Analysis for PhD students (2025); Assignment 1

Problem 1. (a). Prove that for any positive integers N and k,

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma - \sum_{r=1}^{k} \frac{B_r}{r} N^{-r} + \int_N^\infty \widetilde{B}_k(x) \frac{dx}{x^{k+1}},$$

where $\gamma = \frac{1}{2} - \int_1^\infty \widetilde{B}_1(x) \frac{dx}{x^2}$.

(b). Deduce that

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + \frac{1}{2} N^{-1} - \frac{1}{12} N^{-2} + O(N^{-4}), \qquad \forall N \geq 1.$$

(c). Prove also that $\gamma = \int_1^\infty \Big(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \Big)\, dx$.

$(7 + \frac{3}{2} + \frac{3}{2} = 10\text{p})$

Problem 2. Let $0 < \omega_1 \leq \omega_2 \leq \cdots$ be an increasing sequence of positive numbers satisfying

(1) $$\#\{n \in \mathbb{N} \,:\, \omega_n \leq T\} = cT^2 + O(T) \qquad \forall T > 0,$$

where $c > 0$ is some constant. Let $\alpha \leq 2$. Determine an asymptotic formula for $\sum_{\omega_n < T} \omega_n^{-\alpha}$ as $T \to \infty$.

(10p)

Problem 3. Compute the following limits and justify the calculations:

(a) $\displaystyle \lim_{n\to\infty} \int_0^\infty \frac{n \log(1 + \frac{x}{n})}{x(1+x^2)}\, dx$ (b) $\displaystyle \lim_{n\to\infty} \int_0^1 \frac{1 + (nx)^2}{(1+x)^n}\, dx$

(c) $\displaystyle \lim_{n\to\infty} \int_0^\infty \frac{\cos(\frac{x}{n})}{(1+\frac{x}{n})^n}\, dx$ (d) $\displaystyle \lim_{n\to\infty} \int_0^\infty (n+x) e^{-nx}\, dx$

(15p)

Problem 4. Let $E_1, E_2, \ldots$ be measurable subsets of a measure space (X, μ), with $\mu(E_n) < \infty$ for each n. Let $f \in L^1(\mu)$, and assume that $\lim_{n\to\infty} \int_X |f - \chi_{E_n}|\, d\mu = 0$. Prove that $f(x) \in \{0, 1\}$ for μ-almost every $x \in X$.

(10p)

Problem 5. Let $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$. A vector $x \in \mathbb{R}^n$ is said to be of *(linear form) Diophantine type* κ if there exists some $c > 0$ such that for all $q \in \mathbb{Z}^n \setminus \{0\}$ and $m \in \mathbb{Z}$ we have $|qx - m| > c|q|^{-\kappa}$. (Here qx is the standard scalar product of q and x.) Prove that if $\kappa > n$, then almost every $x \in \mathbb{R}^n$ (w.r.t. Lebesgue measure) is of Diophantine type κ.

(15p)

Problem 6. (a) Find an example of a sequence (μ_n) in $M(\mathbb{R})$ such that $\mu_n \to 0$ vaguely, but $\|\mu_n\| \not\to 0$.

(b) Find an example of a sequence (μ_n) in $M(\mathbb{R})$ such that $\mu_n \geq 0$ for every n and $\mu_n \to 0$ vaguely, but there exists some $x \in \mathbb{R}$ such that $\mu_n((-\infty, x]) \not\to 0$.

(c) Let $\mu_n \in M(\mathbb{R})$ be given by $\int_{\mathbb{R}} f\, d\mu_n = \sum_{k=1}^{n} \frac{n-k}{n^2} f(\frac{k}{n})$ for all $f \in C_0(\mathbb{R})$. Prove that the sequence (μ_n) converges vaguely in $M(\mathbb{R})$, and describe the limit measure explicitly.

(15p)

Problem 7. **[Multi-indices]** (a) Prove that for any multi-indices α, β, there is a constant $c_{\alpha,\beta}$ such that

$$\partial^\alpha \left(\frac{1}{x^\beta} \right) = \frac{c_{\alpha,\beta}}{x^{\beta+\alpha}}.$$

Give an explicit formula for $c_{\alpha,\beta}$.

(b) For any multi-index α we write $|\alpha|_\infty := \max(\alpha_1, \ldots, \alpha_n)$. Prove that for any multi-index α, there exist constants $c_{\alpha,m} > 0$ such that

$$\partial^\alpha \exp\Big(\prod_{j=1}^{n} x_j\Big) = \sum_{m=|\alpha|_\infty}^{|\alpha|} c_{\alpha,m} \frac{\prod_{j=1}^{n} x_j^m}{x^\alpha} \exp\Big(\prod_{j=1}^{n} x_j\Big).$$

$\big($Example: $\partial_1^5 \partial_2 \partial_3 \exp(x_1 x_2 x_3) = \big(25 x_2^4 x_3^4 + 11 x_1 x_2^5 x_3^5 + x_1^2 x_2^6 x_3^6\big) \exp(x_1 x_2 x_3).\big)$

(10p)

Problem 8. For any $a > 0$ let $g_a : \mathbb{R} \to \mathbb{R}$ be the function $g_a = a^{-1} \cdot \chi_{(0,a)}$. Let (a_n) be a sequence of positive real numbers and set

$$f_n = g_{a_1} * \cdots * g_{a_n}.$$

(a). Compute $\int_{\mathbb{R}} f_n\, dx$ and $\int_{\mathbb{R}} |f_n|\, dx$.

(b). What is the support of f_n?

(c). Prove that for each $n \geq 2$, $f_n \in C^{n-2}(\mathbb{R})$ but $f_n \notin C^{n-1}(\mathbb{R})$.

(d). Prove that if $\sum_{n=1}^{\infty} a_n = \infty$, then as $n \to \infty$, f_n converges pointwise to 0.

(15p)

[Comment: As (even?) more challenging tasks, you may try to prove that if $\sum_{n=1}^{\infty} a_n < \infty$, then as $n \to \infty$, f_n converges uniformly to a function $f \in C_c^\infty(\mathbb{R})$, $f \not\equiv 0$. Also, when $\sum_{n=1}^{\infty} a_n = \infty$, is the convergence $f_n \to 0$ *uniform* or not?]

To be returned: Tuesday, October 14, before midnight. Please send your solutions by email, as a pdf file. (Either use a scanning app, or write your solutions with LaTeX.)