#13. Distribution Theory (cont'd)

Convolution with C_c^∞-functions

on $\mathbb{R}^n$

Take $\psi \in C_c^\infty$ and $f \in L^1_{loc}$.

Then $f * \psi(x) = \int_{\mathbb{R}^n} f(y)\,\psi(x-y)\,dy$

Action on test functions?

$\forall \phi \in C_c^\infty$: $\int_{\mathbb{R}^n} (f*\psi)(x) \cdot \phi(x)\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\,\psi(x-y)\,dy \cdot \phi(x)\,dx$

$$= \int_{\mathbb{R}^n} f(y) \cdot (\phi * \tilde{\psi})(y)\,dy$$

$\rightsquigarrow$ **DEF:** For $F \in \mathcal{D}'$, $F * \psi \in \mathcal{D}'$ is defined by

$$\boxed{\langle F*\psi, \phi\rangle = \langle F, \phi * \tilde{\psi}\rangle, \quad \forall \phi \in C_c^\infty}$$

Ok def? Yes if the map $\phi \mapsto \phi * \tilde{\psi}$, $C_c^\infty \to C_c^\infty$ is continuous

For any compact $K \subset \mathbb{R}^n$, set $K' = K + \operatorname{supp}(\tilde{\psi})$; this is also a compact subset of $\mathbb{R}^n$, and $\phi \mapsto \phi * \tilde{\psi}$ is continuous $C_c^\infty(K) \to C_c^\infty(K')$.

Indeed, $\forall \alpha$: $\|\partial^\alpha(\phi * \tilde{\psi})\|_u = \|(\partial^\alpha \phi) * \tilde{\psi}\|_u \ll \|\partial^\alpha \phi\|_u$.

Hence ok!

Alternative definition:

For $\psi \in C_c^\infty$, $f \in L^1_{loc}$, note $f*\psi(x) = \int_{\mathbb{R}^n} f(y)\, \psi(x-y)\, dy = \int_{\mathbb{R}^n} f(y) \cdot (\tau_x \tilde{\psi})(y)\, dy$

ALT. DEF: For $F \in \mathcal{D}'$, $F * \psi$ is the function on $\mathbb{R}^n$ given by $F*\psi(x) = \langle F, \tau_x \tilde{\psi} \rangle$ $(x \in \mathbb{R}^n)$

Prop 9.3: In the above situation,

c) the two definitions agree!

a) $F * \Psi \in C^\infty$

b) $\partial^\alpha(F * \Psi) = (\partial^\alpha F) * \Psi = F * (\partial^\alpha \Psi)$, $\forall \alpha \in (\mathbb{Z}_{\geq 0})^n$

Comments about proof:

c) The task is to prove, $\forall \phi \in C_c^\infty$:

$$\langle F, \phi * \widetilde{\Psi} \rangle = \int_{\mathbb{R}^n} \langle F, \tau_x \widetilde{\Psi} \rangle \cdot \phi(x)\, dx$$

- Approximate $\phi * \widetilde{\Psi} = \int_{\mathbb{R}^n} \phi(y) \cdot (\tau_y \widetilde{\Psi})\, dy$ by Riemann sums;

$$S_m = 2^{-nm} \sum_j \phi(y_j) \cdot \tau_{y_j} \widetilde{\Psi} \in C_c^\infty$$

Get $S_m \to \phi * \widetilde{\Psi}$ in C_c^∞, hence

$$\underline{\langle F, \phi * \tilde{\Psi} \rangle} = \lim_{m\to\infty} \langle F, S_m \rangle = \lim_{m\to\infty} 2^{-nm} \sum_j \phi(y_j) \cdot \underbrace{\langle F, \tau_{y_j} \tilde{\Psi} \rangle}$$

continuous "wrt y_j"
(by part (a))

$$= \underline{\int_{\mathbb{R}^n} \phi(y) \cdot \langle F, \tau_y \tilde{\Psi} \rangle \, dy} \; ;$$

done!

