#3. Measure & integration theory

Let $(X, \mathcal{M}, \mu)$ be a measure space, and $L^+ = \{f : X \to [0,\infty] : f\ \mathcal{M}\text{-m-ble}\}$

Thm 2.14. Monotone Convergence Theorem:
If $f_1, f_2, \ldots \in L^+$, $f_1 \le f_2 \le \cdots$, then $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \left(\lim_{n\to\infty} f_n\right) d\mu$

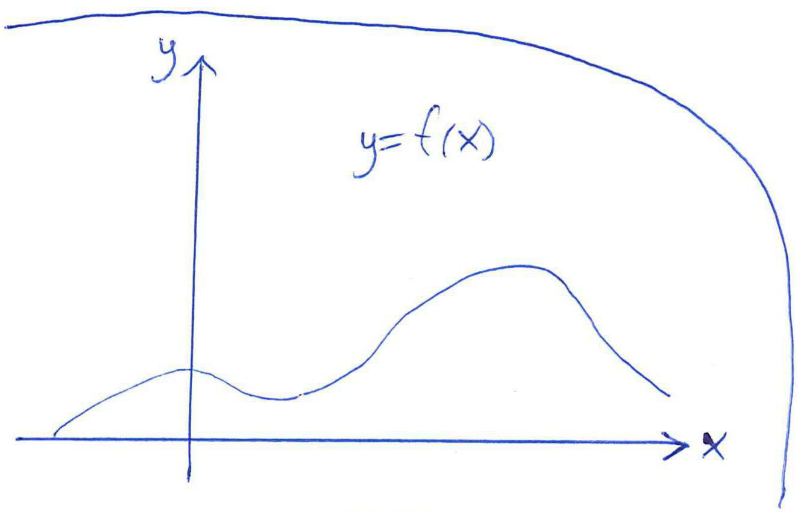
Thm 2.15: For any $f_1, f_2, \ldots \in L^+$, $\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$

proof:
First prove just $\int_X (f_1 + f_2) = \int_X f_1 + \int_X f_2$:

$\geq$: "trivial from def."

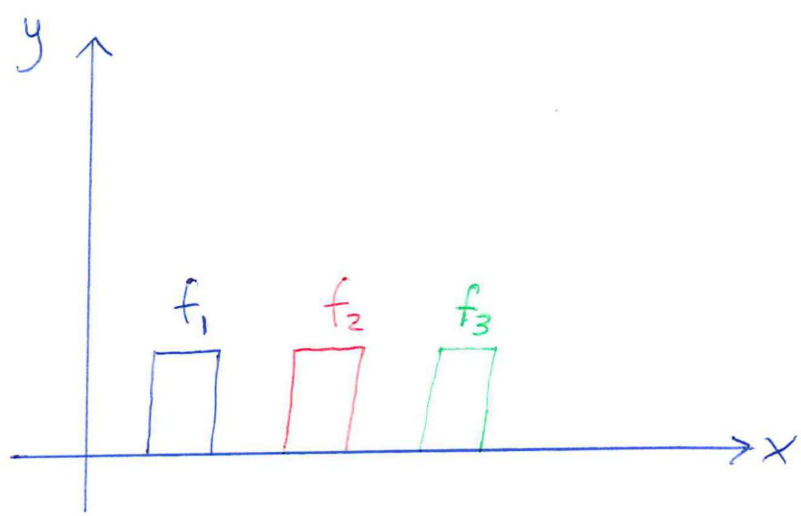
$\leq$: Start by choosing an increasing sequence of simple functions $\phi_1 \le \phi_2 \le \phi_3 \le \cdots$ tending to f_1 pointwise. Similarly for f_2.

Can do, by Thm 2.10

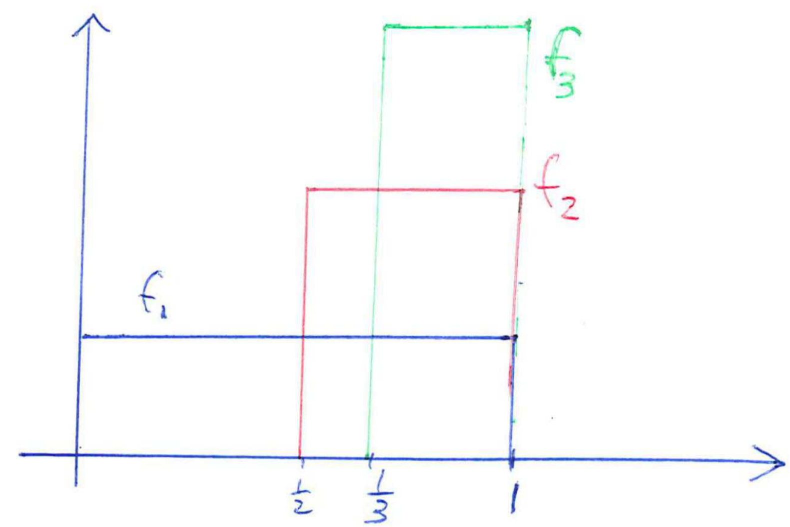


Fatou's Lemma: If $f_1, f_2, \ldots \in L^+$ then $\underline{\underline{\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu}}$.

