenever $x \ge A_0 > 0$. Hence by [19, Prop. 8.2][35] we have, for any $B > A \ge A_0 > 0$:

$$\left| \int_{A}^{B} e^{i\lambda\phi(x)}\,dx \right| \le \frac{3}{\phi'(A_0)\lambda} = \frac{3}{A_0(2+A_0)\lambda} < \frac{3}{A_0^2\lambda}. \tag{9.20}$$

This obviously implies (9.19); one may e.g. take $A_0 = \sqrt{3/(\lambda\varepsilon)}$.

The bound in (9.20) also implies that the Ψ_3-term in (9.18) is $O(\lambda^{-1})$. Indeed, exactly as in the proof of [19, Cor. to Prop. 8.2], the fact that (9.20) holds whenever $B > A \ge A_0 > 0$ implies that, for any $B > A > 0$:

$$\left| \int_{A}^{B} e^{i\lambda\phi(x)}\,\Psi_3(x)\,dx \right| < \frac{3}{A^2\lambda}\biggl(|\Psi_3(B)| + \int_{A}^{B} |\Psi_3'(x)|\,dx \biggr).$$

We have $\Psi_3(x) = 1$ and $\Psi_3'(x) = 0$ for all large x, and we have proved that the generalized integral $\int_{A}^{\infty} e^{i\lambda\phi(x)}\,\Psi_3(x)\,dx$ exists; hence we may take $B \to +\infty$ in the above to conclude that

$$\left| \int_{A}^{\infty} e^{i\lambda\phi(x)}\,\Psi_3(x)\,dx \right| \le \frac{3}{A^2\lambda}\biggl(1 + \int_{A}^{\infty} |\Psi_3'(x)|\,dx \biggr)$$

[35] Applied with $\phi_{\mathrm{new}}(x) := \phi'(A_0)^{-1}\phi(x)$ and $\lambda_{\mathrm{new}} := \phi'(A_0)\lambda$. We also use the fact that "$c_1 = 3$" works in [19, Prop. 8.2].

(and here $\int_A^\infty |\Psi_3(x)|\,dx < \infty$). Here we can fix $A > 0$ so small that $\operatorname{supp}(\Psi_3) \subset (A, +\infty)$; then $\int_A^\infty e^{i\lambda\phi(x)}\,\Psi_3(x)\,dx = \int_0^\infty e^{i\lambda\phi(x)}\,\Psi_3(x)\,dx$, and hence we conclude that the Ψ_3-term in (9.18) is $O(\lambda^{-1})$.

Finally, for the Ψ_2-term in (9.18) we can apply [19, Prop. 8.3] with $k = 2$ and x_0; indeed note that $\phi(0) = \phi'(0) = 0$ while $\phi''(0) = 2$ (since $\phi''(x) \equiv 2 + 2x$). This gives (using also the explicit formula for a_0 in [19, 8.1.3.4]):

$$\begin{aligned}\int_{\mathbb{R}} e^{i\lambda\phi(x)}\,\Psi_2(x)\,dx &= \big(\pi i\big)^{1/2}\Psi_2(0)\lambda^{-\frac{1}{2}} + O(\lambda^{-1})\\ &= \sqrt{\pi}e^{\frac{\pi}{4}i}\lambda^{-\frac{1}{2}} + O(\lambda^{-1}) \qquad \text{as } \lambda \to +\infty.\end{aligned}$$

Adding up our results, we have now proved that

$$\int_{\mathbb{R}} e^{i\lambda\phi(x)}\,dx = \int_{\mathbb{R}} e^{i\lambda(x^2+\frac{1}{3}x^3)}\,dx = \sqrt{\pi}e^{\frac{\pi}{4}i}\lambda^{-\frac{1}{2}} + O(\lambda^{-1}) \qquad \text{as } \lambda \to +\infty.$$

Plugging this into (9.17) gives

$$\operatorname{Ai}(-u) = \frac{\cos\big(\frac{\pi}{4} - \frac{2}{3}u^{3/2}\big)}{\sqrt{\pi}\,u^{\frac{1}{4}}} + O(u^{-1}), \qquad \text{as } u \to +\infty.$$

This is a slightly weaker form of (9.16).

