#9. The J-Bessel function (cf. Lecture Notes, Sec. 8)

DEF: The J-Bessel function is defined by the following formula, for $\nu \in \mathbb{C}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m}$$

The function $f(z) = J_\nu(z)$ solves the ODE

$$f''(z) + \frac{1}{z} f'(z) + \left(1 - \frac{\nu^2}{z^2}\right) f(z) = 0.$$

(Use Frobenius method; $f(z) = z^\nu \sum_{n=0}^{\infty} a_n z^n$, $a_0 \neq 0$.)

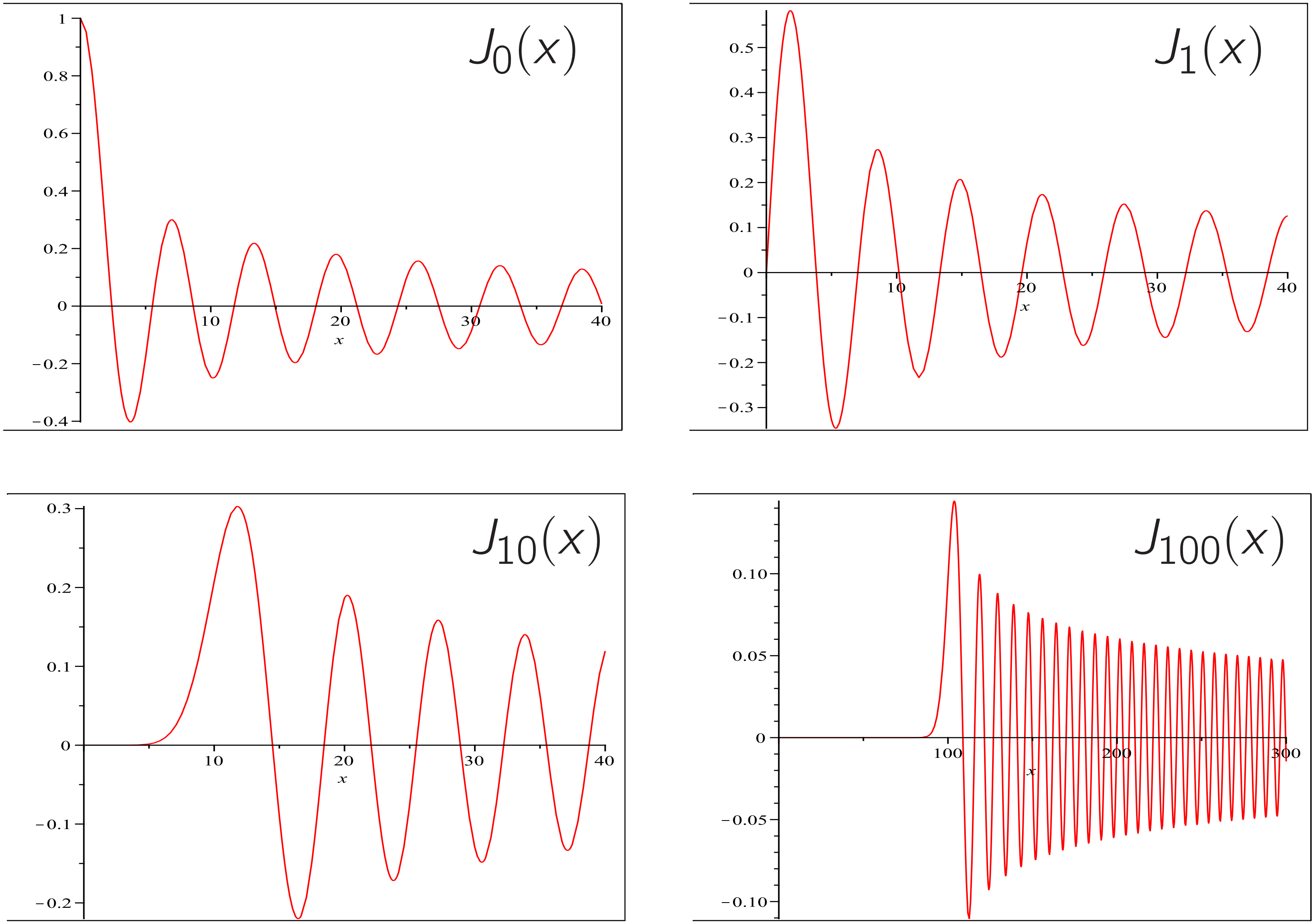
If $\nu \notin \mathbb{Z}$: $\{J_\nu(z), J_{-\nu}(z)\}$ is a fundamental system of solutions.

If $\nu \in \mathbb{Z}$: $J_{-\nu}(z) \equiv (-1)^\nu J_\nu(z)$

Recurrence relations:

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z),$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2 J_\nu'(z).$$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$J_{100}(x)$
0.10
0.05
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−0.05
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x

Alternative formula:

For $\mathrm{Re}(\nu) > -\frac{1}{2}$: $$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{izt} (1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

Proof outline: Plug in $e^{izt} = \sum_{n=0}^{\infty} \frac{(izt)^n}{n!}$ in the right hand side.

