#15. Sobolev spaces

For $k \in \mathbb{N}$ and any $f \in \mathcal{S}'$, set

$$\|f\|_{(k)} := \begin{cases} \sqrt{\sum_{|\alpha| \le k} \int_{\mathbb{R}^n} |\partial^\alpha f|^2 \, dx} & \text{if } \partial^\alpha f \in L^2 \text{ for all } \alpha \text{ with } |\alpha| \le k \\ +\infty & \text{otherwise} \end{cases}$$

Note $\|f\|_{(k)} < \infty \iff \widehat{\partial^\alpha f} \in L^2$ for all $|\alpha| \le k$

$\iff \hat{f} \in L^2$ and $\int_{\mathbb{R}^n} |\xi^\alpha|^2 |\hat{f}(\xi)|^2 \, d\xi < \infty$ for all $|\alpha| \le k$,

and then $\|f\|_{(k)} \asymp \sqrt{\int_{\mathbb{R}^n} \sum_{|\alpha| \le k} |\xi^\alpha|^2 |\hat{f}|^2 \, d\xi} \asymp \sqrt{\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi}$

DEF: For $s \in \mathbb{R}$: $\underline{H_s = \{ f \in \mathcal{S}' : (1+|\xi|^2)^{s/2} \hat{f} \in L^2 \}}$

DEF: For $s \in \mathbb{R}$: $\underline{\Lambda_s : \mathcal{S}' \to \mathcal{S}'}$; $\underline{\Lambda_s f = \left((1+|\xi|^2)^{s/2} \cdot \hat{f} \right)^{\vee}}$

(Then $\underline{H_s = \{ f \in \mathcal{S}' : \Lambda_s f \in L^2 \}}$.)

We make $\underline{H_s}$ into a <u>Hilbert space</u>, with inner product:

$$\underline{\langle f, g \rangle_{(s)}} := \underline{\int_{\mathbb{R}^n} (\Lambda_s f) \cdot \overline{(\Lambda_s g)}\, dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, (1+|\xi|^2)^s\, d\xi};$$

thus norm; the <u>Sobolev (L^2) norm</u>: $\underline{\|f\|_{(s)}} := \underline{\|\Lambda_s f\|_2 = \sqrt{\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s\, d\xi}}$

Facts

* $\mathcal{F}(H_s) = \left\{ f \in L^1_{loc} : \int_{\mathbb{R}^n} |f(\xi)|^2 \cdot (1+|\xi|^2)^s \, d\xi < \infty \right\} = L^2\left(\mathbb{R}^n, (1+|\xi|^2)^s d\xi\right)$

* $\mathcal{S}$ is dense in H_s $\quad (\forall s \in \mathbb{R})$

* For $t < s$: $\quad H_s \subset H_t$; $\quad \|\cdot\|_t \leq \|\cdot\|_s$

* $\Lambda_t : H_s \xrightarrow{\sim} H_{s-t}$

* $H_0 = L^2$; $\quad \|\cdot\|_{(0)} = \|\cdot\|_2$

* $\forall s \in \mathbb{R}, \ \alpha \in (\mathbb{Z}_{\geq 0})^n$: $\quad \partial^\alpha$ is a bounded linear map $H_s \to H_{s-|\alpha|}$.

Prop. 9.16: $\forall s \in \mathbb{R}$: $H_{-s} \cong H_s^*$, a natural Hilbert space isomorphism; $H_{-s} \ni f \mapsto \left[\begin{array}{l}\text{the unique continuous linear extension} \\ \text{of } \phi \mapsto \langle f, \phi\rangle \text{ from } \mathcal{S} \text{ to } H_s\end{array}\right]$

proof of surjectivity:

Take $G \in H_s^*$.

H_s Hilbert space $\Rightarrow$ $\exists!\, v \in H_s : \forall \phi \in H_s : G(\phi) = \langle \phi, v\rangle_{(s)}$.

