Some examples of dynamical systems

DEF : A $\begin{Bmatrix} smooth \\ topological \end{Bmatrix}$ <u>dynamical system</u> (of simplest type!)

is a <u>pair</u> (X, T), where X is a $\begin{Bmatrix} C^\infty \text{ manifold} \\ \text{metric space} \end{Bmatrix}$ and

T is a $\begin{Bmatrix} C^\infty \\ continuous \end{Bmatrix}$ map $X \to X$.

One studies <u>iterations of T</u>, e.g. the <u>orbit</u> of a point $x \in X$:

$$x, \ Tx, \ T^2x, \ T^3x, \dots$$

1

Examples

- Translation on a torus: $X = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$,

$$T : X \to X; \quad Tx = x + \alpha \quad (\alpha \in \mathbb{R}^n \text{ fixed})$$
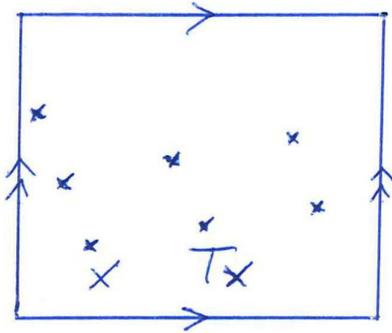
- One-sided full shift: $X = \{1, 2, \ldots, n\}^{\mathbb{N}} = \{1, \ldots, n\} \times \{1, \ldots, n\} \times \ldots$

$$T((a_1, a_2, \ldots)) = (a_2, a_3, \ldots) \quad (\text{"left shift"})$$

- Two-sided full shift: $X = \{1, 2, \ldots, n\}^{\mathbb{Z}}$, $\quad T$ left-shift.

2

- The Gauss map: $Y = [0,1] \setminus \mathbb{Q}$, $\qquad T: Y \to Y; \quad T(x) = \{x^{-1}\} = x^{-1} - \lfloor x^{-1} \rfloor$
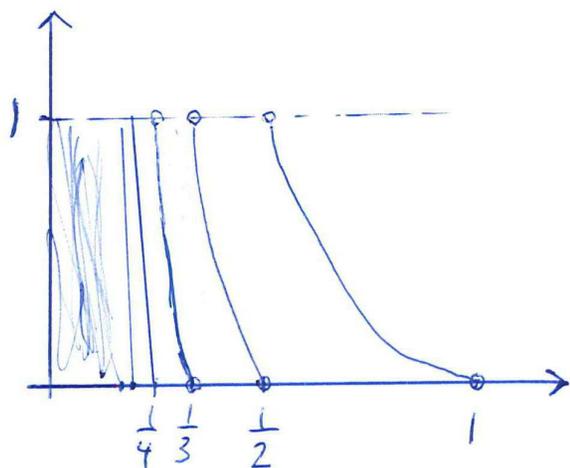
"Coding" of T: $\quad \Psi: Y \to \mathbb{N}^{\mathbb{N}}$;

$\Psi(x)$ gives the continued fraction expansion of x, that is:

$$\Psi(x) = (a_1, a_2, \ldots) \quad \overset{\text{def.}}{\Longleftrightarrow} \quad x = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}}$$

Ψ is a homeomorphism, and $\Psi \circ T \circ \Psi^{-1}$ is the left shift map on $\mathbb{N}^{\mathbb{N}}$.

B

Some details

Consider

$$\mathbb{N}^{\mathbb{N}} \xrightarrow{\ \sigma\ =\ \text{left-shift}\ } \mathbb{N}^{\mathbb{N}}$$

$$\downarrow \Psi^{-1} \qquad\qquad \downarrow \Psi^{-1}$$

$$Y \xrightarrow{\qquad\qquad\qquad} Y$$

?
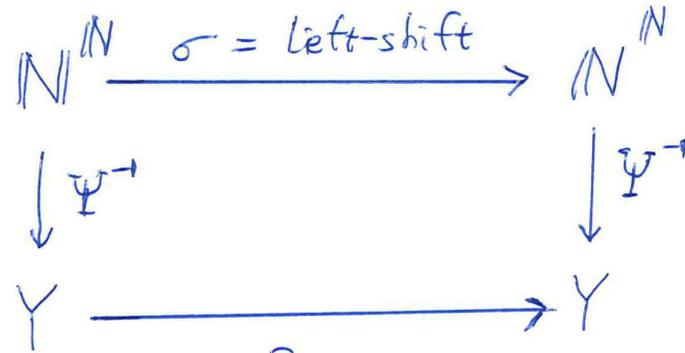
$$X = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \longmapsto \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$

$$\Downarrow$$

$$x^{-1} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}} \quad\Rightarrow\quad \{x^{-1}\} = x^{-1} - \lfloor x^{-1} \rfloor = \ \circledast \quad !$$