Lebesgue Dominated Conv. Thm.

$$\Rightarrow \int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt = \sum_{n=0}^{\infty} \int_{-1}^{1} \frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\,dt$$

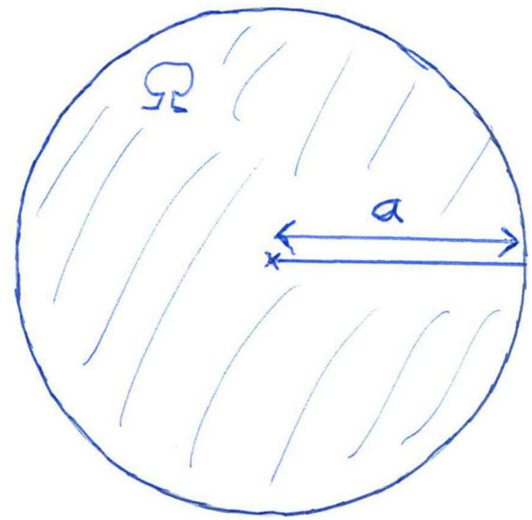
For $n=2m$: $$\int_{-1}^{1} t^{2m}(1-t^2)^{\nu-\frac{1}{2}}\,dt = 2\int_0^1 t^{2m}(1-t^2)^{\nu-\frac{1}{2}} = 2\int_0^1 u^m(1-u)^{\nu-\frac{1}{2}}\frac{du}{2\sqrt{u}}$$

$$= \int_0^1 u^{m-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}\,du = \frac{\Gamma(m+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{\Gamma(m+\nu+1)}$$

Hence get the result.

Application: Dirichlet eigenfunctions in a disk

Let $\Omega = \{x \in \mathbb{R}^2 : |x| < a\}$



We are seeking solutions $\lambda \geq 0$

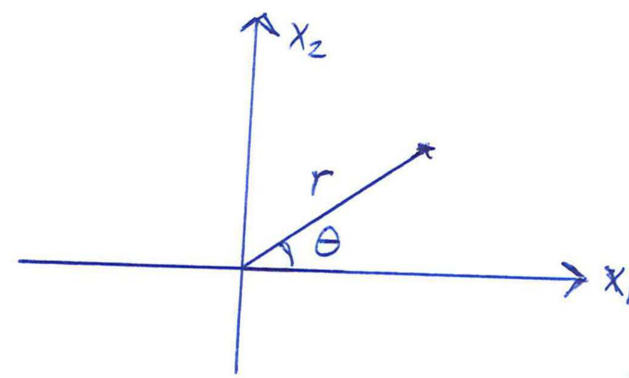
and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ to

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

Recall: $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$

In polar coordinates;

$(x_1, x_2) = r \cdot (\cos\theta, \sin\theta)$



Ansatz: $u(r, \theta) = R(r) \cdot \Phi(\theta)$

Then need: $$\begin{cases} \left(R''(r) + \frac{1}{r}R'(r) + \lambda R(r)\right)\cdot \Phi(\theta) = -\frac{1}{r^2} R(r)\cdot \Phi''(\theta) & 0<r<a \\ R(0+) \text{ exists} \\ R(a-) = 0 \\ \Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi) \end{cases}$$

If non-vanishing solution, then $\exists \mu \in \mathbb{R}$ s.t. $$\begin{cases} R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = \frac{\mu}{r^2} R(r) \quad \text{on } (0,a) \\ \Phi''(\theta) = -\mu \cdot \Phi(\theta) \quad \text{on } \mathbb{R}/2\pi\mathbb{Z} \end{cases}$$

The conditions on Φ imply:

> $\mu = n^2$ for some $n \in \mathbb{Z}_{\geq 0}$
> and $\Phi(\theta) = A\cos(n\theta) + B\sin(n\theta)$
> for some $A, B \in \mathbb{R}$.

Setting $\Psi(s) := R\left(\frac{s}{\sqrt{\lambda}}\right)$, the ODE for R becomes: $\Psi''(s) + \frac{1}{s}\Psi'(s) + \left(1 - \frac{n^2}{s^2}\right)\Psi(s) = 0$

With [$\Psi(0)$ exists], this implies $\Psi(s) = [\text{const}] \cdot J_n(s)$

Finally we also require $\Psi(\sqrt{\lambda}\, a) = 0$.

⇝ The 'general' solution to the Dirichlet eigenvalue problem in Ω is given by

$$\lambda = \left(\frac{j_{n,m}}{a}\right)^2, \quad u_{n,m}(r,\theta) = J_n\left(\frac{j_{n,m}}{a} r\right) \cdot \left(A_n \cos n\theta + B_n \sin n\theta\right)$$

$$\left((n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}\right)$$

where $0 < j_{n,1} < j_{n,2} < \dots$ are the zeros of $J_n(x)$ for $x > 0$.

Application: The Fourier transform of radial functions

Let σ be the standard surface measure on S_1^{n-1} $(\subset \mathbb{R}^n)$.