Now for all $\phi \in \mathcal{S}$:

$$G(\phi) = \langle \phi, v\rangle_{(s)} = \int_{\mathbb{R}^n} \hat{\phi}(\xi)\, \overline{\hat{v}(\xi)} \cdot (1+|\xi|^2)^s \, d\xi$$

$$= \left\langle \overline{\hat{v}(\xi)} \cdot (1+|\xi|^2)^s, \hat{\phi} \right\rangle = \underbrace{\left\langle \widehat{\overline{\hat{v}(\xi)}\,(1+|\xi|^2)^s}, \phi \right\rangle}$$

lies in H_{-s}, and maps to G!

The Sobolev Embedding Theorem (Thm 9.17):

$s > \frac{n}{2} \Rightarrow H_s \subset C_0$ and the inclusion map is continuous.

More generally:

$s > \frac{n}{2} + k \Rightarrow H_s \subset C_0^k$, and — " — — " —

Here $C_0^k = \{ f \in C^k(\mathbb{R}^n) : \partial^\alpha f \in C_0 \text{ for } |\alpha| \leq k \}$,

with norm $f \mapsto \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_k$

proof: To show $H_s \subset C_0$, it suffices to prove $\forall f \in H_s : \hat{f} \in L^1$.

But $f \in H_s \Rightarrow \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty$

$$\Rightarrow \int_{\mathbb{R}^n} |\hat{f}(\xi)| \, d\xi = \int_{\mathbb{R}^n} |\hat{f}(\xi)| \cdot (1+|\xi|^2)^{s/2} (1+|\xi|^2)^{-s/2} \, d\xi$$

$$\leqslant \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi\right)^{\frac{1}{2}} \underbrace{\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} \, d\xi\right)^{\frac{1}{2}}}_{< \infty \text{ iff } 2s > n}$$

Done.

Cor 9.18: If $f \in H_s$ for all $s \in \mathbb{R}$ then $f \in C^\infty$ (and $\partial^\alpha f \in C_0$, $\forall \alpha$).

Localized Sobolev space

DEF: For U open $\subset \mathbb{R}^n$:

$$H_s^{loc}(U) = \{ f \in \mathcal{D}'(U) : \text{For every open set } V \text{ with } \overline{V} \text{ compact} \subset U: \exists\, g \in H_s : f_{|V} = g_{|V} \}$$

Prop. 9.23: $f \in \mathcal{D}'(U)$ is in $H_s^{loc}(U)$ iff $\forall \phi \in C_c^\infty(U): \phi f \in H_s$

proof, outline: $\Leftarrow$ "easy"

$\Rightarrow$ Use the fact that multiplication by any C_c^∞-function preserves H_s. (Thm 9.20, Cor. 9.21)

Elliptic regularity

Let $P(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$ $\left(c_\alpha \in \mathbb{C}, \ D^\alpha = (2\pi i)^{-|\alpha|} \partial^\alpha\right)$.

Assume that $P(D)$ has order m, i.e. $c_\alpha \neq 0$ for some α with $|\alpha| = m$.

DEF: The principal symbol of $P(D)$ is $P_m(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha$.

$P(D)$ is called elliptic if $P_m(\xi) \neq 0, \ \forall \xi \in \mathbb{R}^n \setminus \{0\}$.

(Ex: Δ is elliptic. But $\partial_t - \Delta$ and $\partial_t^2 - \Delta$ (on $\mathbb{R}^{n+1}$) are not.)

Elliptic Regularity Theorem (Thm 9.26): Assume that $L = P(D)$ is elliptic, of order m. Let Ω open $\subset \mathbb{R}^n$, and $u \in \mathcal{D}'(\Omega)$. If $Lu \in H_s^{loc}(\Omega)$ for some $s \in \mathbb{R}$, then $u \in H_{s+m}^{loc}(\Omega)$.

Ex: If $f \in \mathcal{D}'(\Omega)$ and $(\Delta - \lambda)f = 0$ in Ω, then $f \in C^\infty(U)$!

