#12. Distribution Theory

DEF: Let U be an open subset of $\mathbb{R}^n$.
A distribution on U is a continuous linear functional on $C_c^\infty(U)$.

We set $\mathcal{D}'(U)$ = the space of all distributions on U.

Also: $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^n)$

Topology on $C_c^\infty(U)$?

Remark: Any $f \in L^1_{loc}(U)$ is identified with the distribution $\phi \mapsto \int_U f \phi \, dx$ $\left(\phi \in C_c^\infty(U)\right)$. Hence a distribution is "a generalized function"!

Definition of $C_c^\infty(U)$?

DEF: For any Borel set $E \subset \mathbb{R}^n$:

$$C_c^\infty(E) = \{ f \in C_c^\infty(\mathbb{R}^n) : \operatorname{supp}(f) \text{ is compact and } \subset E \}$$

DEF: For K compact $\subset \mathbb{R}^n$, we give $C_c^\infty(K)$ the topology generated by the seminorms $\phi \mapsto \|\partial^\alpha \phi\|_u$ $(\alpha \in (\mathbb{Z}_{\geq 0})^n)$.

This makes $C_c^\infty(K)$ into a Fréchet space

Thm 5.14: Let $\{p_\alpha\}$ be a family of seminorms on a $\mathbb{C}$-vector space X.
For $x \in X$, $\alpha \in A$, $\varepsilon > 0$, set $B_{x,\alpha,\varepsilon} = \{ y \in X : p_\alpha(y-x) < \varepsilon \}$, and let $\mathcal{T}$ be the topology generated by $\{B_{x,\alpha,\varepsilon}\}$.
Then $(X, \mathcal{T})$ is a topological vector space. Also for any net $\langle x_i \rangle_{i \in I}$ in X, we have $x_i \to x$ iff $[p_\alpha(x_i - x) \to 0$ for all $\alpha \in A]$

Prop. 5.15: Assume that $\langle X, \{p_\alpha\}_{\alpha\in A}\rangle$ and $\langle Y, \{q_\beta\}_{\beta\in B}\rangle$ are t.v.s as above, and let $T: X \to Y$ be a linear map. Then T is continuous iff

$$\forall \beta\in B:\ \exists \alpha_1,\dots,\alpha_k\in A,\ \exists C>0 \text{ s.t. } \forall x\in X: q_\beta(Tx) \leq C\sum_{j=1}^{k} p_{\alpha_j}(x).$$

Prop. 5.16: Let $\langle X, \{p_\alpha\}_{\alpha\in A}\rangle$ be a t.v.s. as above.

a) X is Hausdorff iff $\forall x\neq 0: \exists \alpha\in A: p_\alpha(x)\neq 0$.

b) If X is Hausdorff and A countable, then X is metrizable with a translation-invariant metric.

DEF: A t.v.s. as in 5.16(b) which is complete is called a Fréchet space.

Topology on $C_c^\infty(U)$ (for U open $\subset \mathbb{R}^n$)

There is a certain natural topology on $C_c^\infty(U)$ (locally convex, but not metrizable) such that

- a sequence $\phi_1, \phi_2, \dots$ in $C_c^\infty(U)$ converges to ϕ iff there is a compact set $K \subset U$ such that $\phi_1, \phi_2, \dots \in C_c^\infty(K)$ and $\phi_j \to \phi$ in $C_c^\infty(K)$.

- a linear map $F: C_c^\infty(U) \to \mathbb{C}$ is continuous iff $F|_{C_c^\infty(K)}$ is continuous for each compact set $K \subset U$.

$\boxed{\begin{array}{l} \text{Hence: A linear functional } F: C_c^\infty(U) \to \mathbb{C} \text{ is a distribution on } U \\ (\Leftrightarrow \text{ is continuous}) \text{ iff: } \forall \text{ compact } K \subset U: \exists C>0,\ k \in \mathbb{Z}_{\geq 0}: \\ \qquad \forall \phi \in C_c^\infty(K): |F(\phi)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha \phi\|_u \end{array}}$

Ex: For any $f \in L^1_{loc}(U)$:

$\forall\, K$ compact $\subset U$, $\phi \in C_c^\infty(K)$: $\left|\int_U f\phi\, dx\right| = \left|\int_K f\phi\, dx\right| \leqslant \left(\int_K |f|\right)\cdot \|\phi\|_u$

Hence, indeed, $L^1_{loc}(U) \subset \mathcal{D}'(U)$.