Then $$\hat{\sigma}(\xi) = \int_{S_1^{n-1}} e^{-2\pi i \xi \cdot \omega} \, d\sigma(\omega) = \int_{S_1^{n-1}} e^{-2\pi i |\xi| \cdot \omega_1} \, d\sigma(\omega).$$

↑ rotational symmetry

Let $P: \mathbb{R}^n \to \mathbb{R}$; $P(\omega_1, \dots, \omega_n) = \omega_1$. Then we get:

$$= \int_{-1}^{1} e^{-2\pi i |\xi| \cdot \omega_1} \, d(P_* \sigma)(\omega_1) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \int_{-1}^{1} e^{-2\pi i |\xi| \cdot \omega_1} (1-\omega_1^2)^{\frac{n-3}{2}} \, d\omega_1$$

$$= 2\pi \cdot |\xi|^{1-\frac{n}{2}} \cdot J_{\frac{n}{2}-1}(2\pi |\xi|)$$

Thus: $$\boxed{\hat{\sigma}(\xi) = 2\pi \cdot |\xi|^{1-\frac{n}{2}} \cdot J_{\frac{n}{2}-1}(2\pi |\xi|)}$$

Hence for any radial function $F \in L^1(\mathbb{R}^n)$, say $F(\xi) = f(|\xi|)$:

$$\hat{F}(\xi) = \int_{\mathbb{R}^n} F(x) e^{-2\pi i \xi \cdot x} dx = \int_0^\infty \int_{S_1^{n-1}} f(r) e^{-2\pi i \xi \cdot r\omega} r^{n-1} d\sigma(\omega) dr$$

$$= \int_0^\infty f(r) \left(\int_{S_1^{n-1}} e^{-2\pi i \xi \cdot r\omega} d\sigma(\omega) \right) \cdot r^{n-1} dr$$

$$= \int_0^\infty f(r) \hat{\sigma}(r\xi) \cdot r^{n-1} dr$$

$$= 2\pi |\xi|^{1-\frac{n}{2}} \cdot \int_0^\infty f(r) \cdot r^{\frac{n}{2}} \cdot J_{\frac{n}{2}-1}(2\pi r|\xi|) dr =: \tilde{f}(|\xi|)$$

Inversion – SAME formula: $f(r) = 2\pi r^{1-\frac{n}{2}} \int_0^\infty \tilde{f}(y) y^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi y r) dy.$

Asymptotic formula for $J_\nu(x)$ as $x \to \infty$?

First order asymptotic:

$$J_\nu(x) = \underbrace{\sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right.}_{:= F_\nu^{(1)}(x)} \left.+ O(x^{-1})\right) \qquad \text{as } x \to \infty \quad (\nu \ \textit{fixed}).$$

More precise ("second order") asymptotic formula:

$$J_\nu(x) = \underbrace{\sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cdot \frac{\frac{1}{2}\nu^2 - \frac{1}{8}}{x}\right.}_{:= F_\nu^{(2)}(x)} \left.+ O(x^{-2})\right)$$

as $x \to \infty \quad (\nu \ \textit{fixed})$.

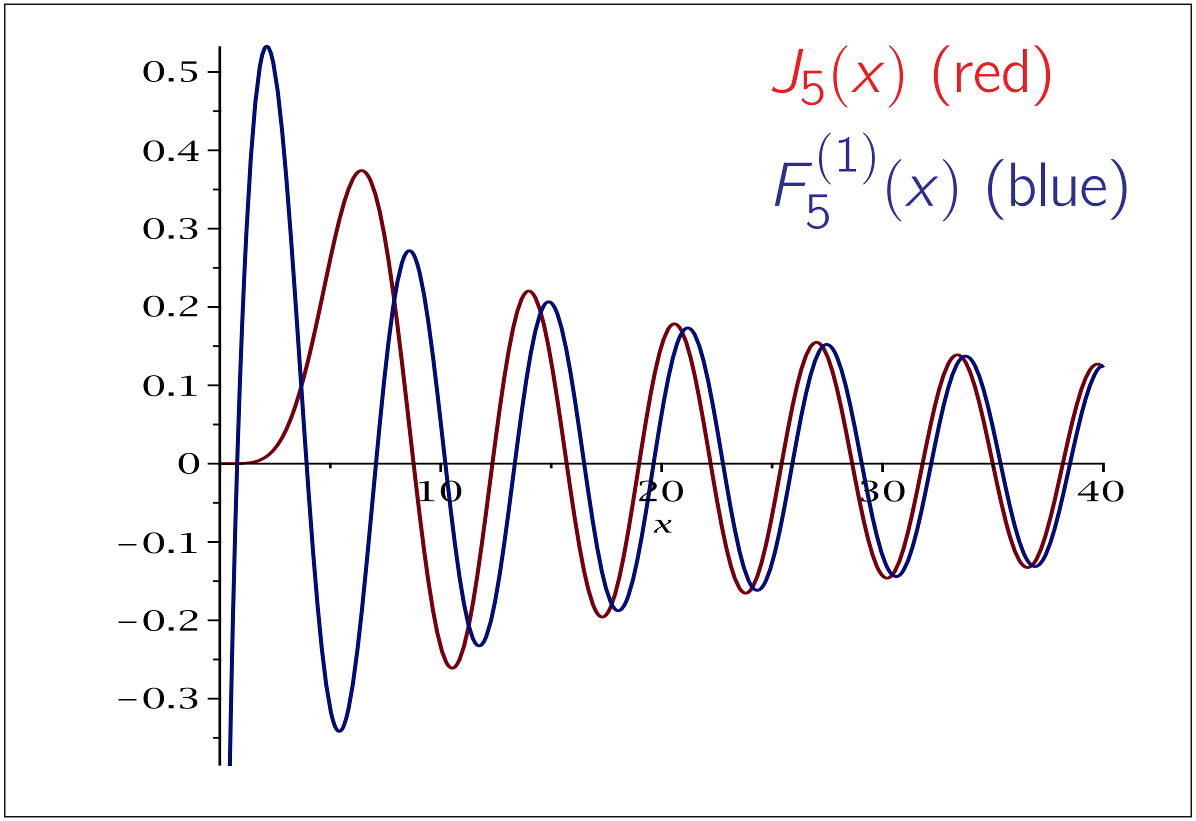
Asymptotic expansion: For any *fixed* $N \in \mathbb{Z}^+$ and $\nu \in \mathrm{C}$,