10. Uniform asymptotic expansion for the J-Bessel function

The purpose of this section is to illustrate, through an example, the notion of *uniform* asymptotic formulas. Here "uniform" refers to some additional parameter(s), apart from the main argument of the function which we are considering.

10.1. Uniform asymptotics for $J_\nu(x)$ for ν large. Using methods for asymptotic expansions of solutions to (ordinary) second order differential equations, Olver [15] (cf. also [16, Ch. 11, (10.18)]) has proved the following formula. For all $\nu \geq 1$ and all $t > 0$:

$$J_\nu(\nu t) = \nu^{-\frac{1}{3}} \Big(\frac{4\zeta}{1-t^2}\Big)^{1/4} \Big\{ \mathrm{Ai}(\xi) + O\Big(\nu^{-1} \frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}}\Big) \Big\} \tag{10.1}$$

where $\zeta = \zeta(t) = (\frac{3}{2}u(t))^{2/3}\,\mathrm{sgn}(1-t)$ and $\xi = \nu^{2/3}\zeta$ with

$$u(t) = \begin{cases} \operatorname{arctanh}\big(\sqrt{1-t^2}\big) - \sqrt{1-t^2} & \text{if } 0 < t \leq 1 \\ \sqrt{t^2-1} - \arctan\big(\sqrt{t^2-1}\big) & \text{if } t \geq 1. \end{cases}$$

We also use the notation $\xi^+ = \max(\xi, 0)$.

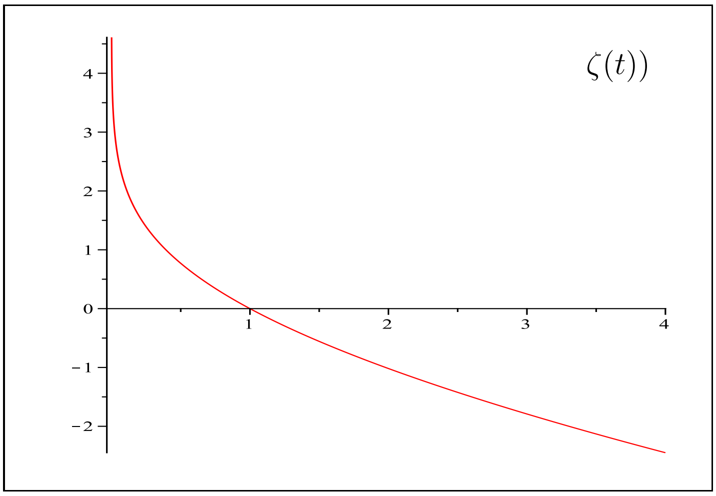

Figure 2 – The auxiliary function $\zeta(t)$.

We stress: In (10.1), the implied constant in the big-O is *absolute!* By contrast, the implied constant in (9.11), with N fixed as $N = 1$, depends on m (=our ν)! See Sec. 10.2 below for a more detailed comparison.

In order to better understand what (10.1) really means, let us derive from it an asymptotic relation involving only elementary functions:

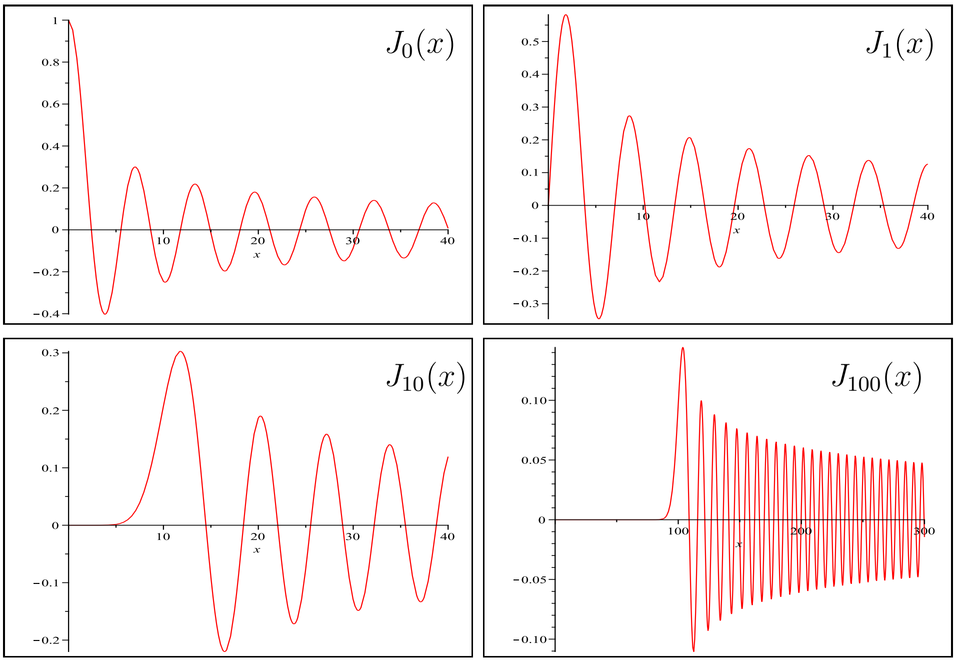

Figure 3 – The J-Bessel function $J_\nu(x)$ for $\nu = 0, 1, 10, 100$. We remark that for every $\nu \geq 1$ the graph of the right hand side in (10.1) (without the error term) is practically indistinguishable from the $J_\nu(x)$-graph; already for $\nu = 1$ the relative error is typically below 0.01!

Proposition 10.1. *Fix an arbitrary $C > 0$. Then for all $\nu \geq 1$ and $x > 0$ we have*

$$J_\nu(x) = \begin{cases} \dfrac{e^{\sqrt{\nu^2-x^2}}}{\sqrt{2\pi}\sqrt[4]{\nu^2-x^2}\left(\frac{\nu}{x}+\sqrt{(\frac{\nu}{x})^2-1}\right)^\nu}\left(1+O\Big(\dfrac{\sqrt{\nu}}{(\nu-x)^{3/2}}\Big)\right) & \\ \qquad\qquad\qquad\qquad \text{if } x \leq \nu - C\nu^{\frac13} & \\ O(\nu^{-\frac13}) \qquad\qquad\qquad \text{if } |x-\nu| \leq C\nu^{\frac13} & \\ \dfrac{\sqrt{2}}{\sqrt{\pi}\sqrt[4]{x^2-\nu^2}}\left\{\cos\Big(\sqrt{x^2-\nu^2}-\nu\arccos\big(\dfrac{\nu}{x}\big)-\dfrac{\pi}{4}\Big)+O\Big(\dfrac{\sqrt{\nu}}{(x-\nu)^{3/2}}+\dfrac{1}{\nu}\Big)\right\} & \\ \qquad\qquad\qquad\qquad \text{if } x \geq \nu + C\nu^{\frac13}. \end{cases} \tag{10.2}$$

Here the implied constant in each "big-O" depends only on C, i.e. it is independent of ν and x. The bound in the case $|x-\nu| \leq C\nu^{\frac13}$ may be complemented by the following fact: There exist absolute constants $C_1, C_2 > 0$ such that

$$J_\nu(x) \gg \nu^{-\frac13}, \qquad \forall\, \nu \geq C_1,\ x \in [\nu - C_2\nu^{\frac13}, \nu + C_2\nu^{\frac13}]. \tag{10.3}$$

Remark 10.2. As will be seen in the proof, for $x < \nu$ we have

$$\frac{e^{\sqrt{\nu^2-x^2}}}{\left(\frac{\nu}{x}+\sqrt{(\frac{\nu}{x})^2-1}\right)^\nu} = \exp\left\{-\nu\left(\operatorname{arctanh}\left(\sqrt{1-\tfrac{x^2}{\nu^2}}\right)-\sqrt{1-\tfrac{x^2}{\nu^2}}\right)\right\},$$

and using this format in the first case in (10.2) we see that the proposition implies that $J_\nu(x) \ll |x+\nu|^{-\frac14}|x-\nu|^{-\frac14}$ whenever $|x-\nu| \geq C\nu^{\frac13}$; thus in particular $J_\nu(x) \ll \nu^{-\frac12}$ whenever $x \geq 1.01\nu$ or $x \leq 0.99\nu$. By contrast, in the comparatively small interval $|x-\nu| \ll \nu^{\frac13}$ the function $J_\nu(x)$ is of order of magnitude $\nu^{-\frac13}$, i.e. much larger than elsewhere!