(a),(b): Nice exercises on using the topology of C_c^∞!

Remark: More generally, if $\underline{F \in \mathcal{D}'(U)}$ (U open $\subset \mathbb{R}^n$) and $\underline{\Psi \in C_c(\mathbb{R}^n)}$, then $\underline{F * \Psi \in \mathcal{D}'(V)}$ where $\underline{V = \{x \in \mathbb{R}^n : x - \mathrm{supp}(\Psi) \subset U\}}$.

Application of convolution:

Prop. 9.5: For U open $\subset \mathbb{R}^n$: $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$.

- here $\mathcal{D}'(U)$ has the weak-$*$ topology, that is, the topology generated by the seminorms $\|F\|_\phi := |\langle F, \phi\rangle|$ $(\phi \in C_c^\infty(U))$.

We have $F_j \to F$ in $\mathcal{D}'(U)$ iff $\langle F_j, \phi\rangle \to \langle F, \phi\rangle$, $\forall\, \phi \in C_c^\infty(U)$.

outline of proof: Given $F \in \mathcal{D}'(U)$, approximate by

$$(\gamma \cdot F) * \psi_t$$

where $\gamma \in C_c(U)$, $\gamma = 1$ on large compact subset of U, and $\psi \in C_c^\infty$ a fixed "bump" function, and t small.

Distributions of compact support (Ch. 9.2)

DEF: For U open $\subset \mathbb{R}^n$: $\mathcal{E}'(U) := \{F \in \mathcal{D}'(U) : \operatorname{supp}(F) \text{ compact}\}$

DEF: $C^\infty(U)$ is given the topological vector space structure generated by the seminorms

$$\phi \longmapsto \sup_K |\partial^\alpha \phi| := \|\phi\|_{[K,\alpha]} \qquad \left(K \text{ compact} \subset U,\ \alpha \in (\mathbb{Z}_{\geq 0})^n\right)$$

- Here it suffices to use a countable family of K's (namely any family with union $= U$).

Facts: $C^\infty(U)$ is a Frechet space.

$C_c^\infty(U)$ is dense in $C^\infty(U)$.

Theorem 9.8: $\mathcal{E}'(U)$ = the dual of $C^\infty(U)$

Precise statement: Any $F \in \mathcal{E}'(U)$ has a unique extension to a continuous linear functional on $C^\infty(U)$, and every continuous linear functional on $C^\infty(U)$ is so obtained.

from the proof: Given $F \in \mathcal{E}'(U)$, take $\psi \in C_c^\infty(U)$ with $\psi = 1$ on $\operatorname{supp}(F)$. Then define the "extended F" by

$$\langle F, \phi \rangle := \langle F, \psi \phi \rangle, \quad \forall\, \phi \in C^\infty(U).$$

Note: $\mathcal{E}'(U) \subset \mathcal{E}'(\mathbb{R}^n)$, but $\mathcal{D}'(U) \not\subset \mathcal{D}'(\mathbb{R}^n)$. (when $U \neq \mathbb{R}^n$)

Operations on $\mathcal{E}'(U)$

Differentiation

Multiplication by C^∞-function

Composition with diffeomorphism

Convolution

① $F \in \mathcal{E}'$, $\Psi \in C_c^\infty \Rightarrow F * \Psi \in \mathcal{E}'$

In fact, $\forall F \in \mathcal{D}'$, $\Psi \in C_c^\infty$: $\operatorname{supp}(F * \Psi) \subset \overline{\operatorname{supp}(F) + \operatorname{supp}(\Psi)}$.

② Can extend to $F \in \mathcal{E}'$, $\Psi \in C^\infty$; "both definitions ok"!

③ Much more general: For $F \in \mathcal{D}'$, $G \in \mathcal{E}'$, define $F * G$ and $G * F$ through $\langle F * G, \phi \rangle = \langle F, \tilde{G} * \phi \rangle$ and $\langle G * F, \phi \rangle = \langle G, \tilde{F} * \phi \rangle$.