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\Bigg(\cos\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) \cdot \sum_{0 \leqslant k < N/2} (-1)^k A_{\nu,2k} x^{-2k}$$
$$- \sin\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) \cdot \sum_{0 \leqslant k < (N-1)/2} (-1)^k A_{\nu,2k+1} x^{-2k-1} + O(x^{-N})\Bigg)$$

as $x \to \infty$ $\quad$ (ν *fixed*), with

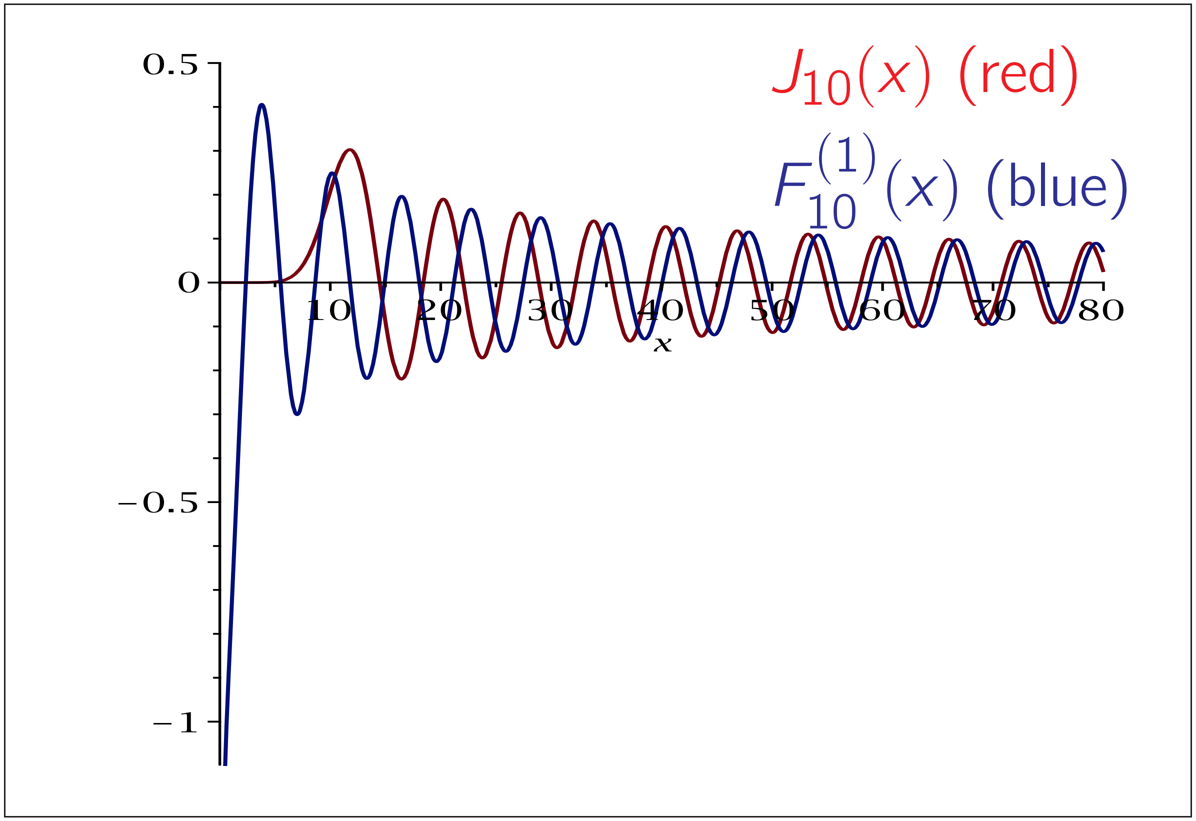
$$A_{\nu,n} := \frac{\prod_{j=-n}^{n-1} (\nu + \frac{1}{2} + j)}{2^n \cdot n!}.$$