4

Coding of the geodesic flow 'on' the modular surface [E & W Ch. 9]

$$M = PSL(2, \mathbb{Z}) \backslash \mathbb{H}; \quad \text{consider geodesic flow on } T_1 M.$$

For a well-chosen cross-section C

we have:
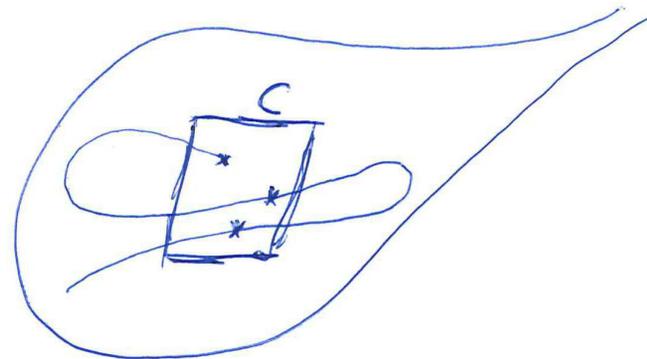
$$\left(C, \begin{array}{c} \text{return map of} \\ \text{geodesic flow} \end{array} \right) \cong \left(\overline{Y}, \overline{T} \right) \cong \left(\mathbb{N}^{\mathbb{Z}}, \sigma \right)$$

$$\overline{Y} = \left\{ (y, z) \in [0,1]^2 ; \quad 0 \leq z \leq \frac{1}{1+y} \right\}$$

$$\overline{T}(y, z) = \left(\{y^{-1}\}, \; y(1 - yz) \right)$$

Invariant measures

Important information about a dynamical system (X, T) comes from knowing one/some/all of its <u>invariant (probability) measures</u>, i.e., <u>measures μ on X</u> satisfying <u>$T_* \mu = \mu$</u> ($\Leftrightarrow$ <u>$\mu(T^{-1}A) = \mu(A)$</u> <u>for all measurable $A \subset X$</u>)

"Purified" situation:

<u>DEF:</u> If <u>$(X, \mathcal{M}, \mu)$ is a measure space,</u> a <u>measurable map $T: X \to X$</u> is called a <u>measure-preserving transformation</u> if <u>$T_* \mu = \mu$</u>. (Also, μ is then said to be a <u>$(T-)$invariant measure</u>.)

We will only consider the case when μ is a <u>probability</u> measure.

Examples

__Translation on torus__: $X = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, $\quad T(x) = x + \alpha$ $\quad (\alpha \in \mathbb{R}^n$ fixed$)$.

Then $\underline{\mu = \text{Lebesgue}}$ is an invariant probability measure.

__Bernoulli shift__: $X = \{1, 2, ..., n\}^{\mathbb{N}}$ __or__ $\{1, 2, ..., n\}^{\mathbb{Z}}$, and $\underline{\underline{T = \text{left-shift}}}$.

Then for $\underline{\text{any}}$ probability measure μ_1 on $\{1, 2, ..., n\}$, $\quad \underline{\mu_1 \times \mu_1 \times ...}$

is a T-invariant probability measure on X.

This measure preserving transformation (X, μ, T) is called a $\underline{\underline{\text{Bernoulli shift}}}$.

(Cf. E&W, Ex. 2.8, 2.9.) $\qquad$ (Also: Folland Thm 7.28.)