Remark 10.3. The result(s) of Proposition 10.1 can also be derived using other methods, such as steepest descent. Cf., e.g., [3, Exercises 7.17, 7.18] and [16, Ch. 4.9].

Proof of Prop. 10.1. We apply (10.1) with $t = x/\nu$. Let us first assume $x \leq \nu - C\nu^{\frac13}$. Then $t < 1$, and we note that for all $0 < t < 1$ we have $\zeta(t) \gg 1-t$ (cf. Figure 2 and Section 10.3); hence our assumption $\nu - x \geq C\nu^{\frac13}$ implies that $\zeta(t) \gg \frac{\nu-x}{\nu} \gg \nu^{-\frac23}$ (here and in the rest of the proof, the implied constant in any $\ll$, $\gg$ or big-O depends only on C) and therefore $\xi = \nu^{2/3}\zeta \gg \nu^{-\frac13}(\nu-x) \gg 1$, i.e. ξ is bounded from below by a positive constant which only depends on C. Hence by (10.1) and (9.15) we have

$$\begin{aligned} J_\nu(x) &= \nu^{-\frac13}\left(\frac{4\zeta}{1-(x/\nu)^2}\right)^{1/4}\cdot\frac{1}{2\sqrt{\pi}}\xi^{-1/4}e^{-\frac23\xi^{3/2}}\left(1+O(\xi^{-\frac32}+\nu^{-1})\right) \\ &= \frac{e^{-\frac23\xi^{3/2}}}{\sqrt{2\pi}\sqrt[4]{\nu^2-x^2}}\left(1+O\left(\frac{\sqrt{\nu}}{(\nu-x)^{3/2}}+\nu^{-1}\right)\right). \end{aligned}$$

The error term may be simplified by using the fact that $\frac{\sqrt{\nu}}{(\nu-x)^{3/2}} > \nu^{-1}$. (It may appear that our treatment of the error term was wasteful in the case of t near 0, since in this case $\zeta(t)$ is of higher order of magnitude than $1-t$. However note that in this case we anyway have $\frac{\sqrt{\nu}}{(\nu-x)^{3/2}} \asymp \nu^{-1}$, i.e. the error term in (10.2) matches the error term in (10.1), i.e. we have not been wasteful.) Finally to express $e^{-\frac23\xi^{3/2}}$ in terms of x, ν, note that

$$\begin{aligned} -\frac23\xi^{3/2} = -\frac23\nu\zeta^{\frac32} &= -\nu\left(\operatorname{arctanh}\left(\sqrt{1-(x/\nu)^2}\right)-\sqrt{1-(x/\nu)^2}\right) \\ &= -\nu\log\left(\tfrac{\nu}{x}+\sqrt{(\tfrac{\nu}{x})^2-1}\right)+\sqrt{\nu^2-x^2}. \end{aligned}$$

Hence we obtain the formula in (10.2), in the case $\nu - x \geq C\nu^{\frac13}$.

Let us next assume $x \geq \nu + C\nu^{\frac13}$. Then $t > 1$. Note that if $1 < t \leq 2$ then $-\zeta(t) \gg t-1$ (cf. Figure 2 and Section 10.3) and thus our assumption $x - \nu \geq C\nu^{\frac13}$ implies that $-\zeta(t) \gg \frac{x-\nu}{\nu} \gg \nu^{-\frac23}$ and $-\xi = -\nu^{\frac23}\zeta \gg \nu^{-\frac13}(x-\nu) \gg 1$. In the remaining case $t > 2$ we have $-\zeta(t) \gg 1$ so that certainly $-\xi \gg 1$ again. Thus our assumption $x \geq \nu + C\nu^{\frac{1}{3}}$ implies that $-\xi$ is bounded from below by a positive constant which only depends on C. Hence by (10.1) and (9.16) we have