$\nu = 5$



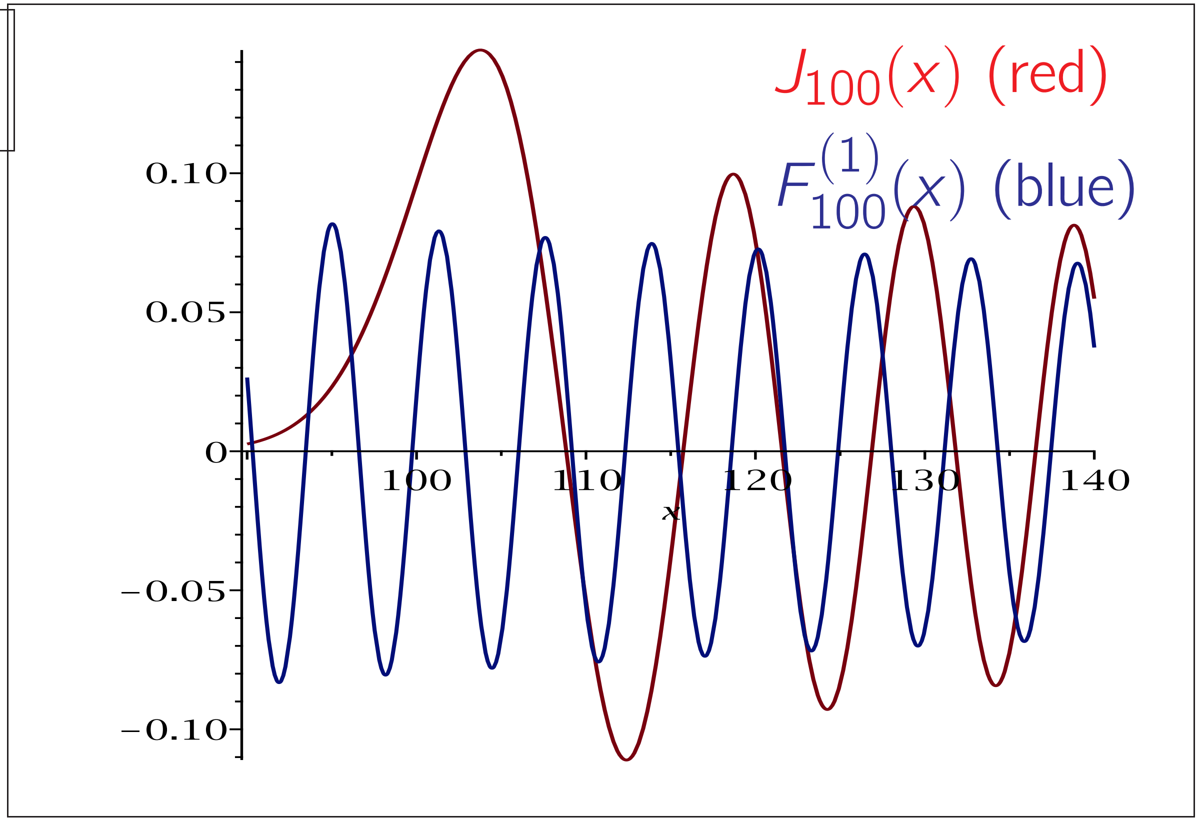
x	$J_5(x)$	$J_5(x) - F_5^{(1)}(x)$	$J_5(x) - F_5^{(2)}(x)$
1	2.497577e-04	-1.70e-01	-9.82e+00
10	-2.340615e-01	-2.87e-01	1.87e-02
10^2	-7.419574e-02	3.02e-03	5.39e-04
10^3	5.025407e-03	3.06e-04	-3.38e-07
10^4	3.638933e-03	-8.78e-06	-2.57e-09
10^5	1.846551e-03	-2.13e-07	-1.30e-11