Note standard ("counter")examples where $\int_X \left(\lim_{n\to\infty} f_n\right) d\mu < \lim_{n\to\infty} \int_X f_n \, d\mu$:



$$\left(\underline{\underline{f_n = \chi_{[n, n+\frac{1}{2}]}}} \quad \text{(eg)} \atop \mu = m \quad \text{(Lebesgue measure)} \right)$$



$\underline{\underline{f_n = n \cdot \chi_{[1-\frac{1}{n}, 1]}}}$, $\mu = m$ (Lebesgue)

Integration of complex functions (Folland Ch. 2.3)

$L^1 := \{ f : X \to \mathbb{C} : f \text{ m'ble and } \int_X |f| \, d\mu < \infty \}$

or "$L^1(\mu)$" or "$L^1(X,\mu)$"

L^1 is a $\mathbb{C}$-vector space. Identify any $f, g \in L^1$ with $f = g$ μ-a.e.

Then L^1 is a normed $\mathbb{C}$-vector space, with norm $\|f\| := \int_X |f| \, d\mu$

For $f \in L^1$, define $\int_X f \, d\mu$ (p.53)

Thm 2.24, Lebesgue Dominated Convergence Theorem:
If $f_1, f_2, \ldots \in L^1$ with (a) $f_n \to f$ μ-a.e. and (b) $\exists$ nonnegative $g \in L^1$ with $|f_n| \le g$ μ-a.e. $(\forall n)$, then $f \in L^1$ and $\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu$

Product measures (Folland Ch. 2.5)

Def: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.
A (measurable) rectangle is any set of the form $A \times B$ where $A \in \mathcal{M}$, $B \in \mathcal{N}$.

The product σ-algebra, $\mathcal{M} \otimes \mathcal{N}$, is defined as the σ-algebra on $X \times Y$ generated by all rectangles.

Thm: If μ and ν are σ-finite then there is a unique measure $\mu \times \nu$ on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ such that $\mu \times \nu (A \times B) = \mu(A)\, \nu(B)$, $\forall A \in \mathcal{M}, B \in \mathcal{N}$.
This measure $\mu \times \nu$ is σ-finite.

Thm 2.37, Fubini – Tonelli: Assume μ, ν are σ-finite.

a) For any $f \in L^+(X \times Y, \mu \times \nu)$,

$$\int_{X \times Y} f(x,y)\, d(\mu \times \nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y).$$

b) Same for $L^1(\mu \times \nu)$.

Example: $$\int_0^\infty \int_0^\infty \int_{xy}^\infty \sum_{n>z} \sum_{m|n} f(n,m,x,y,z)\, dz\, dy\, dx \qquad (*)$$

- can change order to $\sum_m \sum_n \int \int \int f(n,m,x,y,z)\, dy\, dx\, dz$?

Answer: We have

$$(*) = \sum_{m=1}^\infty \sum_{n \in m\mathbb{N}} \int_0^n \int_0^\infty \int_0^{z/x} f(n,m,x,y,z)\, dy\, dx\, dz \qquad (**)$$

provided that f is m'ble and either $(*)$ or $(**)$ with $f \leftarrow |f|$ is finite!

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proof: Apply Fubini to $m \times m \times m \times \text{count} \times \text{count}$ on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$, and the function

$$(x,y,z,n,m) \longmapsto I\big(z > xy \text{ and } n > z \text{ and } m|n\big) \cdot f(n,m,x,y,z).$$

□

The Lebesgue measure on $\mathbb{R}^n$ (Folland Ch. 2.6)

"**Def**": Lebesgue measure m^n (or m) is the completion of the unique measure on $\mathcal{B}_{\mathbb{R}^n}$ which is invariant under all translations and has $m([0,1]^n) = 1$.

Domain of m: $\mathcal{L}^n = \mathcal{L}_{\mathbb{R}^n}$.

Fact: $m\left(\prod_{j=1}^n [a_j, b_j]\right) = \prod_{j=1}^n (b_j - a_j)$ if $a_j \leq b_j$, $j = 1, 2, \dots, n$.