$$J_\nu(x) = \frac{1}{\sqrt{\pi}}\nu^{-\frac{1}{3}}\Big(\frac{4\zeta}{1-(x/\nu)^2}\Big)^{\frac{1}{4}}|\xi|^{-\frac{1}{4}}\Big(\cos\big(\tfrac{2}{3}|\xi|^{3/2}-\tfrac{\pi}{4}\big)+O\Big(|\xi|^{-\frac{3}{2}}+\nu^{-1}\Big)\Big)$$
$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt[4]{x^2-\nu^2}}\Big(\cos(\tfrac{2}{3}|\xi|^{3/2}-\tfrac{\pi}{4})+O\Big(\frac{\sqrt{\nu}}{(x-\nu)^{3/2}}+\frac{1}{\nu}\Big)\Big),$$

where the form of the error term is clear from the previous discussion in the case $1 < t \leq 2$, while in the case $t > 2$ it holds since $|\xi| = \nu^{\frac{2}{3}}|\zeta| \gg \nu^{\frac{2}{3}}$, thus $|\xi|^{-\frac{3}{2}} \ll \nu^{-1}$. Finally to express $\cos(\frac{2}{3}|\xi|^{3/2}-\frac{\pi}{4})$ in terms of x, ν we note that

$$\tfrac{2}{3}|\xi|^{\frac{3}{2}} = \tfrac{2}{3}\nu|\zeta|^{\frac{3}{2}} = \nu\Big(\sqrt{(x/\nu)^2-1}-\arctan\Big(\sqrt{(x/\nu)^2-1}\Big)\Big)$$
$$= \sqrt{x^2-\nu^2}-\nu\arccos(\nu/x),$$

and we again obtain the formula in (10.2).

Finally assume $|x-\nu| \leq C\nu^{\frac{1}{3}}$. Then $|t-1| = \frac{|x-\nu|}{\nu} \leq C\nu^{-\frac{2}{3}}$, and hence if $\nu \geq (2C)^{3/2}$ we have $|t-1| \leq \frac{1}{2}$ and thus $|\zeta(t)| \ll |t-1|$ and $|\xi| = \nu^{\frac{2}{3}}|\zeta| \ll 1$ and using this (10.1) is seen to imply $J_\nu(x) = O(\nu^{-\frac{1}{3}})$. Finally this bound is extended to $\nu \in [1,(2C)^{3/2}]$ by using the continuity of $J_\nu(x)$ (in both variables) and the fact that the set

$$\{(\nu,x) \,:\, \nu \in [1,(2C)^{3/2}],\ |x-\nu| \leq C\nu^{\frac{1}{3}}\}$$

is compact. The lower bound (10.3) can be proved by a similar discussion, using the fact that $\mathrm{Ai}(\xi) \gg 1$ for all ξ sufficiently near 0. (We leave out the details.) □

10.2. **Comparing the uniform and the non-uniform asymptotics.** Let us now compare the non-uniform asymptotics (9.11) with the uniform asymptotics in Prop. 10.1. Note that (9.11) with $N = 1$ implies that for all $\nu > -\frac{1}{2}$ and $x \geq 1$,

$$J_\nu(x) = \Big(\frac{2}{\pi x}\Big)^{1/2}\Big\{\cos\big(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\big)+O_\nu(x^{-1})\Big\}, \tag{10.4}$$

where we write “O_ν” to stress that the implied constant depends on ν. Let us from now on keep ν *large*. By comparing with Prop. 10.1 (the case $x \geq \nu + C\nu^{\frac{1}{3}}$) we can now determine how large x has to be (depending on ν) for the asymptotic formula (10.4) to be *at all* relevant.