Central mechanism:

$$\boxed{\text{Lemma 9.25: If } u \in H_s \text{ and } Lu \in H_s \text{ then } u \in H_{s+m}.}$$

proof (outline): On the "Fourier side", the task is to prove:

$$\boxed{\begin{array}{l}\text{If } \hat{u}(\xi)\cdot(1+|\xi|^2)^s \in L^2 \text{ and } \hat{u}(\xi)\cdot P(\xi)\cdot(1+|\xi|^2)^{s/2} \in L^2, \\ \text{then } \hat{u}\cdot(1+|\xi|^2)^{\frac{s+m}{2}} \in L^2\end{array}}$$

This follows from $\underbrace{\exists C > 0 : \forall \xi \in \mathbb{R}^n : (1+|\xi|^2)^{\frac{m}{2}} \leq C(1+|P(\xi)|)}_{\circledast}$

$\circledast$ Holds since $|P(\xi)| \gg |\xi|^m$ for $|\xi|$ large. To see this, use

$|P_m(\xi)| = |\xi|^m \cdot |P_m(|\xi|^{-1}\xi)| \geq \left(\inf\limits_{|\eta|=1} |P_m(\eta)|\right)\cdot|\xi|^m$, which overwhelms $P - P_m$ for $|\xi|$ large.

Outline of proof of Thm 9.26

Assume $Lu \in H_s^{loc}(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. $\boxed{\text{Want to prove } \phi u \in H_{s+m}.}$

Note $\phi u \in \mathcal{E}' \Rightarrow \widehat{\phi u}$ is slowly increasing

$\Rightarrow \exists\, \sigma \in \mathbb{R}$ such that $\phi u \in H_\sigma$

"Dream": $L(\phi u) \in H_s$ (since $L(\phi u) \approx \phi \cdot Lu$???)

Then Lemma 9.25 $\Rightarrow$ $\phi u \in H_{\min(s,\sigma)+m}$ repeat! $\rightsquigarrow$ $\phi u \in H_{s+m}$

Of course $L(\phi u) \neq \phi \cdot Lu$ in general. But the difference

$$L(\phi u) - \phi L u = \underbrace{[L, \phi]}_{\text{differential operator of order } m-1 \text{ (variable coefficients)}} u$$

Take $\psi \in C_c^\infty(\Omega)$ with $\psi = 1$ on a neighbourhood of $\mathrm{supp}(\phi)$

As before: $\exists\, \sigma \in \mathbb{R}$ such that $\psi u \in H_\sigma$.

Now $L(\phi u) = \phi \cdot Lu + [L, \phi] u$

$$= \phi\, Lu + [L, \phi](\psi \cdot u)$$

$$\in H_s + H_{\sigma-(m-1)} = H_{\min(s, \sigma-(m-1))}$$

Also $\phi u = \phi \psi u \in H_\sigma$ (using Cor. 9.21)

Hence by Lemma 9.25, $\phi u \in H_{\min(s+m, \sigma+1)}$.

$\boxed{\text{Idea: Repeat the above many times before reaching } \phi u!}$

Lower σ $\rightsquigarrow$ may assume $\sigma + k = s + m$, some $k \in \mathbb{N}$.

Pick $\psi_0 = \psi, \psi_1, \psi_2, \dots, \psi_k \in C_c^\infty(\Omega)$ s.t. $\psi_k = \phi$

and $\psi_j = 1$ on neighbourhood of $\mathrm{supp}(\psi_{j+1})$ for $j = 0, 1, \dots, k-1$.

We had $\psi_0 u \in H_\sigma$

This gives $\psi_1 u \in H_{\sigma+1} \Rightarrow \psi_2 u \in H_{\sigma+2} \Rightarrow \dots \Rightarrow \psi_k u \in H_{\sigma+k}$,

that is, $\phi u \in H_{s+m}$.

$\square$