#11. Stationary phase; examples (Lecture notes, Sec. 9)

① Asymptotics of $J_m(r)$ as $r \to +\infty$ for fixed m

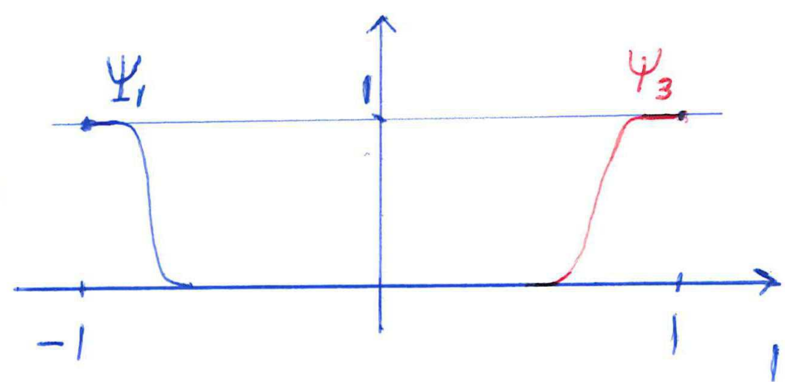
Recall $$J_m(r) = \frac{\left(\frac{r}{2}\right)^m}{\Gamma(m+\frac{1}{2})\,\Gamma(\frac{1}{2})} \cdot \int_{-1}^{1} e^{irx} (1-x^2)^{m-\frac{1}{2}}\, dx.$$

$\forall r \in \mathbb{R}^+$, $m \in \mathbb{C}$ with $Re(m) > -\frac{1}{2}$

Hence wish to understand $\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\,dx$ as $r \to +\infty$.

This is of the form we studied in #10, with $\phi(x) = x$, $\psi(x) = (1-x^2)^{m-\frac{1}{2}}$, $\lambda = r$. No critical points!

Split as $\sum_{j=1}^{3} \int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}} \Psi_j(x)\,dx$ with



Contribution from Ψ_2: Since $x \mapsto (1-x^2)^{m-\frac{1}{2}} \Psi_2(x)$ is C^∞ and has compact support contained in $(-1,1)$,

Prop 1 (in last lecture) gives

$$\int_{-1}^{1} e^{irx} (1-x^2) \Psi_2(x)\, dx = O(r^{-N}) \text{ as } r \to +\infty,$$

for any fixed $N \geq 0$.

Contribution from Ψ_1:

$$\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}}\,\Psi_1(x)\,dx = \boxed{x = x_{new}-1}$$

$$= e^{-ir}\int_0^2 e^{irx}\,x^{m-\frac{1}{2}}\,\underbrace{(2-x)^{m-\frac{1}{2}}\,\Psi_1(x-1)}_{:=\,\psi(x)}\,dx$$

May view $\psi \in C_c^\infty([0,\infty))$

By Lemma 9.1 in lecture notes, this is

$$= e^{-ir}\sum_{j=0}^{N-1} c_j\, r^{-j-m-\frac{1}{2}} + O\left(r^{-N-\frac{1}{2}-Re(m)}\right) \quad \text{as } r \to +\infty.$$

with $c_j = i^{j+m+\frac{1}{2}} \cdot \dfrac{\Gamma(j+m+\frac{1}{2})}{j!} \cdot \psi^{(j)}(0)$

$$\psi^{(j)}(0) = \frac{d^j}{dx^j}\left((2-x)^{m-\frac{1}{2}}\right)\Big|_{x=0}$$

Contribution from Ψ_3: Symmetric! Choose $\Psi_3(x) = \Psi_1(-x)$;

then $\int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}} \Psi_3(x) = \int_{-1}^{1} e^{irx}(1-x^2)^{m-\frac{1}{2}} \Psi_1(x)\,dx$. Etc.!

. . .