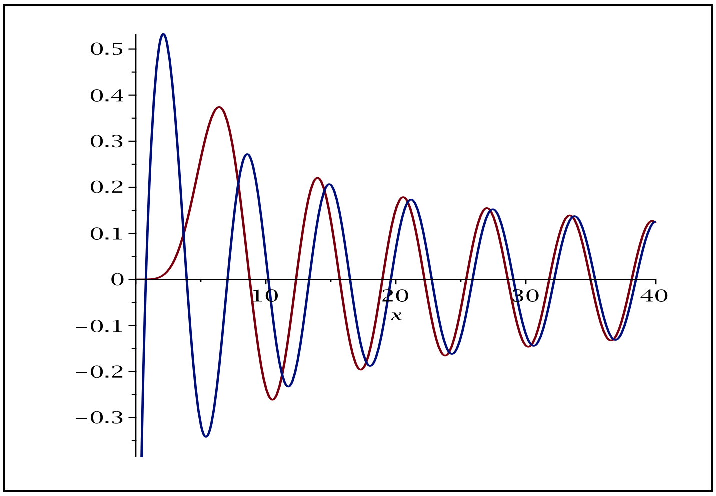

Figure 4 – Graphs of the functions $x \mapsto J_\nu(x)$ (red) and $x \mapsto \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)$ (blue) for $\nu = 5$.

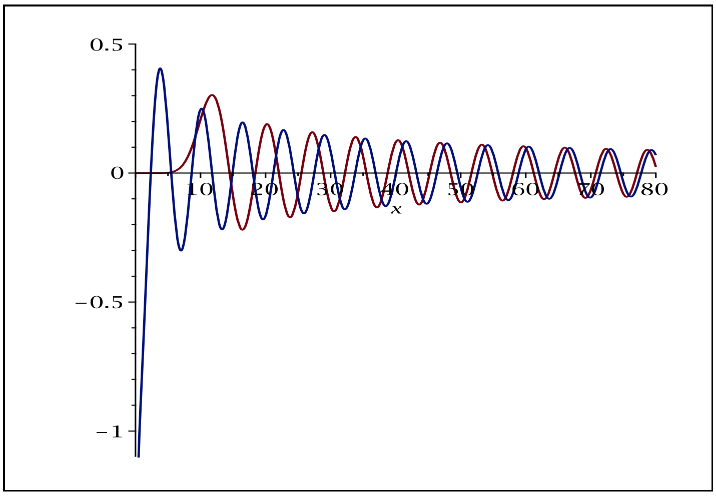

Figure 5 – Graphs of the functions $x \mapsto J_\nu(x)$ (red) and $x \mapsto \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)$ (blue) for $\nu = 10$.

Clearly a necessary condition for the approximation in (10.4) to be anywhere close to $J_\nu(x)$ [36] is that the amplitude in the formula (10.4) is near the amplitude in the formula (10.2), i.e. that $\frac{\sqrt[4]{x^2-\nu^2}}{\sqrt{x}}$ is near one, say $\frac{\sqrt[4]{x^2-\nu^2}}{\sqrt{x}} > 0.99$. This is seen to imply $x > 5\nu$. Next, note that the difference between the two cos-arguments, i.e. between $x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi$ and $\sqrt{x^2-\nu^2} - \nu \arccos(\frac{\nu}{x}) - \frac{\pi}{4}$, tends to 0 as $x \to +\infty$ for any fixed ν. Through differentiation w.r.t. x we see that this absolute difference

[36]Not just 'sporadically', but 'for all x-values in a neighborhood of the x-value under consideration'.

equals $\int_x^\infty (1-\sqrt{1-(\nu/x')^2})\,dx'$; in particular it is a strictly decreasing function of x for fixed ν. Clearly another necessary condition for (10.4) to be relevant for all $x \geq x_0 = x_0(\nu)$ is that this difference is *small* (i.e. less than some fixed small positive constant) for $x = x_0$. Recall that we have already noted that we must have $x_0 > 5\nu$. Using the fact that $1-\sqrt{1-t^2} = \frac{1}{2}t^2 + O(t^4)$ for $0 < t \leq \frac{1}{5}$ we obtain

$$\int_{x_0}^\infty (1-\sqrt{1-(\nu/x')^2})\,dx' = \int_{x_0}^\infty \Big(\frac{\nu^2}{2x'^2} + O\Big(\frac{\nu^4}{x'^4}\Big)\Big)\,dx' = \frac{\nu^2}{2x_0} + O\Big(\frac{\nu^4}{x_0^3}\Big).$$

Hence if $x_0 = \nu^2$ and ν is sufficiently large (i.e. larger than a certain absolute constant) then the above expression is $\in [0.49, 0.51]$; and thus for such large ν the difference under consideration is ≥ 0.49 for all $x_0 \leq \nu^2$, i.e. we must have $x_0 > \nu^2$ for the formula (10.4) to be relevant!