Ex: For any $\mu \in M(\mathcal{B}_U)$:

The functional $\phi \longmapsto \int_U \phi\, d\mu$ is a distribution on U.

$\left(\text{Indeed: } \forall\, \phi \in C_c^\infty(U): \left|\int_U \phi\, d\mu\right| \leqslant \|\mu\|\cdot\|\phi\|_u\right)$

Hence $M(\mathcal{B}_U) \subset \mathcal{D}'(U)$.

Ex: For any $x_0 \in U$ and $\alpha \in (\mathbb{Z}_{\geq 0})^n$:

The functional $\phi \longmapsto \partial^\alpha \phi(x_0)$ is a distribution on U.

DEF: If $V \subset U$ are open subsets of $\mathbb{R}^n$, and $F \in \mathcal{D}'(U)$, then we define $F_{|V} \in \mathcal{D}'(V)$ through $\langle F_{|V}, \phi \rangle = \langle F, \phi \rangle, \ \forall \phi \in C_c^\infty(V)$.

Prop. 9.2: Let $\{V_\eta\}$ be a family of open subsets of U, and set $V = \bigcup_\eta V_\eta$. If $F, G \in \mathcal{D}'(U)$ and $F_{|V_\eta} = G_{|V_\eta}, \ \forall \eta$, then $F_{|V} = G_{|V}$.

proof: Assume $\phi \in C_c^\infty(V)$.

Can then find $\eta_1, \eta_2, \ldots, \eta_m$ s.t. $\operatorname{supp}(\phi) \subset \bigcup_{j=1}^m V_{\eta_j}$.

Now pick $\phi_1, \phi_2, \ldots, \phi_m \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\phi_j) \subset V_{\eta_j}$ and $\sum_{j=1}^m \phi_j = \phi$.

(This uses a partition of unity.)

Now $\langle F, \phi \rangle = \sum_{j=1}^m \langle F, \phi_j \rangle = \sum_{j=1}^m \langle G, \phi_j \rangle = \langle G, \phi \rangle$.

DEF: For $F \in \mathcal{D}'(U)$, $\underline{\underline{\mathrm{supp}(F)}} := \underline{U \setminus \left[\begin{array}{l}\text{The largest open set } V \subset U \\ \text{with } F_{/V} = 0\end{array}\right]}$

Equivalently, $\underline{\underline{\mathrm{supp}(F) = \left\{ x \in U : \ x \text{ has no open neighbourhood } V \subset U \text{ with } F_{/V} = 0 \right\}}}$

Operations on distributions (start)

Differentiation

Note that if $f \in C^1(U)$ and $\phi \in C_c^\infty(U)$ then

$$\int_U (\partial_j f) \cdot \phi \, dx = \underbrace{\text{"}[f \cdot \phi]\text{"}}_{=0} - \int_U f \cdot (\partial_j \phi) \, dx$$

$\rightsquigarrow$ **DEF:** For $F \in \mathcal{D}'(U)$, $\partial_j F \in \mathcal{D}'(U)$ is defined by
$\langle \partial_j F, \phi \rangle := -\langle F, \partial_j \phi \rangle$, $\forall \phi \in C_c^\infty(U)$.

Is really $\partial_j F \in \mathcal{D}'(U)$? Yes; prove by showing $\phi \to \partial_j \phi$ is a continuous map $C_c^\infty(U) \to C_c^\infty(U)$. For this, it suffices to note that $\phi \mapsto \partial_j \phi$ is continuous $C_c^\infty(K) \to C_c^\infty(K)$, for every compact $K \subset U$.

For general multi-index α: $\boxed{\langle \partial^\alpha F, \phi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle}$

Multiplication by smooth functions

DEF: For $F \in \mathcal{D}'(U)$ and $\psi \in C^\infty(U)$:

$\psi F \in \mathcal{D}'(U)$ is defined by $\langle \psi F, \phi \rangle = \langle F, \psi \phi \rangle$, $\forall \phi \in C_c^\infty(U)$.

If $\psi \in C_c^\infty(U)$ then we even have $\psi F \in \mathcal{D}'(\mathbb{R}^n)$!