Final result (cf lecture #9):

$$J_m(r) = \sqrt{\frac{2}{\pi r}} \Bigg(\cos\left(r - \frac{m\pi}{2} - \frac{\pi}{4}\right) \cdot \sum_{0 \le k < \frac{N}{2}} (-1)^k A_{m,2k} \cdot r^{-2k}$$

$$- \sin\left(r - \frac{m\pi}{2} - \frac{\pi}{4}\right) \cdot \sum_{0 \le k < \frac{N-1}{2}} (-1)^k A_{m,2k+1} \cdot r^{-2k-1} + O(r^{-N}) \Bigg)$$

where $A_{m,n} = \frac{\prod_{j=-n}^{n-1} (m + \frac{1}{2} + j)}{2^n \cdot n!}$.

② Special case of ①, via other formula.

Using now $J_m(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin x - imx}\,dx.$

$m \in \mathbb{Z}$

((8.9) in course notes)

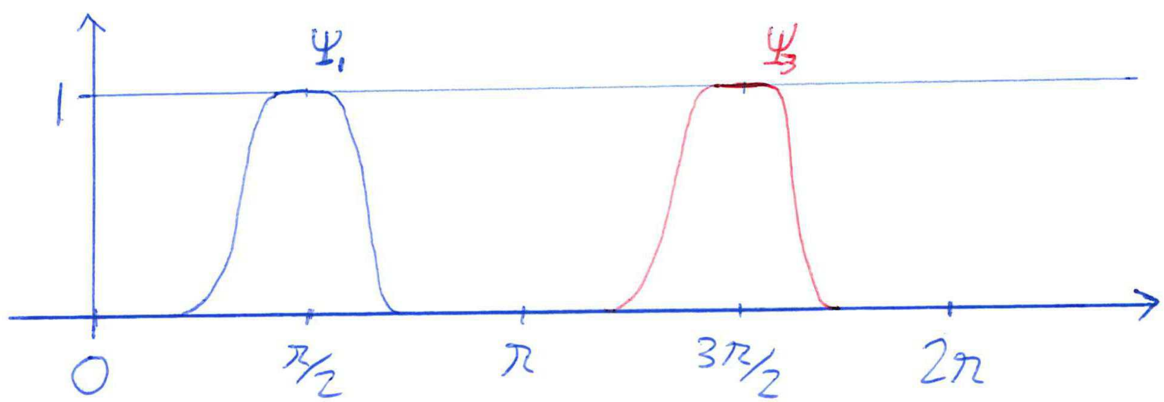
This is of the form studied in #10,

with $\phi(x) = \sin x$; critical points at $\frac{\pi}{2}, \frac{3\pi}{2}$,

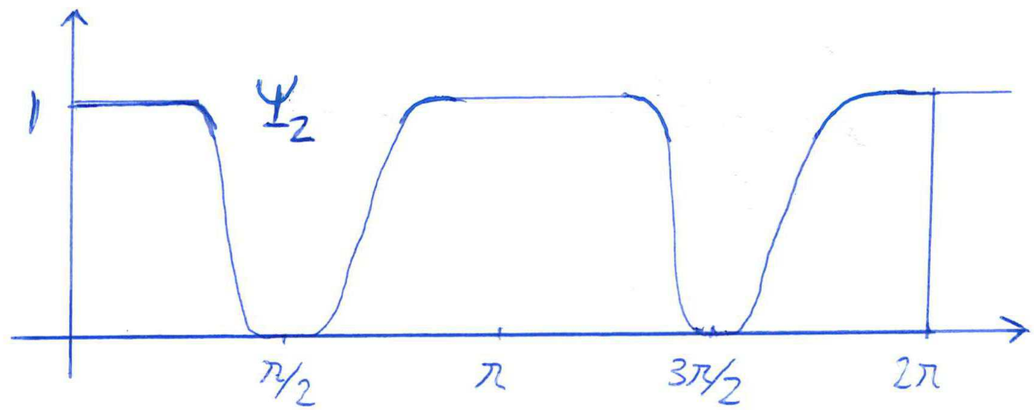
and $\psi(x) = e^{imx}$ and $\lambda = r \to +\infty$.