$\nu = 10$

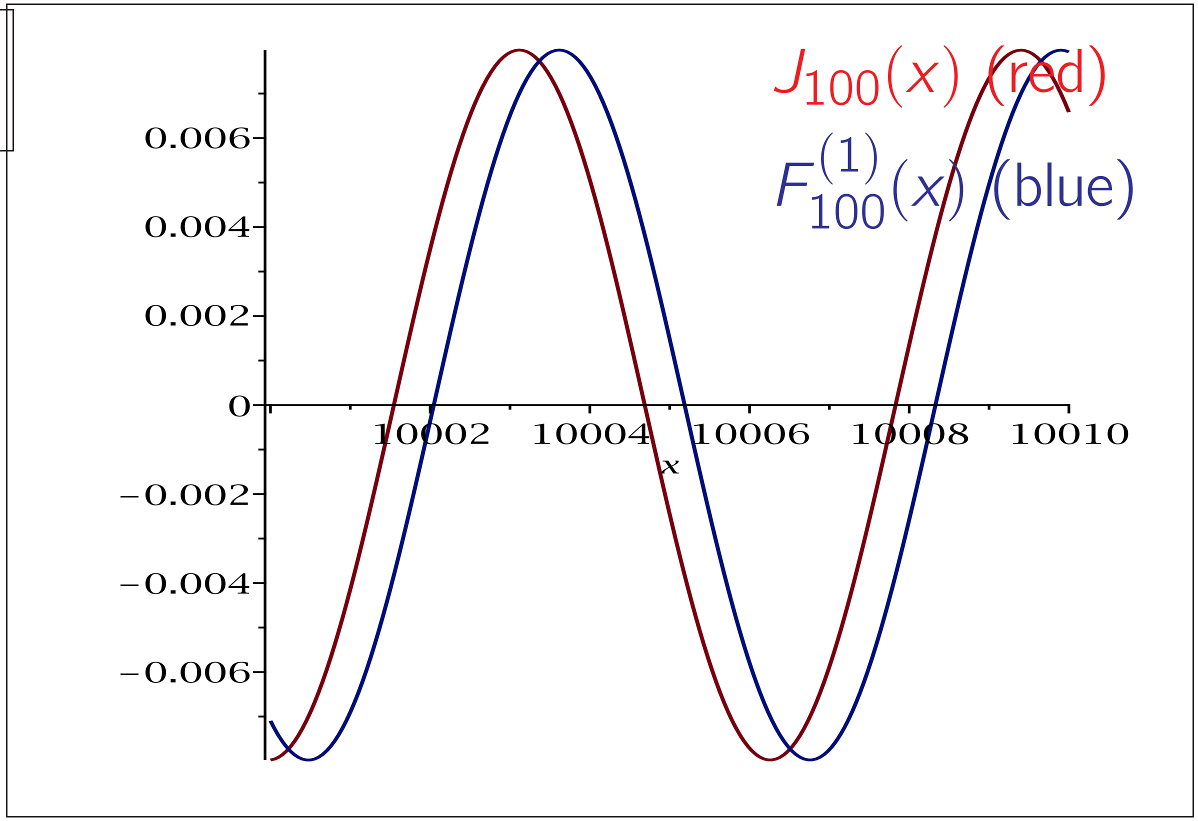


x	$J_{10}(x)$	$J_{10}(x) - F^{(1)}_{10}(x)$	$J_{10}(x) - F^{(2)}_{10}(x)$
1	2.630615e-10	7.80e-01	-7.69e+00
10	2.074861e-01	-3.93e-02	-3.02e-01
10^2	-5.473218e-02	-3.46e-02	3.86e-03
10^3	-2.452062e-02	2.65e-04	3.01e-05
10^4	7.114312e-03	1.81e-05	-8.66e-08
10^5	1.720124e-03	9.21e-07	-2.10e-10

$\nu = 100$



$\nu = 100$



x	$J_{100}(x)$	$J_{100}(x) - F^{(1)}_{100}(x)$	$J_{100}(x) - F^{(2)}_{100}(x)$
1	8.431829e-189	-7.80e-01	8.49e+02
10	6.597316e-89	2.47e-01	2.66e+01
10^2	9.636667e-02	7.63e-02	-3.78e+00
10^3	1.167614e-02	-1.31e-02	1.05e-02
10^4	-7.976516e-03	-8.80e-04	9.44e-04
10^5	-1.809353e-03	-9.01e-05	2.19e-06
10^6	3.346687e-04	3.63e-06	-4.15e-09
10^7	-8.695579e-05	-1.18e-07	1.09e-11

Uniform asymptotics for $J_\nu(x)$, for ν large (Olver, 1954)

$$J_\nu(\nu t) = \nu^{-\frac{1}{3}}\left(\frac{4\zeta}{1-t^2}\right)^{1/4}\left(\mathrm{Ai}(\xi) + O\left(\nu^{-1}\frac{e^{-\frac{2}{3}(\xi^+)^{3/2}}}{(1+|\xi|)^{1/4}}\right)\right) \qquad (\nu \geqslant 1,\ t > 0).$$

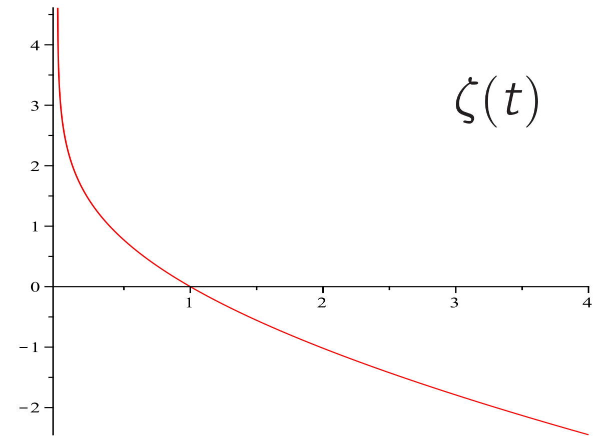
— with an *absolute* implied constant.

Here:

$\xi = \nu^{2/3}\zeta$ and $\xi^+ = \max(\xi, 0)$;