Fact: $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m^n) = (\mathbb{R} \times \cdots \times \mathbb{R}, \mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}, m \times \cdots \times m)$

$(\mathbb{R}^n, \mathcal{L}^n, m^n) = [\text{completion of } \boxtimes] = [\text{completion of } (\mathbb{R} \times \cdots \times \mathbb{R}, \mathcal{L} \otimes \cdots \otimes \mathcal{L}, m \times \cdots \times m)]$

Thm 2.44: Suppose $T \in GL(n, \mathbb{R})$. Then $T : \mathbb{R}^n \to \mathbb{R}^n$ is Borel measurable and $(\mathcal{L}^n, \mathcal{L}^n)$-measurable and $m \circ T^{-1} = |\det T|^{-1} \cdot m$

This is Folland's Thm 2.44(b), but for T^{-1}!

For general measurable map $T : X \to Y$,
where $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, and μ is a measure on X:

push-forward of μ: $T_* \mu := \mu \circ T^{-1}$, a measure on Y.

$\forall f \in L^1(Y, T_*\mu)$: $\int_X (f \circ T)\, d\mu = \int_Y f \, d(T_* \mu)$

Thm 2.47: For any C^1 diffeomorphism $\Phi : \Omega \to \mathbb{R}^n$, with Ω an open subset of $\mathbb{R}^n$,

$$\int_{\Phi(\Omega)} f \, dm = \int_\Omega (f \circ \Phi)(x) \cdot |\det D_x \Phi| \, dm(x), \qquad \forall f \in L^1(\Phi(\Omega))$$

On $\mathbb{R}$: Comparison with the Riemann integral

Thm 2.28: Let $f:[a,b]\to\mathbb{R}$ be a bounded function.

a) $\underline{f \text{ Riemann integrable}} \Rightarrow \underline{f\in L^1([a,b],m)}$ and $\underbrace{\int_a^b f(x)\,dx}_{\text{Riemann integral}} = \int_{[a,b]} f\,dm$

b) f is Riemann integrable iff

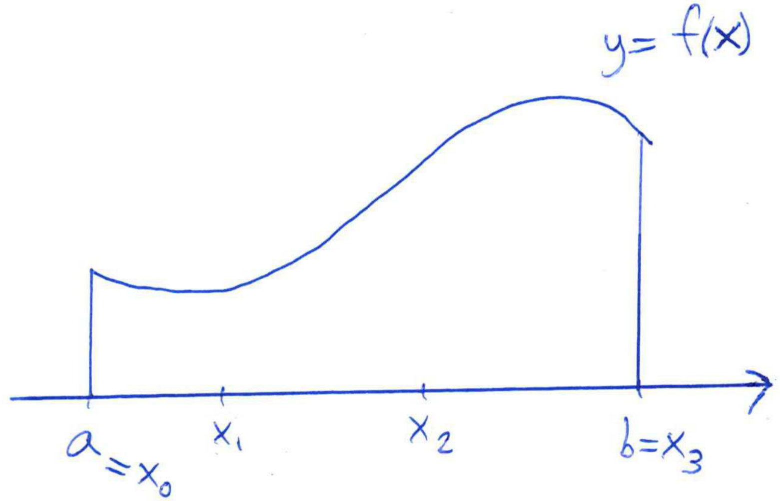
$$m(\{x\in[a,b] : f \text{ discontinuous at } x\}) = 0.$$

proof sketch for (a): Take partitions $P_1\subset P_2\subset\cdots$ of $[a,b]$ with $\text{mesh}(P_k)\to 0$.

Then for any choice of "taggings" $\Xi_1,\Xi_2,\dots,$ $S(P_k,\Xi_k)\to\int_a^b f(x)\,dx$.

This implies $\boxed{\int_{[a,b]} g_{P_k}\,dm \longrightarrow \int_a^b f(x)\,dx}$ and $\boxed{\int_{[a,b]} G_{P_k}\,dm \longrightarrow \int_a^b f(x)\,dx,}$

where, if $P_k=\{x_n\}_{n=0}^N$: $\forall n : \forall x\in[x_{n-1},x_n]$: $\begin{cases} g_{P_k}(x) := \inf_{[x_{n-1},x_n]} f \\ G_{P_k}(x) := \sup_{[x_{n-1},x_n]} f \end{cases}$



Set $g(x) = \lim_{k\to\infty} g_{P_k}(x)$, $G(x) = \lim_{k\to\infty} G_{P_k}(x)$.

Then $g \leqslant f \leqslant G$, and g, G are Lebesgue measurable!

Dom. Conv. Thm. $\Rightarrow \int_{[a,b]} g\,dm = \int_{[a,b]} G\,dm = \int_a^b f(x)\,dx$ $\circledast$

Hence $g(x) = G(x)$ for m-a.e. x.