Split as $\sum_{j=1}^{3} \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin x}\cdot e^{-imx}\cdot \Psi_j(x)\,dx$

with



Contribution from Ψ_2



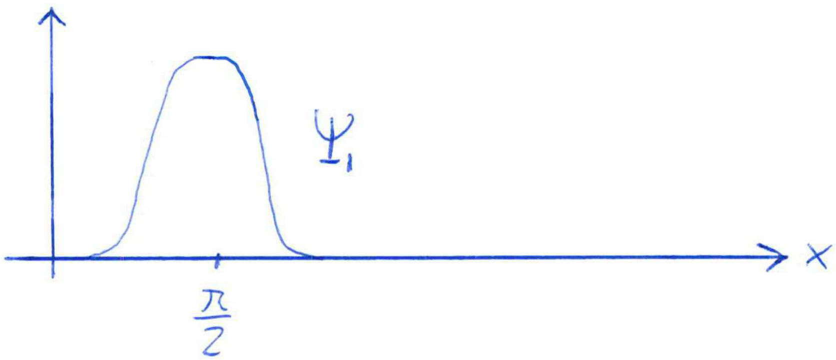
May view as $\displaystyle \frac{1}{2\pi}\int_{\mathbb{R}/2\pi\mathbb{Z}} e^{ir\sin x}\cdot e^{-imx}\cdot \Psi_2(x)\,dx$ (*)

← circle!

Hence Prop. 1 (in last lecture) applies, with no endpoint contributions,

$\therefore$ (*) $= O(r^{-N})$ as $r \to +\infty$, for any fixed $N \geq 0$.

Contribution from Ψ_1



$\phi(x) = \sin x$, with critical point at $x_0 = \frac{\pi}{2}$.

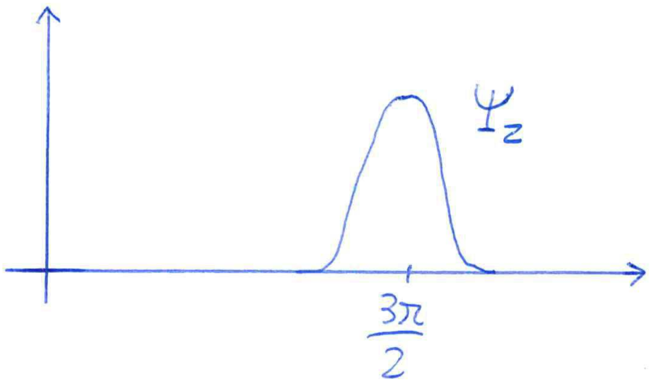
Stein's Prop. 3 applies directly to give

$$\underbrace{\frac{1}{2\pi} \int_0^{2\pi} e^{ir\sin x} \, e^{-imx} \cdot \Psi_1(x)\, dx}_{:= \psi(x)} = \frac{e^{ir}}{2\pi} \int_{-\infty}^{\infty} e^{ir(\sin x - 1)} \, \psi(x)\, dx$$

$$\sim \text{"} \frac{e^{ir}}{2\pi} \cdot r^{-\frac{1}{2}} \cdot \sum_{j=0}^{\infty} a_j \cdot r^{-j/2} \text{"}$$

Stein's § 1.3.4 $\Rightarrow$ $a_0 = \left(\frac{2\pi}{-i\,\phi''(x_0)} \right)^{\frac{1}{2}} \psi(x_0) = \sqrt{-2\pi i} \cdot (-i)^m$.

Contribution from Ψ_2



Take $\Psi_2(x) \equiv \Psi_1(x-\pi)$ and use symmetry!

$$\Rightarrow \frac{1}{2\pi} \int_0^{2\pi} e^{ir\sin x} e^{-imx} \Psi_2(x)\,dx \sim \text{“} \frac{e^{-ir}}{2\pi} r^{-\frac{1}{2}} \sum_{j=0}^{\infty} a_j' \cdot r^{-j/2} \text{”}$$

with $a_0' = \sqrt{2\pi i} \cdot i^m$

Conclusion: $J_m(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi m}{2} - \frac{\pi}{4}\right) + O(r^{-1})$ as $r \to +\infty$

(and also get asymptotic expansion!)