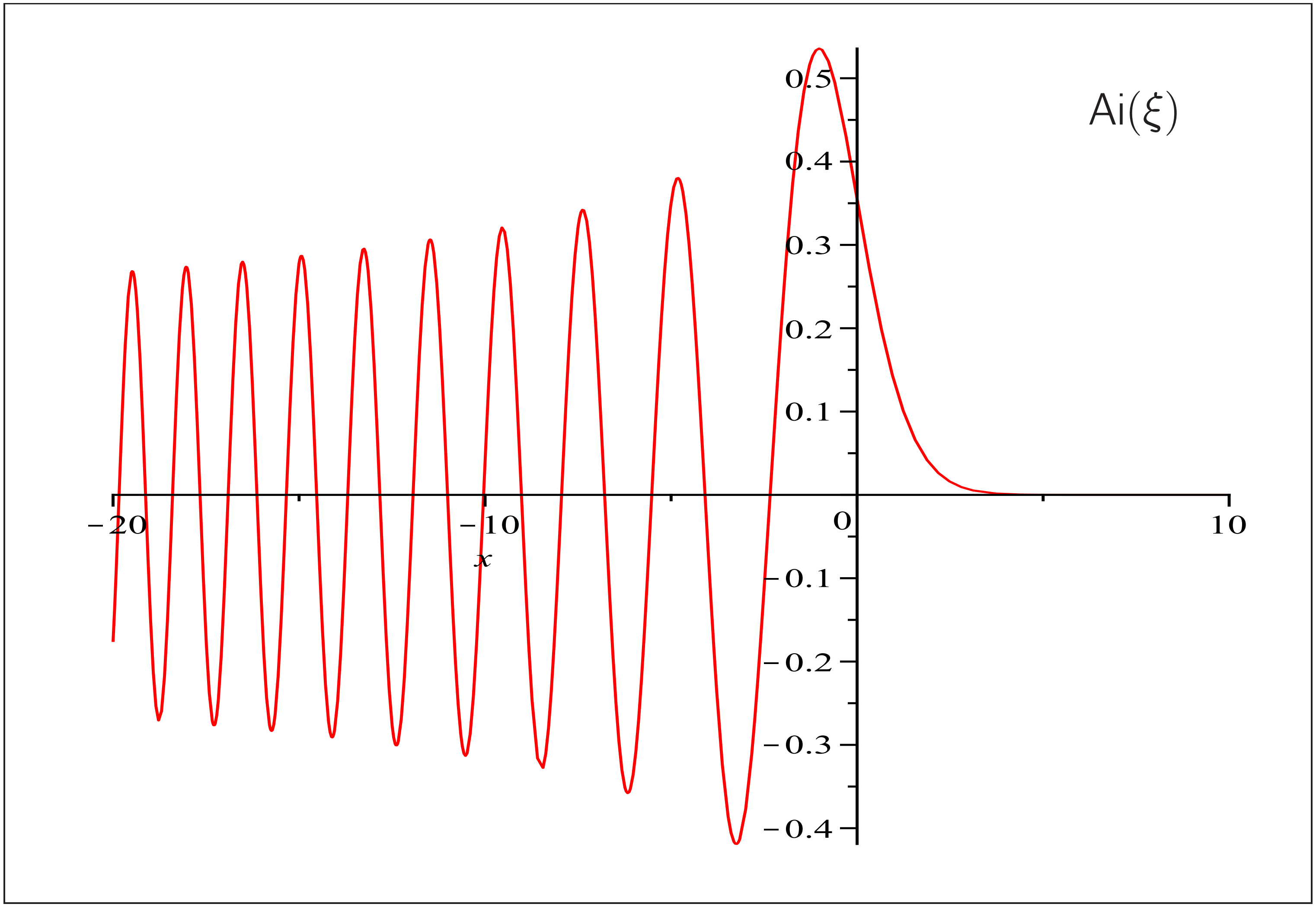
$\zeta = \zeta(t) = (\frac{3}{2}u(t))^{2/3}\,\mathrm{sgn}(1-t)$;

$$u(t) = \begin{cases} \operatorname{arctanh}\left(\sqrt{1-t^2}\right) - \sqrt{1-t^2} & \text{if } 0 < t \leqslant 1 \\ \sqrt{t^2-1} - \arctan\left(\sqrt{t^2-1}\right) & \text{if } t \geqslant 1. \end{cases}$$



$\zeta(t)$

The Airy function:

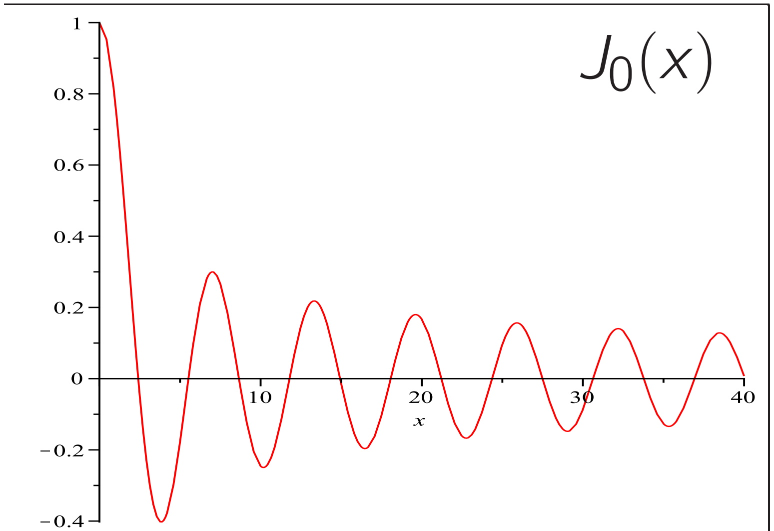


Uniform asymptotics in simplified form:

Fix an arbitrary $C > 0$. Then for all $\nu \geqslant 1$ and $x > 0$ we have

$$J_\nu(x) = \begin{cases} \dfrac{e^{\sqrt{\nu^2-x^2}}}{\sqrt{2\pi}\sqrt[4]{\nu^2-x^2}\left(\frac{\nu}{x}+\sqrt{(\frac{\nu}{x})^2-1}\right)^\nu}\left(1+O\left(\dfrac{\sqrt{\nu}}{(\nu-x)^{3/2}}\right)\right) & \text{if } \ x \leqslant \nu - C\nu^{\frac{1}{3}} \\ O(\nu^{-\frac{1}{3}}) & \text{if } \ |x-\nu| \leqslant C\nu^{\frac{1}{3}} \\ \dfrac{\sqrt{2}}{\sqrt{\pi}\sqrt[4]{x^2-\nu^2}}\left\{\cos\left(\sqrt{x^2-\nu^2}-\nu\arccos\left(\dfrac{\nu}{x}\right)-\dfrac{\pi}{4}\right)+O\left(\dfrac{\sqrt{\nu}}{(x-\nu)^{3/2}}+\dfrac{1}{\nu}\right)\right\} & \\ & \text{if } \ x \geqslant \nu + C\nu^{\frac{1}{3}}. \end{cases}$$

The implied constants *depend only on C.*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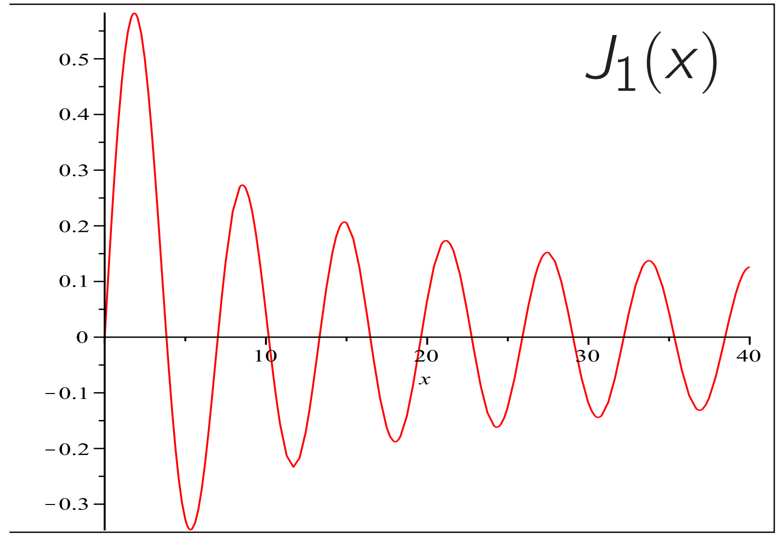
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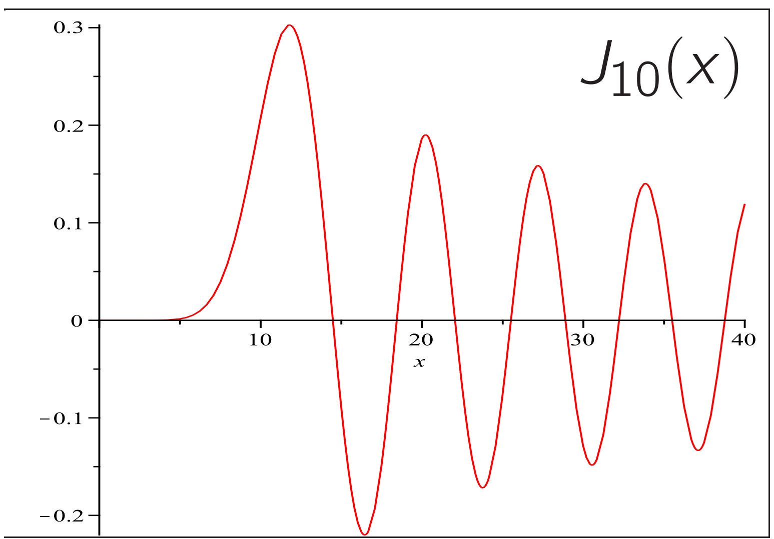
$J_{10}(x)$
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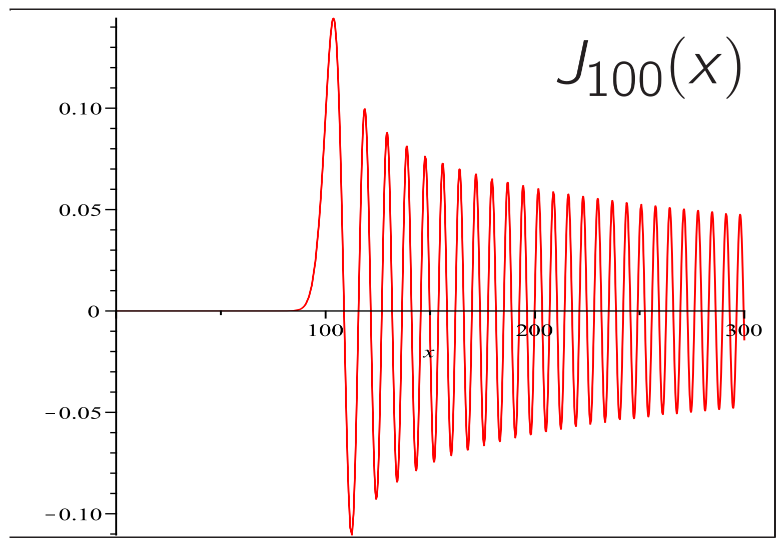
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