#14. Tempered distributions

Fourier transform of distributions?

For $f \in L^1(\mathbb{R}^n)$, $\phi \in C_c^\infty$, we have $\int_{\mathbb{R}^n} \hat{f}\, \phi \, dx \overset{\text{Lemma 8.25}}{=} \int_{\mathbb{R}^n} f \, \hat{\phi} \, dx.$

Here $\hat{\phi}$ is not in C_c^∞ (unless $\phi \equiv 0$);

hence we cannot define "$\langle \hat{F}, \phi \rangle := \langle F, \hat{\phi} \rangle$" for $F \in \mathcal{D}'(\mathbb{R}^n)$.

DEF: The Schwarz space, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n) : \ \|f\|_{(N,\alpha)} < \infty \ \text{ for all } N, \alpha \}$$

where $\|f\|_{(N,\alpha)} := \sup_{x \in \mathbb{R}^n} (1+|x|)^N \, |\partial^\alpha f(x)|$ for $N \in \mathbb{Z}_{\geq 0}$, $\alpha \in (\mathbb{Z}_{\geq 0})^N$.

Prop. 8.2: $\mathcal{S}$ is a Fréchet space with the topology defined by the semi-norms $\|\cdot\|_{(N,\alpha)}$.

Prop 8.11: If $f, g \in \mathcal{S}$ then $f * g \in \mathcal{S}$.

Prop 8.23 & 8.28: The map $f \mapsto \hat{f}$ is an isomorphism of topological vector spaces $\mathcal{S} \xrightarrow{\sim} \mathcal{S}$.

DEF: A tempered distribution on $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{S}$. The space of these: $\mathcal{S}'$.

Prop 9.7, 9.9, ...: We have $C_c^\infty \subset \mathcal{S} \subset C^\infty$, with each embedding map being continuous, and each subspace dense in the larger one(s).

Dually: $\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$

Operations

As before: differentiation, translation, composition with $GL_n(\mathbb{R})$-map.

- these give continuous linear maps $\mathcal{S}' \to \mathcal{S}'$.

For multiplication:

DEF: A function $\psi \in C^\infty$ is said to be slowly increasing if $\forall \alpha \in (\mathbb{Z}_{\geq 0})^n$: $\exists C > 0, N > 0$: $\forall x \in \mathbb{R}^n$: $|\partial^\alpha \psi(x)| \leq C(1+|x|)^N$.

DEF: For $F \in \mathcal{S}'$ and $\psi \in C^\infty$ slowly increasing: $\psi F \in \mathcal{S}'$ is defined by $\langle \psi F, \phi \rangle := \langle F, \psi \phi \rangle$, $\forall \phi \in \mathcal{S}$.

Convolution

DEF: For $F \in \mathcal{S}'$, $\psi \in \mathcal{S}$, define $F * \psi : \mathbb{R}^n \to \mathbb{C}$ by $F * \psi(x) = \langle F, \tau_x \tilde{\psi} \rangle$

Prop 9.10: $F * \psi$ is a slowly increasing C^∞-function, and as an element in $\mathcal{S}'$ it satisfies $\langle F * \psi, \phi \rangle = \langle F, \phi * \tilde{\psi} \rangle$ $(\forall \phi \in \mathcal{S})$

From the proof: $F * \psi$ slowly increasing ?

F continuous $\Rightarrow$ $\exists\, N, k, C$ s.t. $\forall \phi \in \mathcal{S} : |\langle F, \phi \rangle| \leq C \sum_{|\alpha| \leq k} \|\phi\|_{(N,\alpha)}$.

Hence
$$|F * \psi(x)| = \langle F, \tau_x \tilde{\psi} \rangle \leq C \sum_{|\alpha| \leq k} \sup_{y \in \mathbb{R}^n} (1+|y|)^N |\partial^\alpha \psi(y-x)|$$

$$= C \sum_{|\alpha| \leq k} \sup_{y \in \mathbb{R}^n} (1+|x-y|)^N |\partial^\alpha \psi(y)|.$$

Use here $1+|x-y| \leqslant (1+|x|)(1+|y|)$

Get: $|F * \Psi(x)| \leqslant C' \left(\sum_{|\alpha| \leq k} \|\Psi\|_{(N,\alpha)} \right) \cdot (1+|x|)^N$

In the same way, for any β:

$$|\partial^\beta (F * \Psi)(x)| = |\langle F, \tau_x \partial^\beta \tilde{\Psi} \rangle| \leqslant C'' \cdot \left(\sum_{|\alpha| \leq k} \|\partial^\beta \Psi\|_{(N,\alpha)} \right) \cdot (1+|x|)^N$$

$\therefore$ $F * \Psi$ is slowly increasing!

Fourier transform

$\boxed{\textbf{DEF}: \text{ For } F \in \mathcal{S}' \text{ we define } \hat{F} \in \mathcal{S}' \text{ by } \langle \hat{F}, \phi \rangle = \langle F, \hat{\phi} \rangle \ \forall \phi \in \mathcal{S}}$

"All" the basic rules for Fourier transforms extend to $\mathcal{S}'$!