On the other hand for $x_0 > C\nu^2$ with C a not too small constant $C > 1$, the same type of analysis shows that the asymptotic formula (10.4) really *is* starting to be relevant...

Of course, for x *sufficiently* large (as depends on ν), the error term in (10.4) is *better* than the error term in Prop. 10.1! In order to say something more precise, note that it is certainly possible to keep track on the dependence on ν in the computations leading to (9.11), and one result from such an analysis is the following (cf. [11, (B.35)][37]): For all $\nu \geq 0$ and all $x \geq 1+\nu^2$ we have

$$J_\nu(x) = \Big(\frac{2}{\pi x}\Big)^{1/2}\Big\{\cos\big(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\big) + O\Big(\frac{1+\nu^2}{x}\Big)\Big\}, \tag{10.5}$$

where the implied constant is absolute. For large ν the error term here is better than the error term in Prop. 10.1 iff x/ν^3 is sufficiently large!

10.3. **Appendix: Some more details regarding** (10.1)**.** We here give some comments on how to extract (10.1) from Olver [16, Ch. 11.10].

Olver's statement is much more complicated than (10.1) for three reasons: (1) he considers general complex argument in the J-Bessel function; (2) he is interested in allowing to extract *numerical* bounds on the error term; (3) he actually gives an *asymptotic expansion* of $J_\nu(\nu t)$, of which (10.1) is just the main term!

Let us first derive our formula for $\zeta(t)$. Olver's definition is (for $t \in \mathbb{C}$) [16, (10.05)]:

$$\frac{2}{3}\zeta^{3/2} = \log\Big(\frac{1+\sqrt{1-t^2}}{t}\Big) - \sqrt{1-t^2}, \tag{10.6}$$

[37] This reference just gives a statement without a proof or precise reference; I have not yet checked the statement carefully, or located a reference containing a proof. Note however that the statement seems quite reasonable in view of the format of (9.11)–(9.13) (considering higher terms in the expansion).

where the branches take their principal values when $t \in (0,1)$ and $\zeta \in (0,\infty)$, and are continuous elsewhere. Thus for $t \in (0,1)$ (and also for $t = 1$) we have

$$\frac{2}{3}\zeta^{3/2} = \operatorname{arctanh}\big(\sqrt{1-t^2}\big) - \sqrt{1-t^2} = \operatorname{arccosh}(t^{-1}) - \sqrt{1-t^2}.$$

To study the behavior of this function as $t \to 1^-$ we use $\operatorname{arctanh} z - z = \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7 + \ldots$ (true when $|z| < 1$) and (writing $t = 1 - w$, with $w > 0$ near 0): $\sqrt{1-(1-w)^2} = \sqrt{2-w}\sqrt{w} = \sqrt{2w}(1 - \frac{1}{4}w - \frac{1}{32}w^2 - \ldots)$ to conclude:

$$\frac{2}{3}\zeta(1-w)^{3/2} = \frac{2\sqrt{2}}{3}w^{\frac{3}{2}} + \frac{3\sqrt{2}}{10}w^{\frac{5}{2}} + O(|w|^{\frac{7}{2}}).$$

Hence since $\zeta(t)$ is analytic at $t = 1$ and positive for $t < 1$ near 1, we must have $\zeta(1-w) = 2^{\frac{1}{3}}w + \frac{3 \cdot 2^{\frac{1}{3}}}{10}w^2 + \ldots$, i.e.