7

The Gauss map : $\quad Y = [0,1] \setminus \mathbb{Q}, \qquad T(x) = \{x^{-1}\}$

The Gauss measure on Y is $\quad d\mu(x) = \frac{1}{\log 2} \cdot \frac{1}{1+x} \cdot dx.$

It is a T-invariant probability measure on Y.   (E&W Lemma 3.5)

Discussion/proof: Write $\quad d\mu(x) = f(x) \cdot dx, \qquad f \in L^1([0,1], dx).$

By Folland Thm. 1.16, suffices to check:

$$\underbrace{\mu(T^{-1}(0,s)) = \mu((0,s))}_{= \overset{\infty}{\underset{n=1}{\boxed{\bigcup}}} \left(\frac{1}{s+n}, \frac{1}{n} \right)}, \qquad \forall s \in (0,1)$$

$$\Longleftrightarrow \quad \sum_{n=1}^{\infty} \int_{\frac{1}{s+n}}^{\frac{1}{n}} f(x)\, dx = \int_{0}^{s} f(x)\, dx, \qquad \forall s \in (0,1)$$

formally...

$$\sum_{n=1}^{\infty} f\left(\frac{1}{s+n}\right) \frac{1}{(s+n)^2} = f(s) \qquad \qquad (\text{transfer operator})$$

8

Check that $f(x) = \dfrac{C}{1+x}$ satisfies the above:

$$L.H. = C \cdot \sum_{n=1}^{\infty} \frac{s+n}{s+n+1} \cdot \frac{1}{(s+n)^2} = C \sum_{n=1}^{\infty} \left(\frac{1}{s+n} - \frac{1}{s+n+1} \right) = \frac{C}{s+1} = R.H.$$

OK!

Now recall

$$\mathbb{N}^{\mathbb{N}} \xrightarrow{\;\sigma\; = \text{ left-shift }\;} \mathbb{N}^{\mathbb{N}}$$

with vertical maps Ψ upward on the left, Ψ upward on the right, and

$$Y \xrightarrow{\;\;T\;\;} Y$$

It follows that $\underline{\Psi_* \mu}$ is a σ-invariant measure on $\mathbb{N}^{\mathbb{N}}$.

$\underline{\underline{\text{Claim:}}}$ $\Psi_* \mu$ is $\underline{\underline{not}} = \displaystyle\prod_{j=1}^{\infty} \mu_1$, with μ_1 a probability measure on $\mathbb{N}$.

proof: (See E&W Exc. 3.24)

$$(\Psi_* \mu)\{a_1 = 1\} = \int_{(\frac{1}{2}, 1)} d\mu(x) = \frac{\log 2 - \log \frac{3}{2}}{\log 2} = \frac{2\log 2 - \log 3}{\log 2} = 0,415\ldots$$

Thus $\left((\Psi_* \mu)\{a_1 = 1\} \right)^2 = 0,172\ldots$

But $(\Psi_* \mu)\{a_1 = a_2 = 1\} = \displaystyle\int_{(\frac{1}{2}, \frac{2}{3})} d\mu(x) = \frac{\log\left(\frac{5}{3} / \frac{3}{2}\right)}{\log 2} = \frac{\log\left(\frac{10}{9}\right)}{\log 2} = 0,152\ldots$

Not equal! □

10

DEF (E & W, 2.13): A T-invariant probability measure μ on (X, M) is called _ergodic_ if $[\forall A \in M: T^{-1}A = A \Rightarrow [\mu(A) = 0$ or $1]]$.

Note: ergodic $\approx$ "indecomposable" in an appropriate sense!

Ergodic decomposition: Every T-invariant measure can be expressed as a _superposition_ of _ergodic_ T-invariant measures. (E&W Thm 4.8, Thm 6.2.)

Ex: _Translation on torus_: $X = \mathbb{T}^n$, $T(x) = x + \alpha$ ($\alpha \in \mathbb{R}^n$ fixed). If $\alpha = (\alpha_1, \dots, \alpha_n)$ with $1, \alpha_1, \alpha_2, \dots, \alpha_n$ linearly independent over $\mathbb{Q}$, then the T-invariant measure $\mu = $ Lebesgue is _ergodic_ (E&W, Cor. 4.15.)

Ex: Any _Bernoulli shift_ is _ergodic_. (E&W, Prop. 2.15)

Ex: The <u>Gauss measure</u> $\left(d\mu(x) = \dfrac{1}{\log 2} \cdot \dfrac{dx}{1+x} \right)$ for the <u>Gauss map</u>

$T(x) = \{x^{-1}\}$ on $Y = [0,1] \setminus \mathbb{Q}$, is <u>ergodic</u>. $\left(E\&W, \text{ Thm } 3.7 \right)$