Ex: $(\tau_y F)^\wedge = e^{-2\pi i \xi y} \cdot \hat{F}$

Proof (general): Easy 'parsing' through the definitions, or use [$F \mapsto \hat{F}$ is continuous] and [C_c^∞ dense in $\mathcal{S}'$].

Fourier inversion: $F \mapsto \hat{F}$ is an isomorphism of topological vector spaces $\mathcal{S}' \xrightarrow{\sim} \mathcal{S}'$, with inverse $F \mapsto \check{F} := \hat{\tilde{F}} = \tilde{\hat{F}}$

$\langle \tilde{F}, \phi \rangle := \langle F, \tilde{\phi} \rangle$

Examples

$\hat{\delta} = 1$ $\qquad$ ($\delta :=$ Dirac delta at 0)

proof: $\forall \phi \in \mathcal{S}: \langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \int_{\mathbb{R}} \phi(x)\,dx = \langle 1, \phi \rangle$

Alt: $\delta \in \mathcal{E}'$, hence by Prop. 9.11: $\hat{\delta}(\xi) = \langle \delta, E_{-\xi} \rangle = 1, \ \forall \xi \in \mathbb{R}^n$

Hence:

$(\tau_y \delta)^\wedge = e^{-2\pi i y \xi}$ $\qquad$ $(\forall y \in \mathbb{R}^n)$

$(\partial^\alpha \tau_y \delta)^\wedge = (2\pi i \xi)^\alpha e^{-2\pi i y \xi}$ $\qquad$ $(\forall y \in \mathbb{R}^n, \ \alpha \in (\mathbb{Z}_{\geq 0})^n)$

Fourier inversion $\Rightarrow$ $(x^\alpha)^\wedge = (-2\pi i)^{-|\alpha|} \cdot (\partial^\alpha \delta)$

(On $\mathbb{R}$: $\hat{x} = (-2\pi i)^{-1} \delta'$, $\widehat{x^2} = (-2\pi i)^{-2} \delta''$, ...)

Ex: application to functions

For $P : \mathbb{R}^n \to \mathbb{C}$ a polynomial, say $P(\xi) = \sum_\alpha b_\alpha \xi^\alpha$,

define $P(D) := \sum_\alpha b_\alpha D^\alpha$ where $D^\alpha = (2\pi i)^{-|\alpha|} \partial^\alpha$ (cf. §8.7)

(Ex: $P(\xi) = (2\pi i)^2 (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2) \Rightarrow P(D) = \partial_1^2 + \dots + \partial_n^2 = \Delta$

$P(\xi) = (2\pi i)(\xi_1 + i\xi_2) \Rightarrow P(D) = \partial_1 + i\partial_2$)

Exc 9.25: Assume $\forall \xi \in \mathbb{R}^n \setminus \{0\} : P(\xi) \neq 0$.

Then (a) $\forall F \in \mathcal{S}'$: $P(D)F = 0 \Rightarrow F$ is a polynomial.

(b) Every bounded function f satisfying $P(D)f = 0$ is a constant.

solution, (a): Assume $F \in \mathcal{S}'$ and $P(D)F = 0$.

Then $(P(D)F)^\wedge = 0$, i.e. $P(\xi) \cdot \hat{F} = 0$

Also $P \neq 0$ on $\mathbb{R}^n \setminus \{0\}$; hence $\underline{\underline{\operatorname{supp}(\hat{F}) \subset \{0\}}}$.

Hence $\underline{\hat{F}}$ is a finite $\mathbb{C}$-linear combination of $\partial^\alpha \delta$ $(\alpha \in (\mathbb{Z}_{\geq 0})^n)$

$\Rightarrow$ $\underline{\underline{F \text{ polynomial}}}$.

Application to distributions

Prop. 9.14: a) If $F \in \mathcal{E}'$, then $\exists N \in \mathbb{N}$, $c_\alpha \in \mathbb{C}$ (for $|\alpha| \leq N$) and $f \in C_0(\mathbb{R}^n)$ such that $F = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha f$.

b) If $F \in \mathcal{D}'(U)$ and V is open $\subset U$ such that $\overline{V}$ is a compact subset of U, then $\exists N, c_\alpha, f$ as above such that $F = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha f$ on V

proof (a):

$$F \in \mathcal{E}' \Rightarrow \hat{F} \in C^\infty \text{ and slowly increasing}$$

Prop. 9.11

$$\Rightarrow \exists M \in \mathbb{Z}^+ \text{ s.t. } g := (1 + \|\xi\|^2)^{-M} \hat{F} \in L^1$$

$$\Rightarrow \check{g} \in C_0$$

But $\hat{F} = (1 + \|\xi\|^2)^M g \Rightarrow F = \left(I - (4\pi^2)^{-1} \Delta\right)^M \check{g}$