(Prop. 2.16)

$\therefore$ f is Lebesgue m'ble, and $\int_{[a,b]} f\,dm = \circledast$

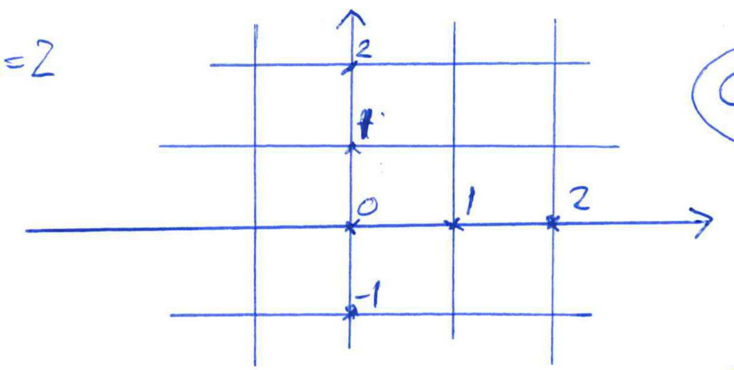
On the other hand: $\boxed{\text{"Generalized Riemann integrable"} \not\Rightarrow \text{Lebesgue integrable}}$

Ex: $\int_1^\infty \frac{\sin x}{x}\,dx$

Jordan content

Defs: For $k \in \mathbb{Z}$, $\mathcal{Q}_k :=$ [the set of closed cubes with side length 2^{-k}, vertices $\in (2^{-k}\mathbb{Z})^n$]

Ex $n=2$



$\mathcal{Q}_0$

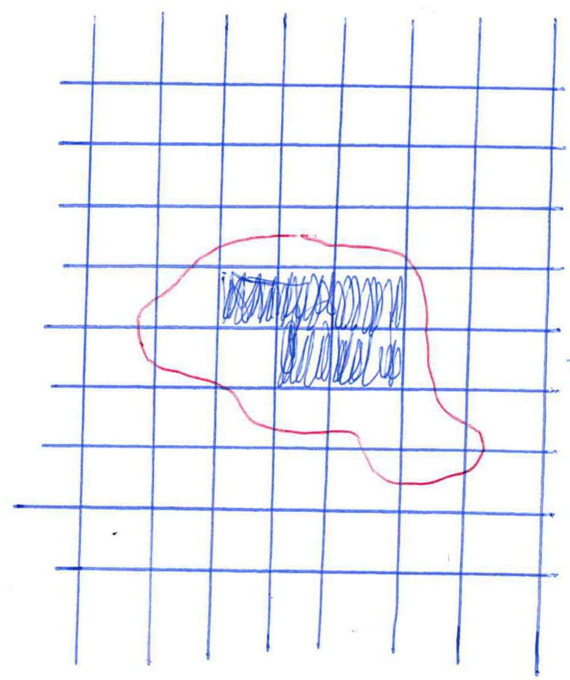
For $E \subset \mathbb{R}^n$, $k \in \mathbb{Z}$:

$$\underline{A}(E,k) = \bigcup \{ Q \in \mathcal{Q}_k ; \ Q \subset E \}$$

$$\overline{A}(E,k) = \bigcup \{ Q \in \mathcal{Q}_k : \ Q \cap E \neq \emptyset \}$$

$$\underline{\kappa}(E) = \lim_{k \to \infty} m(\underline{A}(E,k))$$

$$\overline{\kappa}(E) = \lim_{k \to \infty} m(\overline{A}(E,k))$$



$\underline{A}(E,0)$

If $\underline{\kappa}(E) = \overline{\kappa}(E)$ then $=:$ Jordan content of E.

<u>Theorem:</u> Given any bounded set $E \subset \mathbb{R}^n$, the following are equivalent:

a) $m(\partial E) = 0$

b) ∂E has Jordan content 0.

c) E has Jordan content.

d) $\underbrace{m(\partial_\varepsilon E)}$ $\to 0$ as $\varepsilon \to 0$.

$$\partial_\varepsilon E := \{x \in \mathbb{R}^n : [\exists\, y \in \partial E : |x-y| < \varepsilon]\}$$

<u>proof, outline:</u>

(a) $\Rightarrow$ (b): Use ∂E compact; follow Folland p. 73 (top).

(b) $\Rightarrow$ (c): "See picture".

(b) $\Rightarrow$ (d): Use $\partial_\varepsilon E \subset N_\varepsilon(\bar{A}(\partial E, k))$ $(\forall k \in \mathbb{Z})$, with k large.

"ε - neighborhood of"

(d) $\Rightarrow$ (a): Since "m continuous from above".

$\square$

Ex: If $E \subset \mathbb{R}^n$ is bounded and $m(\partial E) = 0$, then

$$(*) \quad \frac{\#(\mathbb{Z}^n \cap TE)}{T^n} \to m(E) \quad \text{as } T \to \infty.$$

Proof outline: E has Jordan content, hence one can reduce to the case E = a cube!

$\square$

NOTE: $\exists$ bounded open sets $E \subset \mathbb{R}^n$ for which $(*)$ fails!