$$\zeta(1+w) = -2^{\frac{1}{3}}w + \tfrac{3}{10} \cdot 2^{\frac{1}{3}} \cdot w^2 + \ldots \tag{10.7}$$

for all $w \in \mathbb{C}$ near 0. Now consider (10.6) for $t > 1$; we wish to determine which branches to use for the various functions appearing; to start let's assume $\sqrt{1-t^2} = \varepsilon_j i\sqrt{t^2-1}$ where $\varepsilon_1, \varepsilon_2 \in \{1,-1\}$ are for the first and the second appearance of "$\sqrt{1-t^2}$", respectively. Note that $t^{-1}(1 + \varepsilon_1 i\sqrt{t^2-1})$ has absolute value 1 and real part t^{-1}; from this we conclude (since we are using the branch of log which tends to 0 as its argument tends to 1):

$$\log\Big(\frac{1+\varepsilon_1\sqrt{t^2-1}}{t}\Big) = i\varepsilon_1 \arctan\big(\sqrt{t^2-1}\big) = i\varepsilon_1 \arccos(t^{-1}).$$

Hence:

$$\frac{2}{3}\zeta^{3/2} = i\varepsilon_1 \arctan\big(\sqrt{t^2-1}\big) - i\varepsilon_2\sqrt{t^2-1}.$$

Now $\arctan z + z = 2z + O(|z|^3)$ but $\arctan z - z = -\frac{1}{3}z^3 + O(|z|^5)$ as $z \to 0$; hence since we know that the right hand side above must behave like $w^{\frac{3}{2}}$ when $t = 1 + w$, $w \to 0^+$, we conclude that $\varepsilon_1 = \varepsilon_2$; thus

$$\frac{2}{3}\zeta(1+w)^{3/2} = i\varepsilon_1\Big(-\frac{2\sqrt{2}}{3}w^{\frac{3}{2}} + \frac{3\sqrt{2}}{10}w^{\frac{5}{2}} + O(|w|^{\frac{7}{2}})\Big).$$

Comparing with (10.7) we see that we can take either $\varepsilon_1 = 1$ or -1, so long as we take the corresponding correct branch when raising to $\frac{2}{3}$; either way we obtain

$$\zeta(t) = -\big(\tfrac{3}{2}\big(\sqrt{t^2-1} - \arctan\big(\sqrt{t^2-1}\big)\big)\big)^{2/3} \qquad \text{for } t > 0.$$

This completes the proof of the formula for $\zeta(t)$.

Now to get (10.1) we apply [16, (10.18)] with $n = 0$, noticing that Olver's "$A_0(\zeta)$" equals 1:

$$J_\nu(\nu t) = \frac{1}{1+\delta_1}\nu^{-1/3}\Big(\frac{4\zeta}{1-t^2}\Big)^{1/4}\Big\{\mathrm{Ai}(\xi) + \varepsilon_{1,0}(\nu,\zeta)\Big\}$$

(where we use our notation $\xi = \nu^{2/3}\zeta$). Here $|\delta_1| \ll \nu^{-1}$ by [16, (10.20) and p. 422], and from [16, (10.20) and p. 395] we get

$$|\varepsilon_{1,0}(\nu,\zeta)| \ll \nu^{-1}\frac{M(\xi)}{E(\xi)} \ll \nu^{-1}\frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}}$$

(this may be improved for t near 0). Hence for ν larger than a certain absolute constant we have

$$J_\nu(\nu t) = \nu^{-1/3}\Big(\frac{4\zeta}{1-t^2}\Big)^{1/4}\Big\{\mathrm{Ai}(\xi) + O\Big(\nu^{-1}\frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}}\Big)\Big\}\Big(1 + O(\nu^{-1})\Big).$$

Finally the last factor $(1 + O(\nu^{-1}))$ can be multiplied into the expression using (9.15) and (9.16), and we obtain (10.1).

(Note that the above was obtained for ν being lager than a sufficiently large absolute constant $C > 0$. In order to treat the remaining case of $\nu \in [1, C]$ (if $C \geq 1$) one may refer to the easier asymptotic formulas for $J_\nu(x)$ when $x \to 0$ and $x \to \infty$ for ν in a *compact* set.)

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